CHAPTER 1

PRELIMINARIES

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology we refer to Harary [22] or Berge [11].

Definition 1.1. A graph $G$ is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$, called edges. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively.

When there is no possibility of confusion we write $V$ and $E$ for $V(G)$ and $E(G)$, respectively. In general if the graph $G$ is clear from the context, we omit the symbol $G$ when discussing a parameter or a set associated with $G$.

If $e = \{u,v\}$ is an edge, we write $e = uv$, we say that $e$ joins the vertices $u$ and $v$; $u$ and $v$ are adjacent vertices; $u$ and $v$ are incident with $e$.

If two vertices are not joined, then we say that they are nonadjacent.

If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

The order and size of $G$ are defined by $p = |V(G)|$ and $q = |E(G)|$, respectively. A graph of order $p$ and size $q$ is called a $(p,q)$ graph.
Definition 1.2. A graph $G_1$ is isomorphic to a graph $G_2$ if there exists a bijection $\varphi$ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\varphi(u) \varphi(v) \in E(G_2)$. If $G_1$ is isomorphic to $G_2$ we write $G_1 \cong G_2$.

Definition 1.3. A graph $H$ is called a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of $G$ is a subgraph $H$ of $G$ with $V(H) = V(G)$.

For any set $S$ of vertices of $G$, the induced subgraph $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$. The induced subgraph $\langle S \rangle$ is also denoted by $G[S]$.

Let $v$ be a vertex of a graph $G$. The induced subgraph $\langle V(G) - \{v\} \rangle$ is denoted by $G - v$ and it is the subgraph of $G$ obtained by the removal of $v$ and the edges incident with $v$. If $e \in E(G)$, the spanning subgraph with edge set $E(G) - \{e\}$ is denoted by $G - e$ and it is the subgraph of $G$ obtained by the removal of the edge $e$.

Definition 1.4. Let $G = (V, E)$ be a graph and let $v \in V$. Then $N(v) = \{u \in V / uv \in E(G)\}$ is called the open neighborhood of $v$ and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$. The degree of $v$ is
the number of edges of $G$ incident with $v$ and is denoted by $d(v)$. The minimum and maximum degrees of vertices of $G$ are denoted by $\delta$ and $\Delta$ respectively. A vertex of degree zero in $G$ is called an isolated vertex and a vertex of degree one is called a pendant vertex or an end vertex.

A graph $G$ is regular of degree $r$ if every vertex of $G$ has degree $r$. Such graphs are called $r$-regular graphs.

Definition 1.5. A graph $G$ is complete, if every pair of its vertices are adjacent. A complete graph on $p$ vertices is denoted by $K_p$.

Definition 1.6. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2$ and $(V_1, V_2)$ is called a bipartition of $G$. If further every vertex of $V_1$ is joined to every vertex of $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a star.

A graph $G = (V, E)$ is called $r$-partite graph if the vertex set $V$ is partitioned in to $r$ subsets $V_1, V_2, \ldots, V_r$ so that there exists no edge in $G$ connecting two elements of $V_i$ for $i = 1, 2, \ldots, r$. 
A complete $r$-partite graph is an $r$-partite graph in which all possible edges are included. It is denoted by $K_{n_1, n_2, \ldots, n_r}$.

**Definition 1.7.** Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u$-$v$ walk of $G$ is a finite alternating sequence $u = u_0 e_1 u_1 e_2 \ldots u_{n-1} e_n u_n = v$ of vertices and edges beginning with vertex $u$ and ending with vertex $v$, such that $e_i = u_{i-1} u_i$, where $i = 1, 2, \ldots, n$. The number $n$ is called the length of the walk. The walk is said to be open if $u$ and $v$ are distinct vertices. It is closed otherwise. A walk $u_0 e_1 u_1 e_2 \ldots u_{n-1} e_n u_n$ is determined by the sequence $u_0 u_1 u_2 \ldots u_n$ of its vertices and hence we specify this walk by $(u_0, u_1, u_2, \ldots, u_n)$. A walk in which all edges are distinct is called a trail. A walk in which all vertices are distinct is called a path. A closed walk $(u_0 u_1 u_2 \ldots u_n)$ in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a cycle. A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$.

**Definition 1.8.** A graph $G$ is said to be connected if any two distinct vertices of $G$ is joined by a path. A maximal connected subgraph of $G$ is called a component of $G$.

The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$-$v$ path in $G$. 

4
Definition 1.9. A cut-vertex (cut-edge) of a graph \( G \) is a vertex (an edge) whose removal increases the number of components.

A non-separable graph is a connected, nontrivial graph having no cut vertices. A block of a graph \( G \) is a maximal non-separable subgraph of \( G \).

Definition 1.10. A graph having no cycles is called an acyclic graph. A connected acyclic graph is called a tree. A tree which yields a path when its pendant vertices are removed is called a caterpillar. Caterpillar is denoted by \( F \). A spider is a tree which has at most one vertex of degree \( \geq 3 \).

Definition 1.11. A graph is a block-graph of some graph if each block of \( G \) is a complete subgraph.

If \( x = uv \) is an edge of \( G \), and \( w \) is not a vertex of \( G \), then \( x \) is subdivided when it is replaced by the edges \( uw \) and \( wv \). If every edge of \( G \) is subdivided, the resulting graph is called the subdivision graph of \( G \) and is denoted by \( S(G) \).

Definition 1.12 [22]. Let \( G_1 \) and \( G_2 \) be two graphs with disjoint vertex sets \( V_1 \) and \( V_2 \) and edge sets \( E_1 \) and \( E_2 \) respectively. The join of \( G_1 \) and \( G_2 \) is denoted by \( G_1 + G_2 \) and is defined as \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{ \{h, g \} : h \in V(G_1), g \in V(G_2) \} \). The graph \( K_1 + C_{n-1} \) is called the wheel with \( n \) vertices and is denoted by \( W_n \).
**Cartesian product** $G_1 \times G_2$ is defined to be the graph whose vertex set is $V_1 \times V_2$ and edge set is $\{(u_1, v_1), (u_2, v_2)\}/ u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. The **weak direct product** of $G_1$ and $G_2$ is defined to be the graph $G_1 \oplus G_2$ where $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \oplus G_2) = \{(u_1, v_1)(u_2, v_2)| u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. The **strong direct product** of $G_1$ and $G_2$ denoted $G_1 \circ G_2$ has $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \circ G_2) = E(G_1 \times G_2) \cup E(G_1 \oplus G_2)$.

We now present some of the basic definitions and results regarding domination, independence and irredundance in graphs. One of the fastest growing areas within graph theory is the study of domination and related subset problems such as independence, covering and matching. A comprehensive treatment of the fundamental concepts of domination in graphs is given in Haynes et al. [23]. A collection of advanced topics relating to domination in graphs is given in Haynes et al. [24].

**Definition 1.13 [7]**. A set $S \subseteq V$ is said to be a dominating set of $G$ if every vertex in $V - S$ is adjacent to some vertex in $S$. A dominating set $S$ is called a **minimal dominating set** if no proper subset of $S$ is a dominating set of $G$.

The **domination number** of $G$ is the minimum cardinality taken over all dominating sets in $G$ and is denoted by $\gamma$. The **upper domination number**
of $G$ is the maximum cardinality taken over all minimal dominating sets in $G$ and is denoted by $\Gamma$.

**Definition 1.14** [6]. Let $G$ be a graph without isolated vertices. A subset $S$ of $V$ is called a *total dominating set* if every vertex in $V$ is adjacent to at least one vertex in $S$.

If no proper subset of $S$ is a total dominating set, then $S$ is called a *minimal total dominating set* of $G$. The minimum cardinality of a minimal total dominating set of $G$ is called the *total domination number* of $G$ and is denoted by $\gamma$. The maximum cardinality taken over all minimal total dominating sets is called the *upper total domination number* of $G$ and is denoted by $\Gamma$. 

**Definition 1.15** [20]. Let $G = (V, E)$ be a graph, $S \subseteq V$ and $v \in S$. The *private neighbour set* of $v$ is defined by $P_n[v, S] = N[v] - N[S - \{v\}]$. If $P_n[v, S] \neq \emptyset$ for some vertex $v$, then every vertex in $P_n[v, S]$ is called a *private neighbour* of $v$.

We observe that a dominating set $S$ is a minimal dominating set if and only if every vertex $v \in S$ has at least one private neighbour. This observation leads to the concept of irredundance in graphs.

**Definition 1.16** [18,20]. Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called an *irredundant set* if every $v \in S$ has at least one private neighbour.
An irredundant set $S$ is a **maximal irredundant set** if for every vertex $u \in V - S$, the set $S \cup \{u\}$ is not irredundant. The **irredundance number** of $G$ is defined by $\text{ir}(G) = \min \{|S| / S \text{ is a maximal irredundant set of } G \}$. The **upper irredundance number** of $G$ is defined by $\text{IR}(G) = \max \{|S| / S \text{ is a maximal irredundant set of } G \}$.

**Definition 1.17** [3,7]. Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called an **independent set** if no two elements of $S$ are adjacent in $G$. An independent set $S$ is called a **maximal independent set** if it is not properly contained in another independent set. The maximum cardinality of an independent set in $G$ is called the **independence number** of $G$ and is denoted by $\alpha(G)$.

We observe that any maximal independent set is a minimal dominating set. The minimum cardinality of a maximal independent set in $G$ is called the **independent domination number** of $G$ and is denoted by $i(G)$.

Hedetniemi et al. [25] introduced the fractional version of the concept of domination in graph. Let $f$ be the characteristic function of a dominating set $D$ of a graph $G = (V, E)$. Then $f : V \rightarrow \{0, 1\}$ is given by

$$f(u) = \begin{cases} 1 & \text{if } u \in D \\ 0 & \text{otherwise} \end{cases}$$
The dominating property of $D$ can be restated as the following functional inequality. For each $v \in V$, $\sum_{u \in N[v]} f(v) \geq 1$. Similarly $D$ is a total dominating set if and only if $\sum_{u \in V} f(v) \geq 1$. A generalization of the concept of domination arises by allowing functional values in the closed interval $[0, 1]$. For a detailed study on fractional graph theory one may refer to [1,18,19,21,27,28].

**Definition 1.18 [8,9,10,12,14]**. Let $G = (V,E)$ be a graph. A function $f: V \rightarrow [0,1]$ is called a dominating function (DF) of $G$ if $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ for each $v \in V$.

**Definition 1.19 [15]**. Let $f$ and $g$ be functions from $V$ to $[0, 1]$. We define $f < g$ if $f(u) \leq g(u)$ for all $u \in V$, with strict inequality for at least one vertex $u$. A DF $g$ of $G$ is called a minimal dominating function (MDF) if for all $f < g$, $f$ is not a dominating function.

**Definition 1.20**. Let $A$ and $B$ be two subsets of $V$. We say that $B$ dominates $A$ if every vertex in $A - B$ is adjacent to at least one vertex in $B$. If $B$ dominates $A$, then we write $B \rightarrow A$.

$B$ is said to totally dominates $A$ if every vertex in $A$ is adjacent to at least one vertex in $B$ and in this case we write $B \rightarrow_t A$.  

9
Definition 1.21. Let \( f \) be a dominating function. We define the boundary of \( f \) by 
\[
B_i = \left\{ u \in V / f(N[u]) = \sum_{r \in V[i]} f(x) = 1 \right\}.
\]
We define the positive set of \( f \) by 
\[
P_i = \{ u \in V / f(u) > 0 \}.
\]

Theorem 1.22 [15]. A DF \( f \) of \( G \) is an MDF if and only if \( B_i \rightarrow P_i \).

Definition 1.23 [25]. The value of a DF \( f \) is defined to be 
\[
f(V) = \sum_{u \in V} f(u).
\]

The smallest value of a DF of \( G \) is called the fractional domination number of \( G \) and is denoted by \( \gamma_\ast \). The maximum value of an MDF of \( G \) is called the upper fractional domination number and is denoted by \( \Gamma_\ast \).

Definition 1.24[15]. Let \( f \) and \( g \) be two functions from \( V \) to \([0, 1]\) and let \( \lambda \in (0, 1) \). Then the function \( h_\lambda : V \rightarrow [0,1] \) defined by
\[
h_\lambda(v) = \lambda f(v) + (1 - \lambda) g(v)
\]
is called a convex combination of \( f \) and \( g \).

If \( f \) and \( g \) are DF, it is trivial to show that \( h_\lambda \) is also a DF for all \( \lambda \in (0,1) \). However a convex combination of two MDFs may not again be an MDF. The following result shows that the minimality of \( h_\lambda \) is an "all or nothing" situation. In other words if \( f \) and \( g \) are two MDFs, then either all convex combinations of \( f \) and \( g \) are MDFs or none of them is an MDF.
Theorem 1.25 [15]. If \( f \) and \( g \) are MDFs of \( G \) and \( \lambda \in (0,1) \) then \( h_\lambda \) is an MDF of \( G \) if and only if \((B_f \cap B_g) \rightarrow (P_f \cup P_g)\).

Definition 1.26 [14,15]. We define a relation \( R \) on the set of all MDFs of a graph \( G \) by \( gRf \) if and only if any convex combination of \( f \) and \( g \) is again an MDF. A universal minimal dominating function (UMDF) is defined to be an MDF \( g \) such that \( gRf \) for all MDFs \( f \) of \( G \).

Definition 1.27 [9]. A vertex \( v \in V \) is said to absorb a vertex \( u \neq v \) and \( u \) is said to be absorbed by \( v \), if \( N[u] \subset N[v] \). Here \( \subset \) denotes strict inclusion. The vertex \( v \) is called an absorbing vertex and the vertex \( u \) is called an absorbed vertex.

Definition 1.28[9]. Let \( f \) be an MDF of \( G=(V,E) \) and \( A = \{ v \in V / v \text{ is an absorbing vertex} \} \). A vertex \( w \in V \) is called \( f \)-sharp if \( B_f \cap N[w] \subset A \) and \( w \) is called a sharp vertex if \( w \) is \( f \)-sharp for some MDF \( f \) of \( G \).

Theorem 1.29[9]. An MDF \( f \) of \( G=(V,E) \) is a universal MDF if and only if (i) \( V - A \subset B \), and (ii) \( f(w) = 0 \) for each sharp \( w \) vertex \( w \) of \( G \).

Theorem 1.30[9]. The complete \( r \)-partite graph \( G = K_{n_1, n_2, \ldots, n_r} \) has a universal MDF.
Definition 1.31 [11, 25]. Let $G = (V, E)$ be a graph without isolated vertices. A function $f : V \to [0,1]$ is called a total dominating function (TDF) if

$$f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1 \text{ for all } v \in V .$$

The integer valued TDFs are precisely the characteristic functions of total dominating sets.

Definition 1.32[11]. A TDF $f$ is called a minimal total dominating function (MTDF) if for all $g < f$, $g$ is not a TDF.

The smallest value of a TDF of $G$ is called the fractional total domination number of $G$ and is denoted by $\gamma_t^o(G)$. The maximum value of an MTDF of $G$ is called the upper fractional domination number and is denoted by $\Gamma_t^o(G)$.

Definition 1.33 [11]. Let $f$ be a TDF of a graph $G$. The boundary of $f$ is defined by $B_f = \{v \in V / f(N(v)) = \sum_{u \in N(v)} f(u) = 1 \}$. The positive set of $f$ is defined by $P_f = \{v \in V / f(v) > 0 \}$.

Theorem 1.34 [11]. A TDF $f$ is minimal if and only if $B_f \rightarrow P_f$. 

12
Theorem 1.35 [11]. Let \( f \) and \( g \) be MTDFs and let \( h_\lambda = \lambda f + (1 - \lambda)g \) where \( 0 < \lambda < 1 \). Then \( h_\lambda \) is an MTDF if and only if \( B, \cap B_g \rightarrow, P \cup P_g \).

Definition 1.36 [11]. A vertex \( v \in V \) is called a remote vertex if \( v \in N(u) \) for some \( u \in V \) with \( d(u) = 1 \).

Theorem 1.37 [11]. A graph \( G \) has a unique MTDF if and only if every vertex of \( G \) is adjacent to a remote vertex.

Definition 1.38 [31]. Let \( G = (V, E) \) be a graph and let \( V(G) \) and \( w \in V(G) \). We say that \( v \) is split if \( w \) and the edges \( \{wx \mid x \in N(v)\} \) are added to \( G \). The resulting graph is denoted by \( G(v, w) \). In \( G(v, w) \), we have \( N(v) = N(w) \). This operation is called the splitting operation.

The inverse of splitting operation is to delete the vertex \( v \) from \( G \), if there exists another vertex \( w \) such that \( N(v) = N(w) \).

Let \( T(G) \) denote the class of all graphs obtained by applying either splitting operation or inverse splitting operation on \( G \).

Theorem 1.39 [16]. Any caterpillar has \( 0 - 1 \) universal MTDF.

Theorem 1.40 [31]. The graph \( G \) has a universal MTDF if and only if \( G(v, w) \) has a universal MTDF.
Corollary 1.41[31]. The graph $G$ has a universal MTDF if and only if every graph in $T(G)$ has a universal MTDF.

Corollary 1.42[31]. $T(P_n), T(C_n), T(K_n), T(W_n)$ and $T(F)$ all have universal MTDF.

Definition 1.43[20]. Let $G = (V, E)$ be a graph. A function $g : V \rightarrow [0, 1]$ is called irredundant if for every vertex $v \in V$ with $g(v) > 0$ there exists a vertex $w \in N[v]$ such that $g(N[w]) = 1$.

An irredundant function $g$ is called maximal irredundant function if there does not exist another irredundant function $h$ with $h(v) \geq g(v)$ for all $v \in V$ and $h(v) > g(v)$ for some $v \in V$.

The minimum and maximum values taken over all maximal irredundant functions are respectively denoted by $ir_r(G)$ and $IR_r(G)$.

Cockayne et al. [12] introduced the concept of convexity graph with respect to MDFs of a graph $G = (V, E)$.

Definition 1.44. Let $\mathcal{F}_r$ denote the set of all MDFs of a graph $G$. We define a relation $\rho$ on $\mathcal{F}_r$ as follows: $g \rho f$ if and only if $B_f = B_g$ and $P_f = P_g$. Then $\rho$ is an equivalence relation and $\mathcal{F}_r$ is partitioned into a finite number of equivalence classes. Let $X = \{X_1, X_2, \ldots, X_r\}$ denote the set of all equivalence classes. We now construct a graph $G_x$ as follows.
\( V(G) = X = \{ X_1, X_2, \ldots, X_r \} \) and \( X_i \) is adjacent to \( X_j \) if there exists \( f \in X_i \) and \( g \in X_j \) such that the convex combination of \( f \) and \( g \) is an MDF. The graph \( G \) is called the convexity graph of \( G \).

**Proposition 1.45** [24]. For any graph \( G \) we have,

(i) \( \gamma_i(G) \leq \gamma(G) \leq \gamma_r(G) \leq \Gamma_i(G) \).

(ii) \( \gamma_i(G) \leq \gamma_r^0(G) \leq \gamma_r(G) \).

(iii) \( \gamma_r^0(G) \leq \Gamma_r^0(G) \).

(iv) \( i\gamma(G) \leq \gamma(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \).

(v) \( \gamma_i(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_r(G) \).

(vi) \( i\gamma_r(G) \leq \gamma_r(G) \leq \Gamma_r(G) \leq IR_r(G) \).

**Theorem 1.46** [21]. For the complete bipartite graph \( K_{r,s} \), where \( r,s \geq 2 \), we have \( \gamma_i(K_{r,s}) = \frac{r(s-1) + s(r-1)}{rs-1} \).

**Theorem 1.47** [18,21]. For any graph \( G = (V,E) \) with \( n \) vertices, minimum degree \( \delta \) and maximum degree \( \Delta \), we have \( \frac{n}{\Delta + 1} \leq \gamma_r(G) \leq \frac{n}{\delta + 1} \).

**Corollary 1.48** [18,21]. \( \gamma_r(K_n) = 1 \) and \( \gamma_r(C_n) = \frac{n}{r} \).

**Corollary 1.49** [18,21]. If \( G \) is an \( r \)-regular graph with \( n \) vertices,
then $\gamma_r(G) = \frac{n}{r + 1}$.

**Definition 1.50.** A digraph $D = (V, A)$ consists of a finite set $V$ of vertices and an arc set $A \subseteq P$, where $P$ is the set of all ordered pairs of distinct vertices of $V$. If $(x, y) \in A$, then the arc is directed from $x$ to $y$. The vertex $x$ is called the *predecessor* of $y$ and $y$ is called the *successor* of $x$.

**Definition 1.51.** A digraph $D = (V, A)$ is called a *symmetrical digraph* if $(x, y) \in A$ implies $(y, x) \in A$.

**Definition 1.52.** The *complement* of a digraph $D = (V, A)$ is the digraph $\bar{D} = (V, P - A)$.

**Definition 1.53.** The *reversal* of a digraph $D = (V, A)$ is the digraph $D^{-1} = (V, A^{-1})$ where $(v, u) \in A^{-1}$ if and only if $(u, v) \in A$.

**Definition 1.54.** Let $D = (V, A)$ be a digraph and let $u \in V$,

- $O(u) = \{ v \in V / (u, v) \in A \}$ is called the *outset* of $u$.
- $I(u) = \{ v \in V / (v, u) \in A \}$ is called the *inset* of $u$.
- $O[u] = O(u) \cup \{ u \}$ is called the *closed outset* of $u$.

Similarly, $I[u] = I(u) \cup \{ u \}$ is called the *closed inset* of $u$.

The *indegree* and *outdegree* of a vertex $u$ are given by $id(u) = |I(u)|$

and $od(u) = |O(u)|$ respectively.
Definition 1.55. A digraph $D=(V,A)$ is \textit{indegree regular} if $id(u)$ is same for all $u \in V$. Similarly if $od(u)$ is same for all $u \in V$, then $D$ is called \textit{outdegree regular}.

Definition 1.56[3]. Let $D=(V,A)$ be a digraph. A subset $S$ of $V$ is called an \textit{independent set} if $(x,y) \not\in A$ for all $x,y \in S$. An independent set $S$ is called a \textit{maximal independent set} if there does not exist an independent set $S'$ such that $S \subset S'$.

Definition 1.57[3]. Let $D=(V,A)$ be a digraph. A subset $S$ of $V$ is called an \textit{absorbing set} if for every vertex $x \not\in S$, there is a vertex $y \in S$ such that $y$ is a successor of $x$. An absorbing set $S$ is called a \textit{minimal absorbing set} if no proper subset of $S$ is an absorbing set.

A subset $S$ of $V$ is called a \textit{dominating set} if for every vertex $x \not\in S$, $x$ is a successor of some vertex $y \in S$.

A dominating set $S$ is called a \textit{minimal dominating set} if no proper subset of $S$ is a dominating set.

Definition 1.58[2,4]. If a subset $S$ of $V$ is both independent and absorbing, then $S$ is called a \textit{kernel} of $D$. If $S \subset V$ is both independent and dominating, then $S$ is called a \textit{solution} of $D$. 
Remark 1.59. The problem of finding fractional domination number or fractional total domination number is equivalent to the problem of obtaining an optimum solution to an LPP of the following form.

Minimize: \[ Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \] \[(1)\]
Subject to \[ Ax \geq b \] \[(2)\] and \[ x \geq 0 \] \[(3)\]

A solution to the LPP is a vector \( x = (x_1, x_2, \ldots, x_n)^T \) which satisfies the constraints \((2)\). A feasible solution to the LPP is a vector \( x \) which satisfies both constraints \((2)\) and \((3)\). A feasible solution which also optimizes the objective function \((1)\) is called an optimal solution.

Theorem 1.60. The set of all feasible solutions of an LPP is a convex set.

Theorem 1.61. There exists an extreme point of the feasible region at which the optimum value of the LPP occurs.