CHAPTER 2

DOMINATING AND TOTAL DOMINATING FUNCTIONS

In this chapter we introduce the concept of basic minimal total dominating functions and obtain an algorithm to test whether a given minimal total dominating function is a basic minimal total dominating function. We also introduce the concept of convexity graph $G_{c}(G)$ and basic convexity graph $G_{bc}(G)$ with respect to minimal total dominating functions. We prove that the convexity graph of a disconnected graph is the strong direct product of the convexity graph of its components. We also prove similar results for minimal dominating functions of a graph.

The concepts of total dominating function and minimal total dominating function were introduced by Cockayne et al. [6,11]. Since any convex combination of TDFs is again a TDF, it follows that the set of all TDFs forms a convex set. However the convex combination of two MTDFs need not be an MTDF. Cockayne et al. [11] obtained a necessary and sufficient condition for the convex combination of two MTDFs to be again an MTDF (Theorem 1.35). A similar result for MDFs is given in [15](Theorem 1.25).

In the following theorem we generalize these results for the convex combination of $n$ MTDFs and $n$ MDFs.

**Theorem 2.1.** (i) Let $f_1, f_2, \ldots, f_n$ be $n$ distinct MTDFs. Let $g = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$, where $0 < \lambda_i < 1$ and $\sum_{i=1}^{n} \lambda_i = 1$, be a convex
combination of $f_1, f_2, ..., f_n$. Then $g$ is an MTDF if and only if $B_{l_1} \cap B_{l_2} \cap ... \cap B_{l_n} \rightarrow P_{l_1} \cup P_{l_2} \cup ... \cup P_{l_n}$.

(ii) Let $f_1, f_2, ..., f_n$ be $n$ distinct MDFs. Let $g = \lambda_1 f_1 + \lambda_2 f_2 + ... + \lambda_n f_n$, where $0 < \lambda_i < 1$ and $\sum_{i=1}^{n} \lambda_i = 1$, be a convex combination of $f_1, f_2, ..., f_n$. Then $g$ is an MDF if and only if $B_{l_1} \cap B_{l_2} \cap ... \cap B_{l_n} \rightarrow P_{l_1} \cup P_{l_2} \cup ... \cup P_{l_n}$.

**Proof:** (i) We first claim that $B_{x} = B_{l_1} \cap B_{l_2} \cap ... \cap B_{l_n}$ and $P_{x} = P_{l_1} \cup P_{l_2} \cup ... \cup P_{l_n}$.

Let $v \in B_{l_1} \cap B_{l_2} \cap ... \cap B_{l_n}$. Then $\sum_{i=1}^{n} f_i(v) = 1$ for all $i$. Hence $\sum_{i=1}^{n} g(u) \left( (N(v)) = \sum_{i=1}^{n} \lambda_i = 1 \right.$. Hence $v \in B_{x}$, so that $\bigcap_{i=1}^{n} B_{l_i} \subseteq B_{x}$. Also if $v \notin B_{l_i}$ for some $i$, then $f_i(N(v)) > 1$. Hence it follows that $g(N(v)) > 1$. Hence $v \notin B_{x}$. Thus $B_{x} \subseteq \bigcap_{i=1}^{n} B_{l_i}$. Hence $B_{x} = \bigcap_{i=1}^{n} B_{l_i}$.

Now, let $v \in \bigcup_{i=1}^{n} P_{l_i}$. Hence $v \in P_{l_i}$ for some $i$, so that $f_i(v) > 0$. Hence it follows that $g(v) > 0$, so that $v \in P_{x}$. Thus $\bigcup_{i=1}^{n} P_{l_i} \subseteq P_{x}$. Further if $v \notin \bigcup_{i=1}^{n} P_{l_i}$, then $v \notin P_{l_i}$ for each $i$. Hence $f_i(v) = 0$ for each $i$, so that $g(v) = 0$. Thus $P_{x} \subseteq \bigcup_{i=1}^{n} P_{l_i}$. Hence $P_{x} = \bigcup_{i=1}^{n} P_{l_i}$. Now by Theorem 1.35, $g$ is an MTDF if and only if $B_{x} \rightarrow P_{x}$. Hence $g$ is an MTDF if and only if $\bigcap_{i=1}^{n} B_{l_i} \rightarrow \bigcup_{i=1}^{n} P_{l_i}$.
(ii) The proof is similar.

\[ \square \]

**Example 2.2.** Consider the graph $G$ given in Figure 2.1.

Figure 2.1.

Define $f_1$, $f_2$ and $f_3$ as follows:

- $f_1(v_1) = 0$, $f_1(v_2) = 0$, $f_1(v_3) = 1$ and $f_1(v_4) = 1$.
- $f_2(v_1) = 0$, $f_2(v_2) = 1$, $f_2(v_3) = 0$ and $f_2(v_4) = 1$.
- $f_3(v_1) = 1$, $f_3(v_2) = 0$, $f_3(v_3) = 0$ and $f_3(v_4) = 1$.

Clearly, $f_1$, $f_2$ and $f_3$ are MTDFs of $G$. Let $g = \frac{1}{3} f_1 + \frac{1}{3} f_2 + \frac{1}{3} f_3$. Then $g(v_1) = \frac{1}{3}$, $g(v_2) = \frac{1}{3}$, $g(v_3) = \frac{1}{3}$ and $g(v_4) = 1$. Hence $P_x = \{v_1, v_2, v_3, v_4\}$ and $B_x = \{v_1, v_2, v_3, v_4\}$. Clearly $B_x \rightarrow P_x$, so that $g$ is an MTDF. Thus $g$ is a convex combination of three MTDFs.

**Remark 2.3**[29]. Let $S$ be a convex set. An element $x \in S$ is an extreme point of $S$ if there do not exist points $x_1, x_2 \in S$ such that
\[ x = \lambda x_i + (1 - \lambda)x_j, \] where \( 0 < \lambda < 1 \). In other words \( x \) is an extreme point of \( S \) if it cannot be expressed as a convex combination of two distinct elements of \( S \). Since the set of all feasible solutions of a linear programming problem (LPP) forms a convex set and the optimal solution of the LPP occurs at an extreme point of this convex set, it is natural to study the extreme points of the convex set of all TDFs of a graph.

**Definition 2.4.** Let \( G = (V, E) \) be a graph without isolated vertices.

A TDF (MTDF) is called a basic TDF or simply BTDF (basic MTDF or BMTDF), if it cannot be expressed as a proper convex combination of two distinct TDFs (MTDFs).

Similarly we can define basic DF (BDF) and basic MDF (BMDF).

**Remark 2.5.** Let \( G = (V, E) \) be a graph without isolated vertices. Then any BMTDF of \( G \) is obviously a BTDF. However the converse is not true. For example, if \( |V| \geq 3 \), then \( f : V \to [0,1] \) defined by \( f(v) = 1 \) for all \( v \in V \) is a BTDF but not a BMTDF. In fact \( G = K_2 \) is the only connected graph in which every BTDF is also a BMTDF.

Similar result is true for BMDFs.

**Example 2.6.** Consider the graph \( G \) given in Figure 2.1. Define \( f(v_1) = 1, f(v_2) = 0, f(v_3) = 0 \) and \( f(v_4) = 1 \). Clearly \( f \) is an MTDF. We
claim that $f$ is a BMTDF. Suppose these exist distinct MTDFs $f_1, f_2, \ldots, f_n$ such that $f = \lambda_1 f_1 + \ldots + \lambda_n f_n$, where $0 < \lambda_i < 1$ for all $i$ and $\sum \lambda_i = 1$. Since $f(v_i) = 1$ we have $\lambda_1 f_1(v_i) + \lambda_2 f_2(v_i) + \ldots + \lambda_n f_n(v_i) = 1$. Since $0 < \lambda_i < 1$ and $\sum \lambda_i = 1$ it follows that $f_i(v_i) = 1$ for all $i$. Similarly $f(v_i) = 1$ implies $f_i(v_i) = 1$ for $i = 1, 2, \ldots, n$.

Now $f(v_1) = 0$ gives $\lambda_1 f_1(v_2) + \lambda_2 f_2(v_2) + \ldots + \lambda_n f_n(v_2) = 0$. Since $0 < \lambda_i < 1$ this is possible if and only if $f_i(v_2) = 0$ for all $i$. Similarly $f_i(v_i) = 0$ for all $i$. This shows that $f_1 = f_2 = \ldots = f_n = f$. Hence $f$ is a BMTDF.

**Lemma 2.7.** (i) If the convex combination of a collection of MTDFs $g_1, g_2, \ldots, g_n$ is minimal, then the convex combination of any proper sub-collection of $g_1, g_2, \ldots, g_n$ is also minimal.

(ii) If the convex combination of a collection of MDFs $g_1, g_2, \ldots, g_n$ is minimal, then the convex combination of any proper sub-collection of $g_1, g_2, \ldots, g_n$ is also minimal.

**Proof:** Since the convex combination of $g_1, g_2, \ldots, g_n$ is minimal it follows from Theorem 2.1 that $B_{g_1} \cap B_{g_2} \cap \ldots \cap B_{g_n} \rightarrow P_{g_1} \cup P_{g_2} \cup \ldots \cup P_{g_n}$. Let
Let \( \{ g_1', g_2', \ldots, g_m' \} \) be a subcollection of \( \{ g_1, g_2, \ldots, g_n \} \). Then \( \bigcap_{j=1}^m B_{g_j'} \supseteq \bigcap_{i=1}^n B_{g_i} \)
and \( \bigcup_{j=1}^m P_{g_j'} \subseteq \bigcup_{i=1}^n P_{g_i} \). Hence \( \bigcap_{j=1}^m B_{g_j'} \to \bigcup_{j=1}^m P_{g_j'} \), so that convex combination
of \( g_1', g_2', \ldots, g_m' \) is an MTDF by Theorem 2.1.

(ii) The proof is similar. \( \square \)

**Lemma 2.8.** Let \( f \) and \( g \) be two MTDFs. Let \( i \) be any real number. Define
\[ f_i = (1 + i)g - if. \]
Let \( \nu \in V \).

(i) If \( f(v) > g(v) \), then \( f_i(v) \) is a strictly monotonic decreasing
function of \( i \) and is not bounded below.

(ii) If \( f(v) < g(v) \), then \( f_i(v) \) is a strictly monotonic increasing
function of \( i \) and is not bounded above.

(iii) If \( f(v) = g(v) \), then \( f_i(v) \) a constant function.

(iv) If \( f(N(v)) > g(N(v)) \), then \( f_i(N(v)) \) is a strictly monotonic
decreasing function of \( i \) and is not bounded below.

(v) If \( f(N(v)) < g(N(v)) \), then \( f_i(N(v)) \) is a strictly monotonic
increasing function of \( i \) and is not bounded above.

(vi) If \( f(N(v)) = g(N(v)) \), then \( f_i(N(v)) \) is a constant function.

**Proof:**

(i) Let \( f(v) > g(v) \) and \( i > j \). We have \( f_i(v) = (1 + i)g(v) - i \ f(v) \)
\[ g(v) + i[g(v) - f(v)] \] 
\[ \text{Hence } f_i(v) - f_j(v) = (i - j)(g(v) - f(v)) \]

Since \( i - j > 0 \) and \( g(v) - f(v) < 0 \), it follows that \( f_i(v) - f_j(v) \) is negative.

Thus \( f_i(v) < f_j(v) \). Hence \( f_i(v) \) is a strictly monotonic decreasing function of \( i \).

Also \( \lim_{i \to \infty} [g(v) + i(g(v) - f(v))] = -\infty \). This shows that \( f_i(v) \)

is not bounded below.

(ii) Let \( f(v) < g(v) \) and \( i > j \). Then it follows from (2) that

\[ (f_i(v) - f_j(v)) > 0 \]

Hence \( f_i(v) > f_j(v) \), so that \( f_i(v) \) is a strictly monotonic increasing function of \( i \) and is not bounded above.

(iii) If \( g(v) = f(v) \) then it follows from (1) that \( f_i(v) = g(v) \) for all \( i \).

(iv) From (1) and (2) we have

\[ f_i(N(v)) = g(N(v)) + i[g(N(v)) - f(N(v))] \] 

\[ f_i(N(v)) - f_j(N(v)) = (i - j)(g(N(v)) - f(N(v))) \]

Now suppose \( f(N(v)) > g(N(v)) \) and \( i > j \). Then it follows from (4) that

\( f_i(N(v)) < f_j(N(v)) \). Hence \( f_i(N(v)) \) is strictly monotonic decreasing.

Further \( f_i(N(v)) \) is not bounded below.

(v) Similarly if \( f(N(v)) < g(N(v)) \), then it follows from (4) that \( f_i(N(v)) \)

is a strictly monotonic increasing function of \( i \) and is not bounded above.
(vi) If \( f(N(v)) = g(N(v)) \), it follows from (3) that \( f_i(N(v)) = g(N(v)) \) for all \( i \), so that \( f_i(N(v)) \) is a constant function. \( \Box \)

**Lemma 2.9.** Let \( f \) and \( g \) be two MTDFs with \( P_f = P_g \) and \( B_f = B_g \). Then

(i) \( f(v) = 1 \) if and only if \( g(v) = 1 \).

(ii) \( f(v) = 0 \) if and only if \( g(v) = 0 \).

(iii) \( f(N(v)) = 1 \) if and only if \( g(N(v)) = 1 \).

**Proof:** (i) Suppose \( f(v) = 1 \). Then \( v \in P_f \). Since \( f \) is an MTDF, by Theorem 1.34, there exists \( u \in B_f \) such that \( u \) is adjacent to \( v \).

Since \( u \in B_f \), we have \( \sum_{x \in N(u)} f(x) = 1 \). Thus \( f(v) + \sum_{x \in N(u) \setminus \{v\}} f(x) = 1 \) \( \quad (1) \). Since \( f(v) = 1 \), it follows that \( f(x) = 0 \) for all \( x \in N(u) \setminus \{v\} \). Also since \( B_f = B_g \) and \( u \in B_f \), we have \( u \in B_g \). Hence \( \sum_{x \in N(u)} g(x) = g(v) + \sum_{x \in N(u) \setminus \{v\}} g(x) = 1 \).

If \( g(v) < 1 \), there exists \( y \in N(u) \setminus \{v\} \) such that \( g(y) > 0 \). Hence \( y \in P_g = P_f \), so that \( f(y) > 0 \), which is a contradiction. Hence \( g(v) = 1 \). Similarly if \( g(v) = 1 \) then it follows that \( f(v) = 1 \).

(ii) \( f(v) = 0 \iff v \not\in P_f = P_g \iff g(v) = 0 \).

(iii) \( f(N(v)) = 1 \iff v \in B_f = B_g \iff g(N(v)) = 1 \). \( \Box \)

**Remark 2.10.** We can prove results similar to Lemma 2.8 and Lemma 2.9 for MDFs of a graph. It is to be noted that we have to replace the closed
neighbourhood of a vertex instead of open neighbourhood wherever necessary in the proof.

**Notation 2.11.** Let $f$ and $g$ be two MTDFs of a graph $G$, with $B_f = B_g$ and $P_f = P_g$. Let $i$ be any real number and let $h_i = (1 + i)g - if$. Now, let $S$ be the subset of the set of real numbers $\mathbb{R}$ defined by

$$S = \{ i / i \in \mathbb{R}, h_i \text{ is a TDF}, B_{h_i} = B_f \text{ and } P_{h_i} = P_f \}.$$  

Clearly $h_0 = g$ and $h_{-1} = f$ and hence $0, -1 \in S$. In general $h_i$ need not be a TDF, since $h_i$ becomes negative for large values of $i$.

**Lemma 2.12.** If $i \in S$, then $h_i$ is an MTDF of $G$.

**Proof:** It follows from the definition of $S$, that $h_i$ is a total dominating function, $B_{h_i} = B_f$ and $P_{h_i} = P_f$. Also since $f$ is an MTDF $B_f \rightarrow P_f$, and hence $B_{h_i} \rightarrow P_{h_i}$. Hence it follows from Theorem 1.34, that $h_i$ is an MTDF.

**Lemma 2.13.** $S$ is a bounded open interval of $\mathbb{R}$.

**Proof:** Clearly $-1, 0 \in S$. We first prove that $S$ is an interval. Let $m, n \in S$ and $m < p < n$. Let $v \in V$. Because of Lemma 2.8, we may assume without loss of generality, that

$$h_m(v) \leq h_p(v) \leq h_n(v) \quad (1)$$
and \( h_m(N(v)) \leq h_p(N(v)) \leq h_n(N(v)) \). \( \quad (2) \)

Also it follows from the definition of \( S \) that

\[
P_{h_m} = P_{h_n} = P_f
\]

and \( B_{h_m} = B_{h_n} = B_f \). \( \quad (3) \) \( \quad (4) \)

Since \( h_m \) and \( h_n \) are TDFs, it follows from (1) and (2) that \( h_p \) is also a TDF. We now claim that \( B_{h_p} = B_f \) and \( P_{h_p} = P_f \). Let \( v \in P_f \). Then \( f(v) > 0 \) and hence it follows from (3) that \( h_p(v) > 0 \). Thus \( v \in P_{h_p} = P_{h_n} \).

Hence \( h_m(v) > 0 \) and hence by (1) we have \( h_p(v) > 0 \). Thus \( v \in P_{h_p} \). Hence \( P_f \subseteq P_{h_p} \). Now let \( v \in P_{h_p} \). Then \( h_p(v) > 0 \) and hence \( h_n(v) > 0 \). So \( v \in P_{h_p} = P_f \) and thus \( P_{h_p} \subseteq P_f \). Thus \( P_{h_p} = P_f \). By a similar argument, \( B_{h_m} = B_{h_n} = B_f \). Hence \( p \in S \), so that \( S \) is an interval.

Next we claim that \( S \) is bounded. Since \( f \neq g \), there exists a vertex \( v \in V \) such that \( f(v) \neq g(v) \) and \( f(v), g(v) > 0 \). Without loss of generality we assume that \( f(v) > g(v) \).

Now \( h_i(v) = (1 + i)g(v) - if(v) = g(v) + i[g(v) - f(v)] \).

Let \( n_i = \frac{g(v)}{f(v) - g(v)} \). Clearly \( n_i > 0 \). Also \( h_n(v) = 0 \). Since \( h_i(v) \) is a monotonic decreasing function of \( i \), we have \( h_i(v) < 0 \) for all \( j > n_i \). Thus
$j \notin S$ if $j > n_1$ and hence $S$ is bounded above. Now let $n_2 = \frac{(g(v) - 1)}{f(v) - g(v)}$. Then $h_{n_2}(v) = 1$ and $h_j(v) > 1$ for all $j < n_1$. Hence it follows that $j \notin S$, if $j < n_2$, so that $S$ is bounded below. Thus $S$ is bounded.

We now proceed to prove that $S$ is open. To prove this we shall show that given $r \in S$, there exist $x < r$ and $y > r$ such that $x, y \in S$. Since $S$ is an interval and $0, -1 \in S$, it follows that either there exists a real number $x$ such that $x < r$ and $x \in S$ or there exists a real number $y$ such that $y > r$ and $y \in S$.

Without loss of generality, we assume that there exists $y > r$ such that $y \in S$. We now claim that, there exists $x < r$ such that $x \in S$. We shall first prove that there exists $x < r$ such that $h_x$ is a TDF. Suppose $h_x$ is not a TDF for all $x < r$. We claim that there exists a vertex $v \in V$ such that one of the following holds.

(a) $h_x(v) > 1$ and $h_r(v) = 1$
(b) $h_x(v) < 0$ and $h_r(v) = 0$
(c) $h_x(N(v)) < 1$ and $h_r(N(v)) = 1$.

Since $h_x$ is not a TDF, there exists $v \in V$ such that either $h_x(v) < 0$ or $h_x(v) > 1$ or $h_x(N(v)) < 1$. Let $V_i = \{v_1, v_2, ..., v_n\}$ be the set of all vertices
for which \( h_i(v_i) > 1 \). Let \( V_2 = \{ u_1, u_2, ..., u_n \} \) be the set of all vertices in \( V \) for which \( h_i(u_i) < 0 \). Let \( V_3 = \{ w_1, w_2, ..., w_n \} \) be the set of all vertices of \( V \) for which \( h_i(N(w_i)) < 1 \). (The possibility of \( V_1, V_2 \) or \( V_3 \) being equal to empty set is not ruled out). Now suppose \((a), (b) \) and \((c) \) are not true. Then
\[
 h_i(v) < 1 \quad \text{for all} \quad v \in V_1, \quad h_i(v) > 0 \quad \text{for all} \quad v \in V_2 \quad \text{and} \quad h_i(N(v)) > 1 \quad \text{for all} \quad v \in V_3.
\]

Now, \( h_i(v) > 1 > h_i(v') \) for all \( v, v' \in V_1 \) and since \( x < r \) it follows that the sequence \( h_i(v) \) is a strictly monotonic decreasing function for all \( v \in V_1 \).

Let \( a_i \) be a real number such that \( x < a_i < r \) and \( h_i(v_i) = 1 \) for each \( i \).

Let \( a = \max \{ a_1, a_2, ..., a_n \} \). Now, since \( a_i \leq a \) we have \( h_i(v_i) \geq h_a(v_i) \).

Hence \( h_a(v_i) \leq 1 \) for all \( v_i \in V_1 \). Also by definition of \( V_1 \) we have \( h_a(v) \leq 1 \) for all \( v \in V - V_1 \).

Thus \( h_a(v) \leq 1 \) for all \( v \in V \). Now, \( h_i(u_i) < 0 < h_i(u) \) for all \( u_i \in V_2 \) and since \( x < r \), \( h_i(u) \) is a strictly monotonic increasing function for all \( u_i \in V_2 \). Let \( b_i \) be a real number such that \( x < b_i < r \) and \( h_b(u_i) = 0 \). Let \( b = \max \{ b_1, b_2, ..., b_n \} \). Now, since \( b_i \leq b < r \), we have \( h_b(u_i) \leq h_b(u_i) < h_b(u) \). Hence \( 0 \leq h_b(u_i) \) for all \( u_i \in V_2 \). Also by definition of \( V_2 \), we have \( h_b(v) \geq 0 \) for all \( v \in V - V_2 \). Hence \( h_b(v) \geq 0 \) for all \( v \in V \).

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Now, $h_i(N(w_i)) < 1 < h_i(N(w_i))$ for all $w_i \in V_i$ and since $x < r$, $h_i(N(w_i))$ is a strictly monotonic increasing function for $w_i \in V_i$. Let $c_i$ be a real number such that $x < c_i < r$ and $h_i(N(w_i)) = 1$. Let $c = \max\{c_1, c_2, \ldots, c_n\}$. Since $c_i \leq c < r$, we have $h_i(N(w_i)) \leq h_i(N(w_i)) \leq h_i(N(w_i))$ for all $w_i \in V_i$. Hence $h_i(N(w_i)) \geq 1$ for all $w_i \in V_i$. Also by definition of $V_i$, we have $h_i(N(v)) \geq 1$ for all $v \in V - V_i$. Thus $h_i(N(v)) \geq 1$ for all $v \in V$. Now let $\alpha = \max\{a, b, c\}$. Clearly $x < \alpha < r$. We now claim that $h_\alpha$ is a TDF.

If $v \in V_1$ then $h_\alpha(v)$ is a strictly monotonic decreasing function. Further $a \leq \alpha$ and $h_\alpha(v) \leq 1$. Hence it follows that $h_a(v) \leq h_\alpha(v) \leq 1$. Also if $v \notin V_1$, then $h_i(v) \leq 1$ (by definition of $V_1$), $h_\alpha(v) \leq 1$ and $x < \alpha \leq \alpha < r$. Hence it follows that $h_\alpha(v) \leq 1$. Thus $h_a(v) \leq 1$ for all $v \in V$.

If $v \in V_2$ then $h_\alpha(v)$ is a strictly monotonic increasing function. Further $b \leq \alpha$ and $h_\alpha(v) \geq 0$. Hence it follows that $h_a(v) \geq h_\alpha(v) \geq 0$ if $v \in V_2$. If $v \notin V_2$ then $h_i(v) \geq 0$ (by definition of $V_2$). Also $h_i(v) \geq 0$ and $x < b < \alpha < r$. Hence it follows that $h_a(v) \geq 0$ for all $v \in V$.

If $v \in V_3$, then $h_i(N(v))$ is a strictly monotonic increasing function. Also $c \leq \alpha$ and $h_i(N(v)) \geq 1$. Hence it follows that $h_a(N(v)) \geq h_i(N(v)) \geq 1$.

If $v \notin V_3$, then by definition of $V_3$, $h_i(N(v)) \geq 1$. So $h_i(N(v)) \geq 1$ and
Thus $h_a(N(v)) \geq 1$ for all $v \in V$. Hence $h_a$ is a TDF and \alpha < r, which is a contradiction. This shows that (a), (b) or (c) holds.

**Case (i)** Suppose (a) holds.

Then there exists a vertex $v \in V$ such that $h_a(v) > 1$ and $h_a(v) = 1$. Now it follows from Lemma 2.9 that $f(v) = g(v) = 1$ and hence $h_a(v) = (1 + x)g(v) - xf(v) = 1$, which is a contradiction.

**Case (ii)** Suppose (b) holds.

Then there exists a vertex $v \in V$ such that $h_a(v) < 0$ and $h_a(v) = 0$. Now it follows from Lemma 2.9 that $f(v) = g(v) = 0$ and $h_a(v) = (1 + x)g(v) - xf(v) = 0$, which is a contradiction.

**Case (iii)** Suppose (c) holds.

Then there exists a vertex $v \in V$ such that $h_a(N(v)) < 1$ and $h_a(N(v)) = 1$. Also from Lemma 2.9 we have $g(N(v)) = f(N(v)) = 1$. Hence $h_a(N(v)) = (1 + x)g(N(v)) - xf(N(v)) = 1$, which is again a contradiction.

Hence it follows that there exists a real number $x_i < r$ such that $h_{x_i}$ is a TDF. Since any convex combination of two TDFs is again a TDF, it follows that $h_x$ is a TDF for all $x \in [x_i, r]$. We now claim that there exists a real number $x_1$ such that $x_i \leq x_1 < r$ and $B_{x_i} = B_x$ for all $x \in [x_1, r]$. If
$B_n = B_i$ for all $x \in [x_i, r)$, then we take $x_2 = x_i$. Suppose there exists $x \in [x_i, r)$ such that $B_n \neq B_i$. We claim that if $s, t \in [x_i, r)$, and $B_n = B_n$, then $B_n = B_n = B_n$. Let $v \in B_n$. Then $h_i(N(v)) = 1$. Since $B_n = B_i = B_n$, we have $f(N(v)) = g(N(v)) = 1$.

Hence $h_i(N(v)) = (1 + s)g(N(v)) - sf(N(v)) = 1$, so that $v \in B_n$. Hence $B_n \subseteq B_n$. Now, let $v \in B_n$. Then $h_i(N(v)) = h_i(N(v)) = 1$. Hence $(1 + s)g(N(v)) - sf(N(v)) = 1$ and $(1 + t)g(N(v)) - tf(N(v)) = 1$. Hence $(s - t)(g(N(v)) - f(N(v))) = 0$. Since $s \neq t$, we have $g(N(v)) = f(N(v))$ so that $g(N(v)) = f(N(v)) = 1$. Hence $v \in B_i = B_n$ so that $B_n \subseteq B_n$. Thus $B_n = B_n = B_n = B_i$. Since $B_n \subseteq V$ and $V$ is finite, it follows that $B_n = B_i$ for all but a finite number of real numbers $x \in [x_i, r)$. Hence there exists $x_2 \in [x_i, r)$ such that $B_n = B_n$ for all $x \in [x_2, r)$.

Now we claim that $P_n = P_n$ for all $x \in [x_2, r)$. Let $v \in P_n$. Then $h_i(v) > 0$. Suppose $h_i(v) = 0$. Since $x < r$, it follows that $h_i(v)$ is a strictly monotonic decreasing function. Hence $h_i(v) < 0$ for all $j > r$. Thus $j \notin S$ for all $j > r$, which is a contradiction. Hence $h_i(v) > 0$, so that $P_n \subseteq P_n$.

Now, let $v \in P_n$. If $v \notin P_n$, then $h_i(v) > 0$ and $h_i(v) = 0$. Hence $h_i(v)$ is a strictly monotonic increasing function, so that $h_i(v) < 0$ for all $i < x$. In
particular \( h_i(v) < 0 \), which is a contradiction. Hence \( h_i(v) > 0 \), so that
\( v \in P_{h_i} \). Thus \( P_{h_i} \subseteq P_{h_i} \) and hence \( P_{h_i} = P_{h_i} \). Hence it follows that \([x, r] \subseteq S\).
Thus \( S \) is open.

\[ \square \]

**Lemma 2.14.** If \( S = (k, k_2) \), then

(i) \( h_i \) and \( h_i \) are MTDFs.

(ii) \( B_{h_i} \neq B_{h_i} \) or \( P_{h_i} \neq P_{h_i} \).

(iii) \( B_{h_i} \neq B_{h_i} \) or \( P_{h_i} \neq P_{h_i} \).

(iv) \( f \) is a convex combination of \( h_i \) and \( h_i \).

**Proof:** (i) Let \( \{x_n\} \) be a monotonic decreasing sequence in \((k, k_2)\) such
that \( \lim_{n \to \infty} x_n = k_1 \). Then
\[
\lim_{n \to \infty} h_i = \lim_{n \to \infty} ((1 + x_n)g(v) - x_nf(v)) = (1 + k_1)g(v) - k_1f(v) = h_i(v).
\]
Now since \( 0 \leq h_i(v) \leq 1 \) for all \( v \in V \), it follows that
\( 0 \leq h_i(v) \leq 1 \) for all \( v \in V \). Now
\[
h_i(N(v)) = \sum_{u \in N(v)} h_i(u) = \sum_{u \in N(v)} \left( \lim_{n \to \infty} h_i(u) \right) = \lim_{n \to \infty} \left( \sum_{u \in N(v)} h_i(u) \right) \geq 1 \] (Since \( \sum_{u \in N(v)} h_i(u) \geq 1 \)). Thus \( h_i \) is a TDF. Now, let \( s \in S \).
Then \( B_{h_i} \supseteq B_{h_i} \) and \( P_{h_i} \subseteq P_{h_i} \). Since \( h_i \) is an MTDF, we have \( B_{h_i} \to P_{h_i} \).

Hence it follows that \( B_{h_i} \to P_{h_i} \), so that \( h_i \) is a MTDF. By a similar
argument, we can prove that \( h_i \) is also an MTDF. Now since \( k_1, k_2 \in S \), (ii) and (iii) follow.

(iv) We have \( h_i = (1+k_i)g - k_if \) and \( h_i = (1+k_2)g - k_2f \). Solving for \( f \) we get

\[
 f = \left( \frac{1+k_2}{k_2-k_1} \right) h_i - \left( \frac{1+k_1}{k_2-k_1} \right) h_i. \]

Now \( \left( \frac{1+k_2}{k_2-k_1} \right) + \left( \frac{-(1+k_1)}{k_2-k_1} \right) = 1 \).

Since \( k_2 > 0 \) and \( k_1 < -1 \) it follows that \( 0 < \frac{1+k_2}{k_2-k_1} < 1 \). Hence \( f \) is a convex combination of \( h_i \) and \( h_i \).

\[ \square \]

**Remark 2.15.** Let \( f \) and \( g \) be two MDFs of \( G=(V,E) \) such that \( B_i = B_x \) and \( P_i = P_c \). Let \( h_i \) be defined as in Notation 2.11. Let \( S' = \{ i \in \mathbb{R} \mid h_i \text{ is a DF, } B_i = B_1 \text{ and } P_i = P_1 \} \). We can prove the following properties of \( S' \).

(i) If \( i \in S' \) then \( h_i \) is an MDF of \( G \).

(ii) \( S' = (k_1, k_2) \) where \( k_1 \) and \( k_2 \) are finite real numbers.

(iii) \( h_i \) is an MDF for \( i = 1 \) and \( 2 \).

(iv) \( B_{h_i} \neq B_i \) or \( P_{h_i} \neq P_i \) for \( i = 1 \) and \( 2 \).

(v) \( f \) is a convex combination of \( h_i \) and \( h_i \).
The following theorem gives a necessary and sufficient condition for an MTDF to be basic.

**Theorem 2.16.** (i) Let \( f \) be an MTDF. Then \( f \) is a BMTDF if and only if there does not exist an MTDF \( g \) such that \( B_t = B_g \) and \( P_t = P_g \).

(ii) Let \( f \) be an MDF. Then \( f \) is a BMDF if and only if there does not exist an MDF \( g \) such that \( B_t = B_g \) and \( P_t = P_g \).

**Proof:** (i) Suppose \( f \) is a BMTDF and there exists an MTDF \( g \) such that \( B_t = B_g \) and \( P_t = P_g \). Let \( S = \{ i \in \mathbb{R} / h_i = [(i + i)g - if] \} \) be a TDF, \( B_h = B_i \) and \( P_h = P_i \). By Lemma 2.13, \( S \) is a bounded open interval. Let \( S = (k_1, k_2) \). Then \( k_1 < -1 < 0 < k_2 \) and \( h_{k_1} \) and \( h_{k_2} \) are MTDFs. Now \( h_{k_1} = (1 + k_1)g - k_1f \).

Hence \( f = \left( \frac{1 + k_1}{k_1} \right)g - \left( \frac{1}{k_1} \right)h_{k_1} \). Let \( \lambda_1 = \frac{1 + k_1}{k_1} \) and \( \lambda_2 = \frac{1}{k_1} \).

Since \( k_1 < -1 \) we have \( \lambda_1 < 0 \). Also \( \lambda_1 + \lambda_2 = 1 \). Thus \( f \) is a convex combination of the MTDFs \( g \) and \( h_{k_1} \), so that \( f \) is not a BMTDF, which is a contradiction. Hence there does not exist a MTDF \( g \) such that \( B_t = B_g \) and \( P_t = P_g \).

Conversely, suppose \( f \) is not a BMTDF. Then there exist MTDFs \( g_1, g_2, \ldots, g_n \) such that \( f = \sum \lambda_j g_j \), where \( 0 < \lambda_j \) and \( \sum \lambda_j = 1 \). Now, let...
\[ g = \sum_{i=1}^{n} \mu_i g_i, \] where \( 0 < \mu_i < 1 \) and \( \sum \mu_i = 1 \), be another convex combination of \( g_1, g_2, \ldots, g_n \). Then \( B_f = B_x = \bigcap_i B_{x_i} \) and \( P_f = P_x = \bigcup_i P_{x_i} \). Since \( f \) is an MTDF, \( B_f \rightarrow f, P_f \), and hence it follows from Theorem 2.1, that \( g \) is an MTDF. Hence the theorem follows.

(ii) The proof is similar. \( \square \)

**Corollary 2.17.** For any graph \( G = (V, E) \) the number of BMTDFs is finite.

**Proof:** If \( f \) and \( g \) are two distinct BMTDFs, then the pair \((P_f, B_f)\) and \((P_g, B_g)\) are different. Since \( V \) is finite, the number of such pairs of subsets of \( V \) is finite and hence the result follows. \( \square \)

We now proceed to develop a method to check whether a given MTDF is a BMTDF.

**Remark 2.18.** (i) Cockayne et al. [11] have characterized the graphs having unique MTDF (Theorem 1.37). Hence for all other graphs there exist at least two BMTDFs. For such graphs any TDF can be expressed as a convex combination of a set of BTDFs.

(ii) If \( G \) is a graph without isolated vertices, then \( G \) has at least two distinct minimal dominating sets and the corresponding characteristic functions are BMDFs of \( G \). Hence any graph without isolated vertices has at least two distinct BMTDFs.
Lemma 2.19. Let $f$ and $g$ be two distinct MTDFs of graph $G$ with $P_f = P_g$ and $B_f = B_g$. Let $\delta(v) = f(v) - g(v)$, where $v \in V$. Then

(i) If $f(v) = 0$ or $f(v) = 1$, then $\delta(v) = 0$.

(ii) $\sum_{u \in N(v)} \delta(u) = 0$ for all $v \in B_f$.

Proof: (i) Follows from Lemma 2.9.

(ii) Let $v \in B_f$. Then $v \in B_g$. Hence

$$\sum_{u \in N(v)} \delta(u) = \sum_{u \in N(v)} [f(u) - g(u)] = \sum_{u \in N(v)} f(u) - \sum_{u \in N(v)} g(u) = 1 - 1 = 0.$$ 

\[\square\]

Lemma 2.20. Let $f$ and $g$ be two distinct MDF of graph $G$ with $P_f = P_g$ and $B_f = B_g$. Let $\delta(v) = f(v) - g(v)$, where $v \in V$. Then

(i) If $f(v) = 0$ or $f(v) = 1$, then $\delta(v) = 0$.

(ii) $\sum_{u \in N(v)} \delta(u) = 0$ for all $v \in B_f$.

Proof: Similar to that of Lemma 2.19. \[\square\]

Theorem 2.21. Let $f$ be an MTDF of a graph $G = (V, E)$ with $B_f = \{v_1, v_2, \ldots, v_m\}$ and $P_f = \{u \in V : 0 < f(u) < 1\} = \{u_1, u_2, \ldots, u_n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix defined by
\[ a_v = \begin{cases} 
1 & \text{if } v \text{ is adjacent to } u, \\
0 & \text{otherwise}. 
\end{cases} \]

Consider the system of linear equations given by,
\[ \sum_{j=1}^{n} a_v x_j = 0 \quad \text{where } (1 \leq i \leq m). \] \((S_i)\)

Then \( f \) is a BMTDF if and only if \((S_i)\) does not have a nontrivial solution.

**Proof:** Suppose \( f \) is not a BMTDF. Then by Theorem 2.16(i), there exists an MTDF \( g \) such that \( B_f = B_g \) and \( P_f = P_g \). Let \( x_j = f(u_j) - g(u_j), \) where \( 1 \leq j \leq n \). Suppose \( x_j = 0 \) for all \( j = 1, 2, \ldots, n \).

Then \( f(u_j) = g(u_j) \) for all \( u_j \in P_j' \). Also if \( f(v) = 0 \), then \( v \notin P_i = P_g \) and hence \( g(v) = 0 \). If \( f(v) = 1 \), then it follows from Lemma 2.9 that \( g(v) = 1 \).

Hence \( f = g \), which is a contradiction. Thus \( x_j \neq 0 \) for at least one \( j \). Now
\[ \sum_{j=1}^{n} a_v x_j = \sum_{j=1}^{n} (f(u_j) - g(u_j)). \] Since \( v \notin B_f \), it follows from Lemma 2.19, that
\[ \sum_{j=1}^{n} a_v x_j = 0. \] Thus \((S_i)\) has a nontrivial solution.

Conversely, let \( \{x_1, x_2, \ldots, x_n\} \) be a nontrivial solution for \((S_i)\). Define \( g \) on \( V \) as follows;
\[
g(v) = \begin{cases} 
f(v) & \text{if } v \notin P_i', \\
\frac{x_j}{M} & \text{if } v = u_j, 1 \leq j \leq n. 
\end{cases} \]
where $M$ is to be suitably chosen.

Since $\{x_1,x_2,\ldots,x_n\}$ is a nontrivial solution of $(S_i)$, it follows that $g \neq f$.

Since $0 < f(u_i) < 1$, we can choose $M_i > 0$ such that $0 < f(u_i) + \frac{x_i}{M_i} < 1$, for each $j = 1,2,\ldots,n$.------(i). Let $M' = \max\{M_1,M_2,\ldots,M_n\}$. For this choice of $M'$ and for any $v \in V$, we have

$$g[N(v)] = \sum_{u \in N(v)} g(u) = \sum_{u \in N(v) \cap P_i} g(u) + \sum_{u \in N(v) \setminus P_i} g(u)$$

$$= \sum_{u \in N(v) \cap P_i} (f(u) + \frac{x_i}{M'_{x_i}}) + \sum_{u \in N(v) \setminus P_i} f(u)$$

$$= \sum_{u \in N(v) \cap P_i} f(u) + \frac{1}{M'} \sum_{i=1}^n x_i$$

$$= f(N(v)) + \frac{1}{M'} \sum_{i=1}^n x_i$$

If $v \in B_i$, then $\sum_{i=1}^n x_i = \sum_{i=1}^n a_i x_i = 0$ and hence $g(N(v)) = f(N(v)) = 1$.------(2)

Also $f(N(v)) > 1$ for all $v \in B_i$ and hence we can choose $M'' > 0$ such that $g(N(v)) > 1$ for all $v \in B_i$.------(3)

Now let $M = \max\{M',M''\}$. With this choice of $M$ we have $0 \leq g(v) \leq 1$ and $\sum_{u \in N(v)} g(u) \geq 1$ for all $v \in V$. Thus $g$ is a TDF.
Also it follows from (1), (2) and (3), that $P_f = P_g$ and $B_f = B_g$. Since $f$ is an MTDF, we have $B_f \rightarrow P_f$. Hence $B_g \rightarrow P_g$, so that $g$ is also an MTDF and it follows from Theorem 2.16(i) that $f$ is not a BMTDF. \hfill \Box

**Theorem 2.22.** Let $f$ be an MDF of a graph $G = (V, E)$ with $B_f = \{v_1, v_2, \ldots, v_m\}$ and $P_f = \{u \in V / 0 < f(u) < 1\} = \{u_1, u_2, \ldots, u_n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } u_j \text{ or } v_i = u_j, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the system of linear equations given by

$$\sum_{i=1}^{n} a_{ij}x_i = 0, \text{ where } (1 \leq i \leq m). \quad (S_2)$$

Then $f$ is a BMDF if and only if $(S_2)$ does not have a nontrivial solution.

**Proof:** Similar to that of Theorem 2.21. \hfill \Box

**Corollary 2.23.** Let $G = (V, E)$ be a graph without isolated vertices.

(i) Let $S$ be a minimal total dominating set of $G$. Then $f = \chi_S$ is a BMTDF.

(ii) Let $S'$ be a minimal dominating set of $G$. Then $f = \chi_{S'}$ is a BMDF.
Proof: (i) Obviously $f$ is an MTDF. Since $f$ assumes only the values 0 and 1, we have $P_f' = \{v \in V / 0 < f(v) < 1\} = \emptyset$ and hence the result follows from Theorem 2.21.

(ii) The proof is similar. \qed

Example 2.24. Consider the graph $G = K_3$, and let $V(G) = \{v_1, v_2, v_3\}$. Then $f : V \rightarrow [0,1]$ defined by $f(v_1) = f(v_2) = f(v_3) = \frac{1}{2}$ is a MTDF with $B_f = P_f' = \{v_1, v_2, v_3\}$. The corresponding system of linear equations is given by $x_1 + x_2 = 0$, $x_2 + x_3 = 0$ and $x_1 + x_1 = 0$. Since this system of equations has only trivial solution, $f$ is a BMTDF.

Example 2.25. Consider the Hajó's graph given in Figure 2.2.

![Figure 2.2](image)

We define two functions $f_1$ and $f_2$ as follows:

$f_1(v_1) = f_1(v_4) = f_1(v_6) = f_1(v_3) = 0$, $f_1(v_2) = f_1(v_5) = 1$.

$f_2(v_1) = f_2(v_4) = f_2(v_6) = 0$ and $f_2(v_2) = f_2(v_3) = f_2(v_5) = \frac{1}{2}$.
Then \( B_f = \{v_1, v_3, v_4, v_5\} \) and \( P_f = \emptyset \). Also \( B_{f_1} = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( P_{f_1} = \{v_2, v_3, v_5\} \). Trivially \( f_i \) is basic. The system of equations for \( f_2 \) is given below.

For \( v_1 \in B_{f_2} \), \( x_2 + x_5 = 0 \)

For \( v_2 \in B_{f_2} \), \( x_3 + x_5 = 0 \)

For \( v_3 \in B_{f_2} \), \( x_2 + x_5 = 0 \)

For \( v_4 \in B_{f_2} \), \( x_2 + x_5 = 0 \)

For \( v_5 \in B_{f_2} \), \( x_2 + x_5 = 0 \)

For \( v_6 \in B_{f_2} \), \( x_3 + x_5 = 0 \)

These equations reduce to

\[
\begin{align*}
x_2 + x_3 &= 0 \\
x_3 + x_5 &= 0 \\
x_2 + x_5 &= 0.
\end{align*}
\]

This system of equations has only trivial solution and hence \( f_2 \) is also a BMTDF.

Next consider \( g = \frac{1}{2} f_1 + \frac{1}{2} f_2 \). Then \( g(v_1) = g(v_4) = g(v_6) = 0 \), \( g(v_2) = \frac{1}{4} \), \( g(v_3) = \frac{1}{4} \) and \( g(v_5) = \frac{1}{4} \). Also \( B_g = \{v_1, v_3, v_4, v_5\} \) and \( P_g = \{v_2, v_3, v_5\} \). Since \( B_g \to P_g \), it follows that \( g \) is an MTDF.

Corresponding system of equations is given by,
\[ x_2 + x_3 = 0 \quad \text{for} \quad v_i \in B_x \]
\[ x_2 + x_4 = 0 \quad \text{for} \quad v_x \in B_x \]
\[ x_2 + x_4 = 0 \quad \text{for} \quad v_x \in B_x \]
\[ x_2 + x_5 = 0 \quad \text{for} \quad v_x \in B_x \]

This system reduces to two equations: \( x_2 + x_3 = 0 \) and \( x_2 + x_5 = 0 \), which has a nontrivial solution. Hence \( g \) is not basic.

Given a graph \( G = (V, E) \), Cockayne et al. [12] introduced the concept of convexity graph \( G_x \) of MDFs (Definition 1.44).

If \( G \) is a graph without isolated vertices, then the definition of convexity graph can be easily extended to the set of all MTDFs of \( G \).

**Definition 2.26.** Let \( \mathcal{F}_i \) denote the set of all MTDFs of \( G \). We define a relation \( \rho \) on \( \mathcal{F}_i \) as follows. If \( f, g \in \mathcal{F}_i \), then \( f \rho g \) if and only if \( B_f = B_g \) and \( P_f = P_g \). Clearly \( \rho \) is an equivalence relation on \( \mathcal{F}_i \) and since \( G \) is finite, it follows that the number of equivalence classes with respect to \( \rho \) is finite. Let \( Y = \{ Y_1, Y_2, \ldots, Y_r \} \) denote the set of all equivalence classes. We define a graph \( G_i \) as follows. \( V(G_i) = Y \) and \( Y_i \) is adjacent to \( Y_j \) if for every \( f \in Y_i \) and \( g \in Y_j \), the convex combination of \( f \) and \( g \) is an MTDF. The graph \( G_i \) is called the **convexity graph** of the set of all MTDFs of \( G \).
Remark 2.27. It follows from Theorem 2.16 (i) that if $f$ is a BMTDF, then the equivalence class containing $f$ is $\{f\}$ and conversely, if an equivalence class $Y = \{f\}$, then $f$ is a BMTDF.

Definition 2.28. (i) Let $\mathcal{F}_\text{BMTDF}$ denote the set of all BMTDFs of a graph $G$. Then the vertex induced subgraph of the convexity graph $G$, induced by $\mathcal{F}_\text{BMTDF}$ is called the basic convexity graph of $G$ and is denoted by $G_{\text{BMTDF}}$.

(ii) Let $\mathcal{F}_\text{BMD}$ denote the set of all BMDFs of a graph $G$. Then the vertex induced subgraph of the convexity graph $G$, induced by $\mathcal{F}_\text{BMD}$ is called the basic convexity graph of $G$ with respect to MDFs and is denoted by $G_{\text{BMD}}$.

Example 2.29. Consider the graph $G = K_{1,3}$ given in Figure 2.3.

Let $g$ be any MTDF of $G$. Since $N(v_1) = N(v_3) = \{v_2\}$, it follows that $g(v_2) = 1$. Let $g(v_i) = r$. If $0 < r < 1$, then $v_i \in P_x$ and hence $v_i \in B_x$. Hence $g(v_i) + g(v_3) = 1$, which gives $g(v_3) = 1 - r$. If $r = 0$, then $g(v_3) = 1$ and if $r = 1$
then $g(v_3) = 0$. Hence the set $\mathcal{F}_i$ of all MTDFs of $G$ is given by $\mathcal{F}_i = \{g_1, g_2\} \cup \{g, /0 < r < 1\}$ where

$g_1(v_1) = g_1(v_1) = 1$ and $g_1(v_3) = 0$

$g_2(v_2) = g_2(v_1) = 1$ and $g_2(v_3) = 0$

$g_3(v_3) = 1, g_3(v_3) = r$ and $g_3(v_3) = 1 - r$.

Clearly $g_3 = rg_1 + (1-r)g_2$ and hence $g_3$ is not a BMTDF. Since the function values of $g_1$ and $g_2$ are 0 and 1, it follows from Corollary 2.23 (i) that $g_1$ and $g_2$ are BMTDFs. Clearly $\mathcal{F}_i$ is divided into three equivalence classes under the relation $\rho$ given by $Y_1 = \{g_1\}, Y_2 = \{g_2\}$ and $Y_3 = \{g, /0 < r < 1\}$. The convexity graph $G_i$ and the basic convexity graph $G_{III}$ of $K_{1,2}$ are given in Figure 2.4.

![Figure 2.4](image)

The above example is a particular case of the following theorem.
Theorem 2.30. The convexity graph $G_f$ of the star graph $K_{1,n}$ is isomorphic to $K_{2^n-1}$, and the basic convexity graph $G_{B_f}$ is isomorphic to $K_n$.

Proof : Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with $d(v) = n$. Then functions $g_i, 1 \leq i \leq n$, defined by $g_i(v_i) = 1$ and $g_i(v_j) = 0$, if $i \neq j$ are MTDFs. They take only the values 0 and 1 and hence they are BMTDFs. Now, let $g$ be an arbitrary MTDF of $K_{1,n}$. Since $N(v_i) = \{v_i\}$ for all $i$ where $1 \leq i \leq n$, we have $g(v_i) = 1$. Further at least one of the vertices $v_1, v_2, \ldots, v_n$ must be in $P_g$ and hence $v \in B_g$. Hence $\sum g(v_i) = 1$. Let $g(v_1) = r_1, g(v_2) = r_2, \ldots, g(v_n) = r_n$. Clearly $g = r_1g_1 + r_2g_2 + \ldots + r_ng_n$. Thus any MTDF of $K_{1,n}$ is a convex combination of $g_1, g_2, \ldots, g_n$. Hence the set of all MTDFs of $G$ is given by $F_f = \{r_1g_1 + r_2g_2 + \ldots + r_ng_n \mid 0 \leq r_i \leq 1 \text{ and } \sum_{i=1}^{n} r_i = 1\}$. For any non-empty subset $S \subseteq \{1, 2, \ldots, n\}$, $T_S = \{\sum_{i \in S} r_i g_i \mid 0 < r_i \leq 1, \sum_{i \in S} r_i = 1\}$ is a collection of MTDFs of $K_{1,n}$ and each function in $T_S$ has $\{v\} \cup \{v_i \mid i \in S\}$ as its positive set and $V(K_{1,n})$ as its boundary set. Hence $\{T_S \mid S \subseteq \{1, 2, \ldots, n\} \text{ and } S \neq \emptyset\}$ is the set of all equivalence classes of $F_f$ and any two of these equivalence classes are adjacent in the convexity graph $G_f$. Thus $G_f$ is isomorphic to the complete graph $K_{2^n-1}$ and $G_{B_f}$ is isomorphic to the complete graph $K_n$. £
Theorem 2.31. The convexity graph of $G = S(K_{1,n})$ is $K_3$ and its basic convexity graph is $K_2$.

Proof: Let $V(S(K_{1,n})) = \{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$, where $N(u_i) = \{v_i\}$, $N(v_i) = \{v, u_i\}$ and $N(v) = \{v_1, v_2, ..., v_n\}$. Let $g$ be an arbitrary MTDF of $G$.

Since $N(u_i) = \{v_i\}$, we have $g(v_i) = 1$ for all $i, 1 \leq i \leq n$. Now if $g(v) = 0$ we have $g(u_i) = 1$ for $i = 1, 2, ..., n$. If $g(u_i) = 0$ for $i = 1, 2, ..., n$, then $g(v) = 1$.

Suppose $g(u_i) = r > 0$. Then $u_i \in P_v$ and hence $v_i \in B_v$. Hence $g(v) = 1 - r$ and $g(u_i) = r$ for all $i = 1, 2, ..., n$. Now let $f : V \to [0,1]$ be defined by

$$f_i(x) = \begin{cases} 1 & \text{if } x = v_i \text{ for } i = 1, 2, ..., n \\ r & \text{if } x = u_i \text{ for } i = 1, 2, ..., n \text{ where } r \in [0,1] \\ 1 - r & \text{if } x = v \end{cases}$$

Then the family of all MTDFs of $G$ is given by $\mathcal{F}_i = \{f_i / 0 \leq r \leq 1\}$. The functions $f_n$ and $f_i$ assume only the values 0 and 1 and hence are BMTDFs. Hence $\mathcal{F}_i$ is divided into three equivalence classes, namely, $Y_1 = \{f_n\}, Y_2 = \{f_i\}$ and $Y_3 = \{f_i / 0 < r < 1\}$. Thus $G_i$ is isomorphic to $K_1$ and $G_{ii}$ is isomorphic to $K_2$. \qed

Theorem 2.32. The number of BMTDFs of the complete graph $K_n$ is $2^n - n - 1$. 

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Proof: Let $V(K_n) = \{v, v_2, \ldots, v_n\}$. Let $g$ be any BMTDF of $K_n$. Let $S = P_g$.

Clearly $|S| \geq 2$. We claim that $B_x = P_x = S$. Let $v \in B_x$. If $v \notin S$, then $g(v) = 0$. Hence $\sum_{u \in N(v)} g(u) = 1$. Now, for any $x \in S$, we have $\sum_{u \in N(x)} g(u) < 1$, which is a contradiction. Hence $v \in S$, so that $B_x \subseteq S = P_x$.

Now, let $v \in S$. We claim that $v \in B_x$. We consider two cases.

Case (i) $g(v) = 1$.

Since $B_x \subseteq P_x$, it follows that $|S| = 2$ and if $S = \{u, v\}$, then $g(u) = 1$.

Hence $\sum_{x \in N(v)} g(x) = g(u) = 1$, so that $v \notin B_x$.

Case (ii) $g(v) < 1$.

It follows from case (i) that $g(u) < 1$ for all $u \in S$. Hence $P_x = P_x^* = \{x \in V / 0 < g(x) < 1\} = \{x_1, x_2, \ldots, x_n\}$ (say). If $v \notin B_x$, then $|B_{x^*}| < |P_x^*|$. Now consider the system of equations defined in Theorem 2.21, given by $\sum_{i=1}^{n} a_{i} x_i = 0$, $1 \leq i \leq |B_x|$. Since $|B_x| < |P_x^*|$, the number of equations in this system is less than the number of variables. Hence the system of equations has a non-trivial solution, so that by Theorem 2.21, $g$ is not a
BMTDF, which is a contradiction. Hence \( v \in B_x \), so that \( S = P_x \subseteq B_x \). Thus \( P_x = B_x = S \).

Now, let \( S \subseteq V(K_n) \) and \(|S| \geq 2\). Define \( g : V \rightarrow [0,1] \) by

\[
g(v) = \begin{cases} 
0 & \text{if } v \not\in S, \\
\frac{1}{|S|-1} & \text{if } v \in S.
\end{cases}
\]

Clearly for any \( v \in S \), \( \sum_{u \in N(v)} g(u) = \sum_{u \in N(v)} g(u) = 1 \) and for any \( v \not\in S \), \( \sum_{u \in N(v)} g(u) = \sum_{u \in S} g(u) > 1 \). Hence \( g \) is an MTDF with \( B_x = P_x = S \). We claim that \( g \) is a BMTDF. The system of linear equations associated with \( g \) (Theorem 2.21) is given by

\[
\begin{aligned}
x_2 + x_3 + \ldots + x_k &= 0 \\
x_1 + x_3 + \ldots + x_k &= 0 \\
\ldots \ldots \ldots \ldots \ldots \\
x_1 + x_2 + \ldots + x_{k-1} &= 0
\end{aligned}
\]

Here \( k = |S| \). Since this system of linear equations has only trivial solution, \( g \) is a BMTDF. Thus there is a bijection from the set of all BMTDFs of \( G \) to the set of all subsets \( S \) of \( V(K_n) \) with \(|S| \geq 2\). Hence it follows that the number of BMTDFs of \( K_n \) is \( 2^n - n - 1 \).

\( \square \)

**Example. 2.33.** The number of BMTDFs of \( K_3 \) is \( 2^3 - 3 - 1 = 4 \). They are given by

\[
\begin{align*}
&111, &110, &101, &011, &100, &010, &001 \\
&101, &110, &111, &100, &011, &010, &001
\end{align*}
\]
The basic convexity graph $G_{n} (K_{3})$ (Figure 2.5) is isomorphic to $K_{1,3}$.

![Diagram of $G_{n} (K_{3})$]

**Figure 2.5**

**Remark 2.34.** The convexity graph of a graph $G$ is complete if and only if every MTDF of $G$ is a universal MTDF. Also $G$ has a universal MTDF if and only if the convexity graph contains a vertex of full degree.

**Theorem 2.35.** $G_{n} (K_{n}) \cong K_{n-1}$ and $G_{ns}(K_{n}) \equiv K_{n}$.

**Proof:** Let $V(K_{n}) = \{v_{1}, v_{2}, ..., v_{n}\}$. Define $f_{i} : V \to [0,1]$ by,

$$f_{i}(v) = \begin{cases} 1 & \text{if } v = v_{i} \\ 0 & \text{otherwise} \end{cases}$$
Clearly, $f_i$, where $i = 1, 2, ..., n$, are BMDFs. Let $g$ be any MDF of $K_n$. Let $g(v_i) = \lambda_i$, $1 \leq i \leq n$. Since $N[v_i] = V$ for all $i$, and $g$ is an MDF, we have

$$\sum_i g(v_i) = \sum_i \lambda_i = 1.$$ Further $0 \leq \lambda_i \leq 1$. Thus $g = \sum_i \lambda_i f_i$, so that any MDF $g$ of $K_n$ is a convex combination of $f_1, f_2, ..., f_n$. Let $S \subseteq \{1, 2, ..., n\}$ and $g = \{g / g = \sum_i \lambda_i f_i$ where $0 < \lambda_i \leq 1$ and $\sum_i \lambda_i = 1\}$. Every element $g$ of $Y_S$ is a MDF of $G$ with $P_g = \{v_i / i \in S\}$. Thus each nonempty subset of $V(K_n)$ gives a vertex of $G_n(K_n)$ and if $S_i \neq S_2$, then $Y_{S_i}$ and $Y_{S_2}$ are distinct vertices of $G_n(K_n)$. Hence $V(G_n(K_n)) = \{Y_S / S \subseteq \{1, 2, ..., n\}\}$ and $S \neq \phi$ and $|V(G_n(K_n))| = 2^n - 1$. Clearly $G_n(K_n) \cong K_{2^n - 1}$ and $G_{m_n}(K_n) \cong \langle f_1, f_2, ..., f_n \rangle \cong K_n$. 

**Theorem 2.36.** If $G = K_{1,n}$ where $n \geq 2$, then $G_n(K_{1,n}) \cong K_{2}$ and $G_{m_n}(K_{1,n}) \cong K_{2}$.

**Proof:** Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ with $d(v) = n$ and $d(v_i) = 1$ for $i = 1, 2, ..., n$. Let $f_1, f_2 : V \rightarrow [0, 1]$ be defined by,

$$f_1(x) = \begin{cases} 1 & \text{if } x = v \\ 0 & \text{otherwise} \end{cases}$$

and $f_2(x) = \begin{cases} 0 & \text{if } x = v \\ 1 & \text{otherwise} \end{cases}$. 

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Clearly \( f_1 \) and \( f_2 \) are BMDFs of \( G \). Let \( g \) be an arbitrary MDF of \( K_{1,n} \). We claim that \( g = \lambda f_1 + (1 - \lambda) f_2 \), where \( 0 < \lambda < 1 \). If \( g(v) = 1 \), then \( g(v_i) = 0 \) for all \( i \). Hence \( g = f_1 \). Also if \( g(v) = 0 \), then \( g(v_i) = 1 \) for all \( i \). Hence \( g = f_2 \). Suppose \( g(v) = \lambda \) where \( 0 < \lambda < 1 \). We claim that \( v \notin B_x \). Suppose \( v \in B_x \). Then \( g(N[v]) = g(v) + \sum_{i=1}^{n} g(v_i) = 1 \). Hence \( \sum_{i=1}^{n} g(v_i) = 1 - \lambda > 0 \), so that \( g(v_i) > 0 \) for at least one \( i \). Let \( g(v_i) > 0 \). Then \( g(v_2) \leq 1 - \lambda - g(v_i) \). Hence \( g(N[v_2]) = g(v) + g(v_2) \leq \lambda + (1 - \lambda) - g(v_i) = 1 - g(v_i) \), which is a contradiction. Thus \( v \notin B_x \). Since \( g(v_i) \geq 1 - \lambda \) for all \( i \) and \( v \notin B_x \), it follows that \( v_i \in B_x \) for all \( i \). Thus \( g(v_i) = 1 - \lambda \) for all \( i \). Hence \( g = \lambda f_1 + (1 - \lambda) f_2 \), where \( 0 < \lambda < 1 \). Thus there are exactly three equivalent classes of MDFs, namely \([f_1] \), \([f_2] \) and \([g] \). Since \( f_1 \) and \( f_2 \) are BMDFs, \( G_\lambda(K_{1,n}) \cong K_3 \) and \( G_\lambda(K_{1,n}) \cong K_2 \).

**Theorem 2.37.** Let \( G \) be a disconnected graph with \( n \) components \( G_1, G_2, \ldots, G_n \). Then

(i) \( G_\lambda(G) \cong G_\lambda(G_1) \circ G_\lambda(G_2) \circ \cdots \circ G_\lambda(G) \).

(ii) \( G_\lambda(G) \cong G_\lambda(G_1) \circ G_\lambda(G_2) \circ \cdots \circ G_\lambda(G) \).

**Proof:** (i) We prove the theorem for the case \( n \geq 2 \). The proof for the general case is similar. Let \( V(G_1) = V_1 \) and \( V(G_2) = V_2 \). We observe that \( f \) is
an MTDF of $G$ if and only if $f_i = f|V_i$ and $f_2 = f|V_2$ are MTDFs of $G_i$ and $G_2$ respectively. Now let $f, g \in V(G_i(G_2))$. Suppose $f$ and $g$ are adjacent in $G_i(G_2)$. Then $h = \lambda f + (1 - \lambda)g$ (where $0 < \lambda < 1$) is also an MTDF of $G$. Let $f_i = f|V_i$ and $g_i = g|V_i$ for $i = 1, 2$. We consider the following cases.

**Case (i)** $f_1 \neq g_1$ and $f_2 \neq g_2$.

Since $\lambda f + (1 - \lambda)g$ is an MTDF of $G$, it follows that $\lambda f_1 + (1 - \lambda)g_1$ and $\lambda f_2 + (1 - \lambda)g_2$ are MTDFs of $G_i$ and $G_2$ respectively. Hence $f_1g_1 \in E(G_i(G_i))$ and $f_2g_2 \in E(G_i(G_i))$.

**Case (ii)** $f_1 = g_1$ and $f_2 \neq g_2$.

As in case (i) we get $f_2g_2 \in E(G_i(G_2))$.

**Case (iii)** $f_1 \neq g_1$ and $f_2 = g_2$.

In this case $f_1g_1 \in E(G_i(G_i))$.

Thus $fg \in E(G_i(G_i)) \Rightarrow fg \in E(G_i(G_i) \circ G_i(G_2))$.

Now let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ and $fg \in E(G_i(G_i) \circ G_i(G_2))$. We have three possible cases.

**Case (i)** $f_1 = g_1$ and $f_2g_2 \in E(G_i(G_2))$.

Then $\lambda f_2 + (1 - \lambda)g_2$ is an MTDF of $G_2$. Also $f_1$ is an MTDF of $G_1$. Hence $fg \in E(G_i(G_i))$.  

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Case (ii) \( f_2 = g_2 \) and \( f_1 g_1 \in \mathcal{E}(G_{j}(G_i)) \).

Case (iii) \( f_1 g_1 \in \mathcal{E}(G_{j}(G_i)) \) and \( f_2 g_2 \in \mathcal{E}(G_{j}(G_2)) \).

In both these cases we can prove by a similar argument that \( fg \in \mathcal{E}(G_{j}(G)) \).

Hence \( G_{j}(G) \cong G_{j}(G_1) \circ G_{j}(G_2) \).

(ii) Proof is similar. \( \square \)

**Theorem 2.38.** Let \( G \) be a disconnected graph with \( n \) components \( G_1, G_2, \ldots, G_n \). Then

(i) \( G_{HF}(G) \cong G_{HF}(G_1) \circ G_{HF}(G_2) \circ \ldots \circ G_{HF}(G_n) \).

(ii) \( G_{HS}(G) \cong G_{HS}(G_1) \circ G_{HS}(G_2) \circ \ldots \circ G_{HS}(G_n) \).

**Proof:** (i) We prove the theorem for \( n = 2 \). The proof for the general case is similar. Let \( f \) be a BMTDF of \( G \). Let \( V(G_1) = V_1, V(G_2) = V_2, f_1 = f \big| V_1 \) and \( f_2 = f \big| V_2 \). We claim that \( f_1 \) is a BMTDF of \( G_1 \). Suppose not. Then there exist two MTDFs \( g \) and \( h \) of \( G_1 \), such that \( f_1 = \lambda g + (1 - \lambda) h \). Now \( (g, f_2) \) and \( (h, f_2) \) are MTDFs of \( G \) and \( f = \lambda(g, f_2) + (1 - \lambda)(h, f_2) \). Hence \( f \) is a convex combination of two different MTDFs, which is a contradiction.

Thus \( f_1 \) is a BMTDF of \( G_1 \). Similarly \( f_2 \) is a BMTDF of \( G_2 \).

Conversely let \( f_1 \) and \( f_2 \) be BMTDFs of \( G_1 \) and \( G_2 \) respectively. We claim that \( f = (f_1, f_2) \) is a BMTDF of \( G \). Otherwise there exist two
MTDFs $g$ and $h$ of $G$ such that $f = \lambda g + (1 - \lambda)h$ where $0 < \lambda < 1$. Then $f_i = \lambda g_i + (1 - \lambda)h$, where $i = 1$ and $2$, so that either $f_1$ or $f_2$ is not a BMTDF, which is a contradiction. Hence $f$ is a BMTDF of $G$. Now as in Theorem 2.37, we can prove that $G_{\mu_1} (G) \equiv G_{\mu_2} (G_1) \circ G_{\mu_3} (G_2)$.

(ii) Proof is similar.

**Theorem 2.39.** (i) Let $f$ be an MTDF of a graph $G = (V, E)$. Let $f_1, f_2, \ldots, f_n$ be BMTDFs of $G$ such that $f$ is a convex combination of $f_1, f_2, \ldots, f_n$. Then the subgraph of $G_{\mu_1} (G)$ induced by $f_1, f_2, \ldots, f_n$ is a complete graph.

(ii) Let $f$ be an MDF of a graph $G = (V, E)$. Let $f_1, f_2, \ldots, f_n$ be BMDFs of $G$ such that $f$ is a convex combination of $f_1, f_2, \ldots, f_n$. Then the subgraph of $G_{\mu_1} (G)$ induced by $f_1, f_2, \ldots, f_n$ is a complete graph.

Proof: (i) Since $f$ is a convex combination of $f_1, f_2, \ldots, f_n$, we have $B_f = \bigcap B_{f_i}$ and $P_f = \bigcup P_{f_i}$. Since $f$ is an MTDF of $G$, we have $B_f \rightarrow P_f$. Now $B_f \cap B_{f_i} \supseteq \bigcap B_{f_i} = B_i$ and $P_f \cup P_{f_i} \subseteq \bigcup P_{f_i} = P_i$. Hence $B_i \cap B_{f_i} \rightarrow P_i \cup P_{f_i}$, so that convex combination of $f_i$ and $f_j$ is an MTDF. Hence $f_i$ and $f_j$ are adjacent in $G_{\mu_1} (G)$.

(ii) Proof is similar to that of (i).
Lemma 2.40. Let the MTDF (MDF) \( f \) be a convex combination of \( n \) BMTDFs (BMDFs) \( f_1, f_2, \ldots, f_n \) and the MTDF (MDF) \( g \) be a convex combination of \( m \) BMTDFs (BMDFs) \( g_1, g_2, \ldots, g_m \). Then any convex combination of \( f \) and \( g \) is a convex combination of \( f_1, f_2, \ldots, f_n \) and \( g_1, g_2, \ldots, g_m \).

Proof: Let \( f = \sum_{i} \lambda_i f_i \), \( 0 < \lambda_i < 1, \sum_{i} \lambda_i = 1 \) and \( g = \sum_{j} \mu_j g_j \), \( 0 < \mu_j < 1, \sum_{j} \mu_j = 1 \).

Let \( h_\alpha = \alpha f + (1 - \alpha) g \) where \( 0 < \alpha < 1 \).

Now, \( h_\alpha = \alpha f + (1 - \alpha) g \)
\[= \alpha (\sum_i \lambda_i f_i) + (1 - \alpha)(\sum_j \mu_j g_j) \]
\[= \sum_i \alpha \lambda_i f_i + \sum_j (1 - \alpha) \mu_j g_j. \]

Clearly \( \sum_i \alpha \lambda_i + \sum_j (1 - \alpha) \mu_j = 1 \).

Further \( 0 < \alpha \lambda_i < 1 \) for all \( i \)
and \( 0 < (1 - \alpha) \mu_j < 1 \) for all \( j \).

Hence the result follows. \( \square \)

Theorem 2.41. (i) Let \( f \) be an MTDF of a graph \( G = (V, E) \). Then \( f \) is either a BMTDF or a convex combination of a set of BMTDFs.
(ii) Let \( f \) be an MDF of a graph \( G = (V, E) \). Then \( f \) is either a BMDF or a convex combination of a set of BMDFs.

**Proof :** (i) If \( f \) is a BMTDF, then we are done. Suppose that \( f \) is an MTDF but not a BMTDF. Then \( f \) is not a basicTDF. Hence there exists a set of BTDFs \( f_1, f_2, ..., f_n \) such that \( f = \sum \lambda_i f_i \) where \( \sum \lambda_i = 1 \) and \( 0 < \lambda_i < 1 \). Now we claim that \( f_i \) is an MTDF for all \( i \). Since \( f \) is an MTDF, we have \( B_i \to P_i \). Also \( B_i \supseteq B \) and \( P_i \subseteq P \) and hence \( B_i \to P_i \). Thus \( f_i \) is an MTDF. Thus \( f \) is both minimal and basic and \( f \) is the convex combination of the BMTDFs \( f_1, f_2, ..., f_n \).

(ii) The proof is similar. \( \square \)

**Theorem 2.42.** The convexity graph \( G_c(G) \) is connected if and only if the basic convexity graph \( G_{bc}(G) \) is connected.

**Proof** Suppose \( G_{bc}(G) \) is connected. By definition, \( G_{bc}(G) \) is a subgraph of \( G_c(G) \). Now let \( f \in V(G_c(G)) - V(G_{bc}(G)) \). Then \( f \) is an MTDF but not a BMTDF. By Lemma 2.14 there exist two MTDFs \( g_i \) and \( h_i \) such that \( f \) is a convex combination of \( g_i \) and \( h_i \) and either \( B_i \neq B \) or \( P_i \neq P \). Hence \( f \) and \( g_i \) are distinct vertices of \( G_c \). Similarly, \( f \) and \( h_i \) are distinct vertices of \( G_c \). We now claim that \( g_i \) and \( h_i \) are distinct vertices of \( G_c \).
Otherwise we have \( B_{<r} = B_{h_i} \) and \( P_{<r} = P_{h_i} \). Hence \( B_f = B_{<r} \cap B_{h_i} = B_{h_i} \) and \( P_f = P_{<r} \cup P_{h_i} = P_{h_i} \), which is a contradiction. Thus \( f, g, \) and \( h_i \) are distinct vertices of \( G_r \) and these three vertices induce \( K_1 \). If \( g_i \) or \( h_i \) is a BMTDF of \( G \), then it follows that \( f \) is adjacent to a vertex of \( G_{\text{Bj}} \). Suppose \( g_i \) and \( h_i \) do not belong to \( V(G_{\text{Bj}}) \). Then \( g_i \) can be expressed as a convex combination of two MTDFs \( g_2 \) and \( h_2 \) and at least one of these MTDFs; say \( g_2 \), is different from both \( f \) and \( g_1 \); and either \( B_{g_i} \neq B_{g_2} \) or \( P_{g_i} \neq P_{g_2} \).

We now claim that \( f \) and \( g_2 \) are adjacent. Let \( c = \lambda f + (1 - \lambda)g_2 \) (where \( 0 < \lambda < 1 \)) be a convex combination of \( f \) and \( g_2 \). Since \( B_f \subseteq B_{g_i} \subseteq B_{g_2} \), it follows that \( B_c = B_f \cap B_{g_2} = B_f \) and since \( P_f \supseteq P_{g_i} \supseteq P_{g_2} \), \( P_c = P_f \cup P_{g_2} = P_f \). Since \( f \) is an MTDF, \( B_f \rightarrow P_f \). Hence \( B_c \rightarrow P_c \), so that \( c \) is an MTDF. Hence it follows that \( f \) and \( g_2 \) are adjacent in \( G_r \). If \( g_2 \in V(G_{\text{Bj}}) \) then it follows that \( f \) is adjacent to a vertex of \( G_{\text{Bj}} \).

Suppose we have constructed functions \( g_1, g_2, \ldots, g_k \) such that each \( g_i \) does not belong to \( V(G_{\text{Bj}}) \), \( g_k \) is distinct from \( f, g_1, g_2, \ldots, g_{k-1} \), \( f \) is adjacent to \( g \), and \( B_{g_i} \neq B_{g_k} \) or \( P_{g_i} \neq P_{g_k} \) for \( i = 1, 2, \ldots, k \). Since \( g_k \not\in V(G_{\text{Bj}}) \), there exist \( g_{k+1} \) and \( h_{k+1} \) such that \( g_k \) is the convex combination of \( g_{k+1} \) and \( h_{k+1} \), \( g_{k+1} \neq f, g_1, g_2, \ldots, g_k \) and \( f \) is adjacent to
$g_{i+1}$ in $G_f$. Suppose $g_{k+1} = g_i$ for some $i$, where $1 \leq i \leq k$. Since

$B_i \subseteq B_{k_1} \subseteq B_{k_2} \subseteq \ldots \subseteq B_{k_i} \subseteq B_{k_{i+1}}$ and $P_{k_1} \supseteq P_{k_2} \supseteq \ldots \supseteq P_{k_i} \supseteq P_{k_{i+1}}$. We have $B_{k_i} = B_{k_{i+1}}$ and $P_{k_i} = P_{k_{i+1}}$. Hence $B_{k_i} = B_{k_{i+1}} = \ldots = B_{k_n}$ and $P_{k_i} = P_{k_{i+1}} = \ldots = P_{k_n}$, which is a contradiction. Thus $f, g_1, \ldots, g_k$ and $g_{k+1}$ are all distinct vertices of $G_f$. Since $V(G_f)$ is finite there exists $r$ such that $g_r \in V(G_{hr})$ and $f$ is adjacent to $g_r$. Thus it follows that every vertex of $V(G_f) - V(G_{hr})$ is adjacent to a vertex of $V(G_{hr})$. Further $G_{hr}$ is connected, and hence $G_f$ is also connected.

Conversely, suppose $G_f$ is connected and $G_{hr}$ is not connected. Let $C_1$ and $C_2$ be two components of $G_{hr}$. Let $y_1$ and $y_2$ be two vertices belonging to $C_1$ and $C_2$ respectively. Since $G_f$ is connected, there exists a path $y_1, x_1, x_2, \ldots, x_n, y_2$ in $G_f$ connecting $y_1$ and $y_2$. Let $Y_i = \{x_1, x_2, \ldots, x_n\}$ be the set of all BMTDFs so that $x_i$ is a convex combination of the elements of $Y_i$. Since $y_1, x_1 \in E(G_f)$, by Theorem 2.39 (i), the induced subgraph $\langle \{y_1\} \cup Y_i \rangle$ is a complete graph in $G_{hr}$. Hence $\{y_1\} \cup Y_i \subseteq V(C_i)$. Similarly, $Y_2, Y_1, \ldots, Y_n \subseteq V(C_i)$. Since $x_n, y_2 \in E(G_f)$ we have $\{y_2\} \cup Y_n \subseteq V(C_2)$. Thus $Y_2 \subseteq V(C_1)$ and $Y_n \subseteq V(C_2)$, which is a contradiction. □
Theorem 2.43. The convexity graph $G_s(G)$ with respect to MDFs of $G$ is connected if and only if the basic convexity graph $G_{bs}(G)$ is connected.

Proof: The proof is similar to that of Theorem 2.42.

Conclusion and scope. Basic minimal dominating functions (basic minimal total dominating functions) which we have introduced in this chapter correspond to extreme points in the convex set of dominating functions (total dominating functions). Theorem 2.21 (Theorem 2.22) provide a necessary and sufficient condition for an MTDF (MDF) to be a BMTDF (BMDF). We have also investigated the convexity graph and basic convexity graph with respect to dominating functions as well as total dominating functions. The following are some interesting problems for further investigation.

(i) Does there exist a graph $G$ such that its convexity graph or basic convexity graph with respect to MDFs or MTDFs is disconnected?

(ii) Determine the convexity graph and basic convexity graph with respect to MDFs and MTDFs for standard graphs such as path $P_n$, cycle $C_n$ and $K_{m,n}$.

(iii) Another interesting area for further research is to develop properties of the convexity graph and basic convexity graph with respect to MDFs and MTDFs.
(iv) Extend the concept of dominating functions to edge set of a graph and develop the theory of edge dominating functions and investigate the structure of convexity graph and basic convexity graph with respect to edge dominating functions.