CHAPTER 3
INDEPENDENT FUNCTIONS

In this chapter we introduce the concept of maximal independent functions, fractional independent domination number $i_r$, fractional independence number $\beta_0$, universal maximal independent function (UMIF) and basic maximal independent function (BMIF) of a graph $G$.

Haynes et al. ([24], Chapter 3, page 85) raised the following question. Can we define the concept of a fractional independent function as a function $g : V \rightarrow [0, 1]$ in such a way that,

(i) the characteristic function of every independent set of vertices is an independent function,

(ii) every maximal independent set of vertices corresponds to a maximal independent function,

(iii) every maximal independent function is also a minimal dominating function.

In this Chapter we present an acceptable definition of fractional independent function and study its properties. Let $G = (V, E)$ be a graph. Let $S \subseteq V$ be an independent set in $G$. Let $f = \chi_s : V \rightarrow \{0, 1\}$ be the characteristic function of $S$. We observe that if $v \in S$, then $f(v) = 1$ and since no neighbour of $v$ is in $S$, we have $\sum_{u \in N(v)} f(u) = 1$. Further if $S$ is a maximal
independent set, then for any \( v \in S \), at least one neighbor of \( v \) is in \( S \), so that

\[
\sum_{u \in N[v]} f(u) \geq 1.
\]

These observations motivate the following definition.

**Definition 3.1.** Let \( G = (V, E) \) be a graph. A function \( f : V \to [0,1] \) is called an *independent function* if for every vertex \( v \) with \( f(v) > 0 \), we have

\[
\sum_{u \in N[v]} f(u) = 1.
\]

An independent function \( f \) is called a *maximal independent function*, if for every \( v \in V \) with \( f(v) = 0 \), we have \( \sum_{u \in N[v]} f(u) \geq 1 \).

We observe that if \( S \) is (a maximal) an independent set in \( G \), then \( f = \chi_S \) is (a maximal) an independent function.

**Remark 3.2.** A function \( f : V \to [0,1] \) is an independent function if and only if \( P_i \subseteq B_j \).

**Theorem 3.3.** Every maximal independent function is a minimal dominating function.

**Proof:** Let \( f \) be a maximal independent function. It follows from the definition that \( \sum_{u \in V} f(u) \geq 1 \) for every \( v \in V \). Hence \( f \) is a dominating function.

Further \( P_i \subseteq B_j \), so that \( B_j \rightarrow P_i \). Hence by Theorem 1.22, \( f \) is a minimal dominating function. \( \square \)
Definition 3.4. The *fractional independence number* $\beta_{fr}(G)$ and the *fractional independent domination number* $i_{fr}(G)$ are defined as follows.

$$\beta_{fr}(G) = \max \{|g| / g \text{ is a maximal independent function of } G\}$$

$$i_{fr}(G) = \min \{|g| / g \text{ is a maximal independent function of } G\}.$$ 

Theorem 3.5. If $f$ is a maximal independent function of $G = (V, E)$ then $B_f$ is a dominating set of $G$.

Proof: Let $v \in V - B_f$. Since $P_f \subseteq B_f$, it follows that $v \notin P_f$ and hence $f(v) = 0$. Since $f(N[v]) \geq 1$, there exists some vertex $u \in N(v)$, such that $f(u) > 0$, and $u \in P_f \subseteq B_f$. Hence $B_f$ dominates all vertices in $V$. Thus $B_f$ is a dominating set of $G$. \[ \square \]

Remark 3.6. Since every maximal independent function is a minimal dominating function, we have $\gamma_f \leq i_f \leq \beta_{fr} \leq \Gamma_f$. Hence we obtain the following fractional domination chain which is analogous to the well-studied domination chain ( [24], page 61):

$$ir_f \leq \gamma_f \leq i_f \leq \beta_{fr} \leq \Gamma_f \leq IR_f.$$ 

Example 3.7. The convex combination of two independent functions may or may not be an independent function.
For example, consider the path $P_1=(v_1,v_2,v_3)$. Let $f_1, f_2$ and $f_3$ be functions defined by, $f_1(v_1) = f_1(v_3) = 0$ and $f_1(v_2) = 1$; $f_2(v_1) = f_2(v_3) = 1$ and $f_2(v_2) = 0$ and $f_3(v_1) = f_3(v_2) = f_3(v_3) = \frac{1}{2}$. Clearly $B_1 = \{v_1,v_2,v_3\}$, $P_1 = \{v_2\}$, $B_2 = \{v_1,v_3\} = P_2$ and $B_3 = \{v_1,v_3\}$, $P_3 = \{v_1,v_2,v_3\}$.

Since $P_1 \subseteq B_1$ and $P_2 \subseteq B_2$, it follows that $f_1$ and $f_2$ are independent functions. Also since $P_3 \supseteq B_3$, it follows that $f_3$ is not an independent function. Further $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$, which is a convex combination of $f_1$ and $f_2$. Thus a convex combination of two independent functions need not be an independent function. We observe that $f_1$ and $f_2$ are both MIFs. Hence a convex combination of two MIFs may not be an independent function.

**Example 3.8.** Consider the graph $G = K_3$, given in Figure 3.1.

![Figure 3.1](image)

Define $f_i : V \rightarrow [0,1]$ by

$$
f_i(v_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ where } i = 1, 2 \text{ and } 3.
$$
Clearly \( P_\lambda = \{ v \} \) and \( B_{\lambda} = \{ v_1, v_2, v_3 \} = V \). Since \( P_\lambda \subseteq B_{\lambda} \), it follows that each \( f_\lambda \) is an independent function.

Now, let \( g : V \rightarrow [0,1] \) be defined by \( g(v_1) = g(v_2) = g(v_3) = \frac{\lambda}{3} \). Then
\[
g = \frac{\lambda}{3} f_1 + \frac{\lambda}{3} f_2 + \frac{\lambda}{3} f_3,
\]
which is a convex combination of \( f_1, f_2 \) and \( f_3 \).

Further \( P_\gamma = B_\gamma = \{ v_1, v_2, v_3 \} \), so that \( g \) is also an independent function. In fact \( g \) is an MIF.

**Remark 3.9.** Let \( f \) and \( g \) be two independent functions and \( 0 < \lambda < 1 \).

Then \( h_\lambda = \lambda f + (1 - \lambda)g \) is an independent function if and only if \( P_\lambda \cup P_\mu \subseteq B_\lambda \cap B_\mu \). Hence either all convex combinations of \( f \) and \( g \) are independent functions or none of them are independent functions. The following lemma shows that a similar result is true for MIFs.

**Lemma 3.10.** Let \( f \) and \( g \) be two MIFs. Then either all convex combinations of \( f \) and \( g \) are MIFs or none of them are MIFs.

**Proof:** Let \( h_\lambda = \lambda f + (1 - \lambda)g \) where \( 0 < \lambda < 1 \). Suppose that \( h_\lambda \) is a MIF and let \( \lambda \neq \lambda_\mu \). We claim that \( h_\lambda \) is an MIF. Let \( v \in V \). Suppose \( h_\lambda(v) = 0 \).

Then \( f(v) = g(v) = 0 \). Since \( f \) and \( g \) are MIFs, we have \( f(N[v]) \geq 1 \) and \( g(N[v]) \geq 1 \). Hence it follows that \( h_\lambda(N[v]) \geq 1 \).
Now suppose \( h_\lambda(v) > 0 \). Then either \( f(v) > 0 \) or \( g(v) > 0 \). Hence \( h_\lambda(v) > 0 \). Since \( h_\lambda \) is an MIF, \( h_\lambda(N[v]) = 1 \), so that \( f(N[v]) = g(N[v]) = 1 \). Hence \( h_\lambda(N[v]) = 1 \). Thus \( P_{h_\lambda} \subseteq B_{h_\lambda} \) and \( h_\lambda(N[v]) \geq 1 \) for all \( v \in V \). Hence \( h_\lambda \) is an MIF.

\[\square\]

**Lemma 3.11.** Let \( f \) and \( g \) be two MIFs of a graph \( G \) and let \( h_\lambda = \lambda f + (1 - \lambda)g \) where \( 0 < \lambda < 1 \). If \( h_\lambda \) is an independent function then \( h_\lambda \) is an MIF.

**Proof:** Since \( f \) and \( g \) are MIFs we have \( f(N[v]) \geq 1 \) and \( g(N[v]) \geq 1 \) for all \( v \in V \). Hence it follows that \( h_\lambda(N[v]) \geq 1 \) for all \( v \in V \), so that \( h_\lambda \) is an MIF.

\[\square\]

**Definition 3.12.** A maximal independent function \( f \) is said to be a universal maximal independent function (UMIF) if the convex combination of \( f \) with any other MIF is again an MIF.

**Lemma 3.13.** A maximal independent function \( f \) of a graph \( G \) is universal if and only if any convex combination of \( f \) with any other MIF is an independent function.

**Proof:** Let \( h_\lambda = \lambda f + (1 - \lambda)g \) where \( 0 < \lambda < 1 \). If \( f \) is universal, obviously \( h_\lambda \) is an independent function.
Conversely, suppose $h_2$ is an independent function. Then it follows from Lemma 3.11 that $h_2$ is a maximal independent function.

**Example 3.14.** Consider the graph $G = K_3$, given in figure 3.1. Let $f : V \rightarrow [0,1]$ be defined by $f(v_1) = 1$ and $f(v_2) = f(v_3) = 0$. We claim that $f$ is a universal MIF. Let $g$ be any MIF of $K_3$. Then $g(v) > 0$ for at least one $v \in V$, so that $g(N[v]) = g(v_1) + g(v_2) + g(v_3) = 1$. Since $N[v_1] = N[v_2] = N[v_3] = \{v_1, v_2, v_3\}$, it follows that $g(N[v_1]) = g(N[v_2]) = g(N[v_3]) = 1$ and hence $P_f \cup P_g \subseteq B_f \cap B_g = V$. Hence $h_2 = \lambda f + (1 - \lambda) g$ is an independent function for all $\lambda$, such that $0 < \lambda < 1$. Hence it follows from Lemma 3.13 that $f$ is a universal MIF.

**Lemma 3.15.** If $g$ is a universal MIF of a graph $G = (V, E)$ then $B_g = V$.

**Proof:** Let $v \in V$ and let $S$ be a maximal independent set of $G$, such that $v \in S$. Clearly $f = \chi_S$ is an MIF of $G$ and $v \in P_f$. Since $g$ is a universal MIF, it follows that any convex combination of $g$ and $f$ is an independent function. Hence $P_f \cup P_g \subseteq B_f \cap B_g$. Hence $v \in B_g$, so that $B_g = V$.

**Definition 3.16.** An MIF $f$ of a graph $G$ is called a *basic maximal independent function* (BMIF) if there does not exist two MIFs $f_1$ and $f_2$ such that $f = \lambda f_1 + (1 - \lambda) f_2$ where $0 < \lambda < 1$. 

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Example 3.17. For the complete graph $K_3$, with vertex set $V = \{v_1, v_2, v_3\}$, the function $f$ defined by

$$f(v_1) = 1 \quad \text{and} \quad f(v_2) = f(v_3) = 0$$

is a basic MIF.

Theorem 3.18. Any basic MIF of a graph $G$ is a basic MDF of $G$.

Proof: Suppose there exists a basic MIF $f$, such that $f$ is not a basic MDF. Then there exist two MDFs $f_i$ and $f_j$ of $G$, such that, $f = \lambda f_i + (1 - \lambda) f_j$ where $0 < \lambda < 1$. Hence $P_i = P_i \cup P_j$ and $B_i = B_i \cap B_j$. Also since $f$ is an MIF, $P_i \subset B_i$. Hence $P_i \subset P_i \subset B_i \subset B_i$. Thus $P_i \subset B_i$ and by similar argument $P_j \subset B_j$. Hence it follows that $f_i$ and $f_j$ are independent functions. Further since $f_i$ and $f_j$ are MDFs, it follows that $f_i$ and $f_j$ are MIFs. Hence $f$ is a convex combination of two MIFs, which is a contradiction. 

Lemma 3.19. Any MDF $f$ of a graph $G$ with $B_i = V$ is an MIF of $G$.

Proof: Since $B_i = V$ it follows that $P_i \subset B_i$ and hence $f$ is an independent function. Also since $f$ is an MDF, $f(N[v]) \geq 1$ for all $v \in V$. Hence $f$ is an MIF.

Corollary 3.20. If $G$ is a graph such that $B_i = V$ for every MDF $f$ of $G$, then $\gamma_f = i_f$ and $\Gamma_f = \beta_{o_f}$.

Proof: It follows from Lemma 3.19 that set of all MIFs of $G$ is equal to the set of all MDFs of $G$. Hence the result follows.
Example 3.21. For the complete graph $K_n$, we have $\gamma_i = i_i = \Gamma_i = \beta_{0_i} = 1$.

Theorem 3.22. If there exist two MIFs $f$ and $g$ of $G$, such that $B_i \cap B_g = \emptyset$, then $G$ has no universal MIF.

Proof: Suppose $G$ has a universal MIF $h$. Then any convex combination of $h$ and $f$ is an independent function and hence $P_h \cup P_f \subseteq B_h \cap B_f$. Similarly $P_h \cup P_g \subseteq B_h \cap B_g$. Hence $P_h \subseteq B_f$ and $P_h \subseteq B_g$. Since $B_f \cap B_g = \emptyset$ we have $P_h = \emptyset$, which is a contradiction. Hence $G$ has no universal MIF. \qed

Corollary 3.23. Any bipartite graph with $\delta \geq 2$ has no universal MIF.

Proof: Let $V_1$ and $V_2$ be the bipartition of $G$. Define $f$ and $g$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ 0 & \text{if } v \in V_2 \end{cases}$$

$$g(v) = \begin{cases} 0 & \text{if } v \in V_1 \\ 1 & \text{if } v \in V_2 \end{cases}$$

Clearly $f$ and $g$ are MIFs with $P_f = B_f = V_1$ and $P_g = B_g = V_2$. Clearly $B_f \cap B_g = V_1 \cap V_2 = \emptyset$. Hence it follows from Theorem 3.22, that $G$ has no universal MIF. \qed

Example 3.24. $i_i (K_{1,n}) = 1$ and $\beta_{0_i} (K_{1,n}) = n$. 

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Proof: Let \( V(K_{1,n}) = \{v_1, v_2, \ldots, v_n\} \), \( d(v) = n \) and \( d(v_i) = 1 \) for all \( i \). Now \( f_1 = \chi_{v_1} \) and \( f_2 = \chi_{v_2, \ldots, v_n} \) are MIFs of \( K_{1,n} \). Let \( g \) be any MIF of \( K_{1,n} \). We claim that \( g = f_1 \) or \( g = f_2 \). If \( g(v) = 0 \) then \( g(v_i) = 1 \) for all \( i \). Hence \( g = f_2 \).

Now suppose \( g(v) = 1 \). If \( g(v_i) = \delta > 0 \) for some \( i \), then \( g(N[v]) = 1 + \delta < 1 \) and hence \( v \notin B_x \). Thus \( P_x \) is not a subset of \( B_x \), and hence \( g \) is not an independent function, which is a contradiction. Hence \( g(v_i) = 0 \) for all \( i \), so that \( g = f_1 \).

Suppose \( g(v) = \lambda \), where \( 0 < \lambda < 1 \). Now, \( g(N[v_i]) = g(v_i) + g(v) \)
\( = g(v_i) + \lambda \geq 1 \). Therefore \( g(v_i) \geq (1 - \lambda) \). By a similar argument, we have \( g(v_i) \geq (1 - \lambda) \) for all \( i = 1, 2, \ldots, n \). Hence \( v \notin B_x \). But \( v \in P_x \subset B_x \), which is a contradiction. Thus \( f_1 \) and \( f_2 \) are the only two MIFs of \( K_{1,n} \). Hence \( i_1(K_{1,n}) = 1 \) and \( \beta_{0f}(K_{1,n}) = n \).

Theorem 3.25. For any two graphs \( G \) and \( H \), we have,

(i) \( i_1(G + H) \leq \min \{i_1(G), i_1(H)\} \).
(ii) \( \beta_{0f}(G + H) \geq \max \{\beta_{0f}(G), \beta_{0f}(H)\} \).

Proof:

(i) let \( f \) and \( g \) be MIFs of \( G \) and \( H \) respectively such that \( |f| = i_1(G) \) and \( |g| = i_1(H) \). Define \( f_1 : V(G) \cup V(H) \rightarrow [0,1] \) by
\[ f_i(v) = \begin{cases} f(v) & \text{if } v \in V(G) \\ 0 & \text{if } v \in V(H). \end{cases} \]

Clearly, \( P_i = P \) and \( B_i = B \). Since \( f \) is an MIF of \( G \), it follows that \( P_i \subseteq B_i \) and hence \( P_i \subseteq B_i \). Thus \( f_i \) is an independent function of \( G \). Since \( f \) is an MIF of \( G \) it follows that \( f_i \) is an MIF of \( G+H \). Hence \( i,(G+H) \leq |f_i| = |f| = i,(G) \). Similarly \( i,(G+H) \leq i,(H) \). Hence \( i,(G+H) \leq \min \{ i,(G),i,(H) \} \).

(ii) Proof of (ii) is similar to that of (i).

\[ \square \]

**Lemma 3.26.** Let \( f \) be an MIF of the complete multipartite graph \( G = K_{r_1, r_2, \ldots, r_n} \) and let \( V_1, V_2, \ldots, V_n \) be the partition of \( V(G) \) with \( |V_i| = r_i \) \((1 \leq i \leq n)\). Then \( f \mid V_i \) is a constant function for all \( i \).

**Proof:** Let \( v_i \in V_i \).

**Case (i) \( f(v_i) = 0 \).**

Suppose there exists a vertex \( v_1 \in V_i \) such that \( f(v_1) > 0 \). Then \( v_2 \) belongs to \( P_i \subseteq B_i \). Hence \( \sum_{v \in V(V_i)} f(x) = f(v_1) + \sum_{v \in V(V_2)} f(x) = 1 \). Since \( f(v_1) > 0 \), it follows that \( \sum_{v \in N(v_1)} f(x) < 1 \). However \( N(v_1) = N(v_2) \) and hence
\[ \sum_{x \in N[v]} f(x) = f(v_1) + \sum_{x \in N(v_1)} f(x) = f(v_1) + \sum_{x \in N[v_1]} f(x) < 1, \] which is a contradiction.

Hence \( f(v) = 0 \) for all \( v \in V_i \).

**Case (ii) \( f(v_i) > 0 \).** It follows from case (i) that \( f(v) > 0 \) for all \( v \in V_i \).

Hence \( f(N[v]) = 1 \) for all \( v \in V_i \). So \( f(v_i) + \sum_{x \in N[v_1]} f(x) = f(v) + \sum_{x \in N[v_i]} f(x) = 1 \) for all \( v \in V_i \).

Since \( N(v_i) = N(v) \), it follows that \( f(v_i) = f(v) \) for all \( v \in V_i \). Hence the result follows. \( \square \)

**Example 3.27.** Consider the graph \( G = K_{2,3,4} \). Let \( V_1, V_2, V_3 \) be a partition of \( G \) with \( |V_1| = 2, |V_2| = 3 \) and \( |V_3| = 4 \). Let \( V_1 = \{v_1, v_2\}, V_2 = \{u_1, u_2, u_3\} \) and \( V_3 = \{w_1, w_2, w_3, w_4\} \). Let \( f \) be an arbitrary MIF of \( G \). Then by Lemma 3.26, \( f(v_i) = f(v_1) = x, f(u_i) = f(u_2) = f(u_3) = y \) and \( f(w_i) = f(w_1) = f(w_3) = f(w_4) = z \).

We have the following cases.

**Case (i) \( x \neq 0, y \neq 0 \) and \( z \neq 0 \).**

Then \( f(N[v]) = 1 \) for all \( v \in V \). Corresponding system of equations is given by,

\[
\begin{align*}
x + 3y + 4z &= 1 \\
2x + y + 4z &= 1 \\
2x + 3y + z &= 1
\end{align*}
\]
Solving for $x, y$ and $z$, we get $x = \frac{y}{2}, y = \frac{y}{3},$ and $z = \frac{y}{3}$.

**Case (ii)** $x = 0, y \neq 0$ and $z \neq 0$.

Then $y + 4z = 1$

$3y + z = 1$.

Solving these equations, we get $x = 0, y = \frac{y}{3},$ and $z = \frac{y}{3}$.

**Case (iii)** $y = 0, x \neq 0$ and $z \neq 0$.

Then $x + 4z = 1$

$2x + z = 1$.

Solving these equations, we get $x = \frac{y}{2}, y = 0$ and $z = \frac{y}{2}$.

**Case (iv)** $z = 0, x \neq 0$ and $y \neq 0$.

Then $x + 3y = 1$

$2x + y = 1$.

Solving these equations, we get $x = \frac{y}{3}, y = \frac{y}{3}$ and $z = 0$.

**Case (v)** $x = y = 0$.

Then $z = 1$.

**Case (vi)** $x = z = 0$.

Then $y = 1$.

**Case (vii)** $y = z = 0$.

Then $x = 1$.

Thus there are exactly seven MIFs which are given below.
\[ f_1(u_i) = \frac{1}{3} \quad f_2(v_i) = \frac{1}{3} \quad f_3(w_i) = \frac{1}{3} \quad \text{with} \quad |f_1| = \frac{1}{3}. \]
\[ f_2(u_i) = 0 \quad f_2(v_i) = \frac{1}{3} \quad f_3(w_i) = \frac{1}{3} \quad \text{with} \quad |f_2| = \frac{1}{3}. \]
\[ f_3(u_i) = \frac{1}{3} \quad f_3(v_i) = 0 \quad f_3(w_i) = \frac{1}{3} \quad \text{with} \quad |f_3| = \frac{1}{3}. \]
\[ f_4(u_i) = \frac{1}{3} \quad f_4(v_i) = \frac{1}{3} \quad f_4(w_i) = 0 \quad \text{with} \quad |f_4| = \frac{1}{3}. \]
\[ f_5(u_i) = 0 \quad f_5(v_i) = 0 \quad f_5(w_i) = 1 \quad \text{with} \quad |f_5| = 4. \]
\[ f_6(u_i) = 0 \quad f_6(v_i) = 1 \quad f_6(w_i) = 0 \quad \text{with} \quad |f_6| = 3. \]
\[ f_7(u_i) = 1 \quad f_7(v_i) = 0 \quad f_7(w_i) = 0 \quad \text{with} \quad |f_7| = 2. \]

Hence \( i_1(G) = \min \{f_1, f_2, \ldots, f_7\} = \min \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, 4, 3, 2\} = \frac{1}{3} \) and
\[ B_{01}(G) = \max \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, 4, 3, 2\} = 4. \]

**Example 3.28.** If \( r, s \geq 2 \) then

(i) \( i_f(K_{r,s}) = \frac{r(s-1)}{(rs-1)} + \frac{s(r-1)}{(rs-1)} \) and

(ii) \( B_{01}(K_{r,s}) = \max \{r, s\} \).

**Proof:**

(i) By Theorem 1.46, \( \gamma_f(K_{r,s}) = \frac{r(s-1)}{rs-1} + \frac{s(r-1)}{rs-1} \).

Hence it is enough to prove that \( i_f(K_{r,s}) = \gamma_f(K_{r,s}) \). Obviously \( \gamma_f(K_{r,s}) \leq i_f(K_{r,s}) \). Further if \( A = \{a_1, a_2, \ldots, a_r\} \) and \( B = \{b_1, b_2, \ldots, b_s\} \) is a bipartition of \( K_{r,s} \), then \( g : V \rightarrow [0,1] \) is defined by,
\[ g(v) = \begin{cases} 
\frac{(s - 1)}{rs - 1} & \text{if } v \in A \\
\frac{(r - 1)}{rs - 1} & \text{if } v \in B 
\end{cases} \]

is an MIF.

Also \( \sum_{v \in A \cup B} g(v) = \frac{r(s - 1)}{rs - 1} + \frac{s(r - 1)}{rs - 1} = \gamma_1(K_{rs}). \) Hence \( i_1(K_{rs}) \leq \gamma_1(K_{rs}). \)

Thus \( i_1(K_{rs}) = \gamma_1(K_{rs}). \)

(ii) Let \( f \) be any MIF of \( K_{rs} \). It follows from Lemma 3.26 that \( f \mid A \) and \( f \mid B \) are both constant functions. If \( f(v) = 0 \) for all \( v \in A \), then \( f(v) = 1 \) for all \( v \in B \). Also if \( f(v) = 0 \) for all \( v \in B \), then \( f(v) = 1 \) for all \( v \in A \). Now suppose \( f(x) > 0 \) for all \( x \in A \cup B \) and let \( f(v) = a \) for all \( v \in A \) and \( f(v) = b \) for all \( v \in B \). Since \( \sum_{x \in A \cup B} f(x) = 1 \) for all \( x \in A \cup B \), it follows that

\[ a + sb = 1 \quad \text{and} \quad ra + b = 1. \]

Hence \( a = \frac{s - 1}{rs - 1} \) and \( b = \frac{r - 1}{rs - 1} \).

Thus there are exactly three MIFs \( f_1, f_2, \) and \( f_3 \) of \( K_{rs} \) and they are given by

\[ f_1(v) = \begin{cases} 
0 & \text{if } v \in A \\
1 & \text{if } v \in B 
\end{cases} \]
\begin{align*}
f_2(v) &= \begin{cases} 
1 & \text{if } v \in A \\
0 & \text{if } v \in B
\end{cases} \\
\text{and} \\
f_3(v) &= \begin{cases} 
\frac{s-1}{rs-1} & \text{if } v \in A \\
\frac{r-1}{rs-1} & \text{if } v \in B.
\end{cases}
\end{align*}

Also \(|f_1| = s, |f_2| = r\) and \(|f_3| < 2\).

Hence it follows that \(\beta_{0/1}(K_{r,s}) = \max \{r, s\}\). \hfill \Box

**Conclusion and scope.** The basic domination, independence and irredundance parameters of a graph \(G\) satisfy the chain of inequality

\[
ir(G) \leq \gamma(G) \leq i(G) \leq \beta_6(G) \leq \Gamma(G) \leq IR(G).
\]

This is called the domination chain of the graph \(G\), which has been the focus of more than 100 research papers. In the case of fractional version of these parameters we have the following chain of inequalities.

\[
ir_f(G) \leq \gamma_f(G) \leq \Gamma_f(G) \leq IR_f(G).
\]

The question of formulating an acceptable definition of fractional independent function has been raised as an interesting open problem ( [24], p85 ). In this chapter we have introduced the concept of independent functions and maximal independent functions, thus answering the question raised in [24]. This leads to the fractional independent domination number
and the fractional independence number $\beta_{0,\gamma}(G)$, thus giving the fractional version of the domination chain:

$$ir_f(G) \leq \gamma_f(G) \leq i_f(G) \leq \beta_{0,\gamma}(G) \leq \Gamma_f(G) \leq IR_f(G).$$

We have also introduced the concept of basic maximal independent function and determined $i_f$ and $\beta_{0,\gamma}$ for several classes of graphs. The following are some interesting problems for further investigation.

(i) Develop properties of universal maximal independent functions (UMIFs). Which graphs possess a UMIF? Obtain a characterization of UMIF of a graph.

(ii) In Corollary 3.20, we have proved that if $B_f = V$ for every MDF $f$ of $G$, then $\gamma_f = i_f$ and $\Gamma_f = \beta_{0,\gamma}$. Is the converse true?

(iii) Characterize the classes of graphs for which

(a) $\gamma_f = i_f$.

(b) $\Gamma_f = \beta_{0,\gamma}$.

(c) $\gamma_f = i_f$ and $\Gamma_f = \beta_{0,\gamma}$.

(iv) In Theorem 3.25, we have proved that

$$i_f(G + H) \leq \min\{i_f(G), i_f(H)\} \quad \text{and}$$

$$\beta_{0,\gamma}(G + H) \geq \max\{\beta_{0,\gamma}(G), \beta_{0,\gamma}(H)\}.$$
Is a similar result true for other graph operations such as cartesian product, weak direct product and strong direct product?

(v) Determine \( i \) and \( \beta_\gamma \) for standard families of graphs such as paths, cycles, regular graphs etc.

(vi) Given six rational numbers \( a, b, c, d, e \) and \( f \) with \( a \leq b \leq c \leq d \leq e \leq f \), find a necessary and sufficient condition for the existence of a graph \( G \) such that \( ir_\gamma(G) = a, \gamma_\gamma(G) = b, i_\gamma(G) = c, \beta_\gamma(G) = d, \Gamma_\gamma(G) = e \) and \( IR_\gamma(G) = f \).

(vii) Another interesting area for further research is to define the edge analogue of independent functions and develop its theory.