CHAPTER 4

DOMINATING FUNCTIONS IN DIRECTED GRAPHS

In this chapter we extend the notion of dominating functions to directed graphs and define fractional domination number of a directed graph. We also introduce related concepts such as irredundant functions and independent functions for digraphs. We also extend the concept of universal dominating functions to directed graphs and discuss the existence of such functions.

Although domination and related concepts have been extensively studied in graphs, the respective analogues on directed graphs have not received much attention. A survey of the existing results on domination in directed graphs is given in Chapter 15 of Haynes et al. [24]. In this chapter we initiate a study of dominating functions in directed graphs.

Let $S$ be a dominating set of a directed graph $D = (V, A)$. Then the characteristic function $f = \chi_S : V \rightarrow \{0, 1\}$ satisfies the condition $\sum_{x \in S} f(x) \geq 1$ for all $v \in V$. If we replace the set $\{0, 1\}$ by $[0, 1]$, we get the linear relaxation of $f$, which leads to the concept of dominating functions of $D$.

**Definition 4.1.** Let $D = (V, A)$ be a directed graph. A function $f : V \rightarrow [0, 1]$ is called a **dominating function** if $\sum_{x \in S} f(x) \geq 1$ for all $v \in V$. 

A dominating function $f$ is called a \textit{minimal dominating function} if there is no other dominating function $g$ such that $g(v) \leq f(v)$ for all $v \in V$, with strict inequality for at least one vertex.

**Example 4.2.** Consider the graph $D$ given in Figure 4.1

![Figure 4.1](image)

Let $f$ and $g$ be functions from $V$ to $[0,1]$ defined by $f(v_1) = f(v_2) = f(v_3) = 1$; $g(v_1) = 0$ and $g(v_2) = g(v_3) = 1$. Clearly $f$ and $g$ are dominating functions of $D$. Since $g < f$, it follows that $f$ is not an MDF of $D$. It can be easily verified that $g$ is an MDF of $D$.

**Definition 4.3.** Let $D = (V, A)$ be a directed graph and let $f$ be a dominating function of $D$. Then the \textit{positive set} $P_i$ and the \textit{boundary set} $B_i$ are defined by

$$P_i = \{v / v \in V \text{ and } f(v) > 0\}$$

and

$$B_i = \left\{v / v \in V \text{ and } \sum_{x \in \{v\}} f(x) = 1\right\}.$$
Definition 4.4. Let \( D = (V, A) \) be a directed graph. Let \( X \) and \( Y \) be two subsets of \( V \). We say that \( X \) absorbs \( Y \) [denoted by \( X \leftarrow \_ \_ Y \)] if \( O[u] \cap X \neq \phi \) for all \( u \in Y \).

Cockayne et al. [11] studied the conditions required for a total dominating function \( f \) to be minimal for undirected graphs. In the following theorem we obtain a similar result for dominating functions in directed graphs.

**Theorem 4.5** A dominating function \( f \) of a directed graph \( D = (V, A) \) is minimal if and only if \( B, \_ \_ P, \_ \_ \).

**Proof:** Let \( f \) be a dominating function with \( B, \_ \_ P, \_ \_ \). Suppose \( f \) is not minimal. Then there exists a dominating function \( g \) such that \( g(x) \leq f(x) \) for all \( x \in V \) and \( g(x) < f(x) \) for at least one \( x \in V \). Let \( v \) be the vertex such that \( g(v) < f(v) \). Hence \( v \in P, \_ \_ \). Since \( B, \_ \_ P, \_ \_ \), there exists \( u \in O[v] \) such that \( f(I[u]) = 1 \). Hence \( \sum_{x \in [u] \cup \{v\}} f(x) + f(v) = 1 \). Since \( g(x) \leq f(x) \) for all \( x \in V \) and \( g(v) < f(v) \), it follows that \( g(I[u]) = \sum_{x \in [u] \cup \{v\}} g(x) + g(v) < 1 \). Hence \( g \) is not a dominating function, which is a contradiction. Thus \( f \) is an MDF.

Conversely, let \( f \) be an MDF of a directed graph \( D \). Suppose that \( B, \_ \_ \) does not absorb \( P, \_ \_ \). Then there exists \( u \in P, \_ \_ \) such that \( B, \_ \_ \cap O[u] = \phi \). Let
\( O[u] = \{ v_1, v_2, \ldots, v_n \} \). Then \( f(I[v_i]) = \sum_{x \in I[v_i]} f(x) > 1 \) for all \( v_i \in O[u] \). Let 
\[ \alpha_i = f(I[v_i]) - 1 \] and \( \alpha = \min \{ f(u), \alpha_1, \alpha_2, \ldots, \alpha_n \} \).

Clearly \( \alpha > 0 \). Now consider the function \( g : V \rightarrow [0,1] \) defined by,
\[
g(v) = \begin{cases} f(u) - \alpha & \text{if } v = u \\ f(v) & \text{otherwise} \end{cases}
\]

Then \( g(I[v_i]) \geq 1 \) for \( i = 1, 2, \ldots, n \) and \( g(I[v]) = f(I[v]) \geq 1 \) if \( v \in V - O[u] \). Hence \( g \) is a dominating function and \( g < f \). This shows that \( f \) is not minimal, which is a contradiction. \( \square \)

**Lemma 4.6.** Let \( D = (V, A) \) be a directed graph and let \( f \) and \( g \) be two DFs of \( D \). Then any convex combination of \( f \) and \( g \) is again a DF.

**Proof:** Let \( h = \lambda f + (1 - \lambda) g \) where \( 0 < \lambda < 1 \). Since \( f \) and \( g \) are DFs, we have \( f(I[v]) \geq 1 \) and \( g(I[v]) \geq 1 \) for all \( v \in V \). Hence it follows that \( h(I[v]) \geq 1 \) for all \( v \in V \), so that \( h \) is a DF of \( D \). \( \square \)

**Remark 4.7.** Convex combination of two MDFs of a directed graph \( D \) need not be an MDF.

For example, consider the graph \( D = (V, A) \) given in Figure 4.2.
Figure 4.2.

Let \( f \) and \( g \) be defined by \( f(v_1) = f(v_2) = 1, f(v_3) = 0 \); \( g(v_1) = 0 \) and \( g(v_2) = g(v_3) = 1 \). Clearly \( f \) and \( g \) are dominating functions of \( D \). Also \( B_f = \{v_1, v_3\} \) and \( B_g = \{v_1, v_2\} \). \( P_f = \{v_1, v_2\} \) and \( P_g = \{v_2, v_3\} \). Clearly \( B_f \leftrightarrow P_f \) and \( B_g \leftrightarrow P_g \). Hence \( f \) and \( g \) are MDFs of \( D \). Now, let \( h = \frac{1}{2} f + \frac{1}{2} g \). Then \( B_h = \{v_1\} \) and \( P_h = \{v_1, v_2, v_3\} \). Clearly \( B_h \) doesn't absorb \( P_h \) and hence \( h \) is not an MDF of \( D \). \( \square \)

The following theorem gives a necessary and sufficient condition for the convex combination of two MDFs to be minimal.

**Theorem 4.8.** Let \( D = (V, A) \) be a directed graph and let \( f \) and \( g \) be two MDFs of \( D \). Then any convex combination of \( f \) and \( g \) is again an MDF of \( D \), if and only if \( B_f \cap B_g \leftrightarrow P_f \cup P_g \).
Proof: Let \( h_\lambda = \lambda f + (1-\lambda)g \) where \( 0 < \lambda < 1 \). By Theorem 4.5, \( h_\lambda \) is an MDF if and only if, \( B_{h_\lambda} \leftarrow P_{h_\lambda} \). Further \( P_{h_\lambda} = P_f \cup P_g \) and \( B_{h_\lambda} = B_f \cap B_g \). Hence the result follows.

E.J. Cockayne et al. [11] characterized graphs having unique MTDFs. We now proceed to characterize directed graphs having unique minimal dominating functions.

Lemma 4.9. Let \( D = (V, A) \) be a directed graph. Let \( X = \{ v \in V / I[v] = \{v\} \} \) and \( C = \{ v \in V / f(v) = 1 \) for all MDFs \( f \) of \( D \} \). Then \( X = C \).

Proof: Let \( v \in C \). Suppose \( v \notin X \). Let \( f \) be an MDF of \( D = (V, A) \). Since \( v \notin X \), \( I(v) \neq \phi \).

Now, define \( g : V \to [0,1] \) by

\[
g(x) = \begin{cases} 0 & \text{if } x = v \\ 1 & \text{if } x \in I(v) \cup O(v) \\ f(x) & \text{otherwise} \end{cases}
\]

Since \( I(v) \neq \phi \), we have \( g(I[v]) \geq 1 \). Also \( g(I[x]) \geq g(x) = 1 \) for all \( x \in I(v) \cup O(v) \). Now suppose \( x \in V - (I[v] \cup O[v]) \). If \( I[x] \cap (I[v] \cup 0[v]) = \phi \), let \( y \in (I[x] \cap (I[v] \cup 0[v])) \). If \( y \neq v \), then \( g(I[x]) \geq g(y) = 1 \). If \( y = v \), then \( g(I[x]) \geq g(x) = 1 \). If \( I(x) \cap (I[v] \cup 0[v]) = \phi \), then \( g(I[x]) = f(I[x]) \geq 1 \). Thus \( g \) is a DF. Hence there exists an MDF \( g' \) such that \( g' \leq g \). Since \( g(v) = 0 \),
we have $g'(v) = 0$ and hence $v \not\in C$, which is a contradiction. Thus $v \in X$, so that $C \subseteq X$.

Now let $v \in X$. Then $I(v) = \emptyset$ and hence $f(v) = 1$ for all MDFs $f$ of $D$. Thus $v \in C$, so that $X \subseteq C$. Hence $X = C$. \[\square\]

**Theorem 4.10.** Let $D = (V, A)$ be a directed graph. Then $D$ has a unique MDF if and only if $I[v] \cap X \neq \emptyset$, for all $v \in V$.

**Proof:** Suppose $I[v] \cap X \neq \emptyset$ for all $v \in V$. Let $g$ be any MDF of $D$. It follows from Lemma 4.9 that $g(v) = 1$ for all $v \in X$. We now prove that $g(v) = 0$ for all $v \not\in X$. Suppose there exists a vertex $v \not\in X$ such that $g(v) > 0$. Choose a vertex $u \in I[v] \cap X$. By Lemma 4.9, $g(u) = 1$ and hence $v \not\in B_g$. Since $g$ is an MDF, there exists a vertex $w \in B_g \cap O[v]$. Again since $I[w] \cap X \neq \emptyset$, there exists $x \in I[w] \cap X$ such that $g(x) = 1$. Since $x, v \in I[w]$, $g(I[w]) \geq g(x) + g(v) > 1$, which is a contradiction, since $w \in B_g$.

Hence $g(v) = 0$.

Thus $g(v) = \begin{cases} 1 & \text{if } v \in X \\ 0 & \text{if } v \not\in X \end{cases}$.

Hence $D$ has a unique MDF.
Conversely, Let \( g \) be the unique MDF of \( D = (V, A) \). Let \( C = \{ v \in V / g(v) = 1 \} \). It follows from Lemma 4.9 that \( C = X \) and \( g(v) = 0 \) for all \( v \in X \). Since \( g(I[v]) \geq 1 \), it follows that \( I[v] \cap X \neq \emptyset \) for all \( v \in V \).

**Example 4.11.** Let \( D = (V, A) \) be the directed graph obtained by orienting the edges of the graph \( K_{n,n} \) in such a way that all its pendant vertices, \( v_1, v_2, \ldots, v_n \), have indegree 0 (Figure 4.3).

![Figure 4.3](image)

Let \( X = \{v_1, v_2, \ldots, v_n\} \). Clearly, \( I[v_i] \cap X = \{v_i\} \) for \( i = 1, 2, \ldots, n \) and \( I[v] \cap X = X \). Hence \( D \) has a unique MDF \( f \) given by,

\[
    f(x) = \begin{cases} 
        0 & \text{if } x = v \\
        1 & \text{if } x = v_i, \ i = 1,2,\ldots,n.
    \end{cases}
\]

**Theorem 4.12.** Let \( D = (V, A) \) be a directed graph. Then \( D \) has either a unique MDF or infinite number MDFs.

**Proof:** Suppose \( D \) has more than one MDF. Then by Theorem 4.10, there exists a vertex \( u \) such that \( I[u] \cap X = \emptyset \). Let \( I[u] = \{u_1, u_2, \ldots, u_r\} \) and let \( r \) be any real number with \( r > 1 \).
Define $g_r : V \rightarrow [0,1]$ by

$$g_r(v) = \begin{cases} 
\frac{1}{r} & \text{if } v = u \\
\frac{(1-1/r)}{n} & \text{if } v = u, \text{ for } i = 1,2,...,n \\
1 & \text{if } v \in V - I[u] 
\end{cases}$$

Now $g_r(I[u]) = g_r(u) + \sum_{i=1}^{n} g_r(u_i) = \frac{1}{r} + \frac{n(1-\frac{1}{r})}{n} = 1$.

Also since $I[u] \cap X = \emptyset$, $|I[u]| \geq 2$ for all $i = 1,2,...,n$ and hence $g_r(I[u]) > 1$. Obviously $g_r(I[v]) > 1$ for all $v \in V - I[u]$.

Thus $g_r$ is a dominating function of $D$. Hence there exists an MDF $h_r$ such that $h_r \leq g_r$. Clearly $h_r(v) = \frac{1}{r}$ and $h_r(u_i) = \frac{(1-\frac{1}{r})}{n}$ for all $i$. Thus for each $r > 1$, there exists an MDF $h_r$ such that $h_r(v) = \frac{1}{r}$, and hence $D$ has infinite number of MDFs.

**Definition 4.13.** Let $D = (V, A)$ be a directed graph. An MDF $f$ of $D$ is called a universal MDF (UMDF) if its convex combination with any other MDF is an MDF.

**Example 4.14.** Consider the digraph $D = (V, A)$ given in Figure 4.4.
Let $f$ be any MDF of $D$. Then $f(v) > 0$ for at least two vertices. We have the following cases.

**Case (i) $f(v) = 0$ for exactly one vertex.**

Suppose $f(v_1) = 0$. Since $f(I[v_1]) \geq 1$ and $0 \leq f(v_1) \leq 1$, we have $f(v_1) = 1$. By a similar argument we can prove that $f(v_2) = 1$. Let the corresponding MDF of $D$ be $f_1$.

Then $f_1(v_1) = 0$, $f_1(v_2) = f_1(v_3) = 1$.

Similarly assuming $f(v_2) = 0$ we get the MDF $f_2$ given by $f_2(v_1) = 0$ and $f_2(v_1) = f_2(v_3) = 1$.

If $f(v_3) = 0$ we get the MDF $f_3$ given by $f_3(v_1) = 0$ and $f_3(v_1) = f_3(v_2) = 1$.

**Case (ii) $f(v) > 0$ for all $v \in V$.**
Since \( f(v_1) > 0 \), either \( v_1 \in B_i \) or \( v_2 \in B_i \). Similarly either \( v_2 \in B_i \) or \( v_3 \in B_i \) and either \( v_1 \in B_i \) or \( v_3 \in B_i \). Thus \( |B_i| \geq 2 \). Now four cases are possible.

(i) \( B_i = \{v_1, v_2, v_3\} \).

Then \( f(v_1) + f(v_2) = 1 \)
\( f(v_1) + f(v_3) = 1 \)
\( f(v_2) + f(v_3) = 1 \).

Solving these equations, we get \( f(v_i) = \frac{1}{2} \) for all \( i \).

We denote this function by \( f_i \).

(ii) \( B_i = \{v_1, v_2\} \).

Then \( f(v_1) + f(v_2) = 1 \)
\( f(v_1) + f(v_3) = 1 \).

Hence \( f(v_2) = f(v_3) = (1 - f(v_1)) \). Also since \( f(I[v_3]) = f(v_3) + f(v_2) \geq 1 \), we have \( 2(1 - f(v_1)) \geq 1 \) and hence \( f(v_1) \leq \frac{1}{2} \). Hence \( 0 \leq f(v_i) \leq \frac{1}{2} \). Now, let \( \lambda = 2f(v_1) \). Clearly \( 0 \leq \lambda \leq 1 \). We claim that \( f = \lambda f_i + (1 - \lambda) f_i \),

\[
(\lambda f_i + (1 - \lambda) f_i)(v_1) = \lambda f_i(v_1) + (1 - \lambda) f_i(v_1)
= \lambda \frac{1}{2} + (1 - \lambda) 0
= f(v_1).
\]

Also, \( (\lambda f_i + (1 - \lambda) f_i)(v_2) \)
\[
\begin{align*}
= \lambda f_4(v_2) + (1 - \lambda) f_i(v_2) \\
= 2 f(v_1) \frac{1}{2} + (1 - 2 f(v_1))! \\
= 1 - f(v_1) \\
= f(v_1)
\end{align*}
\]

Similarly \((\lambda f_4 + (1 - \lambda) f_i)(v_3) = f(v_3)\).

Thus \(f = \lambda f_4 + (1 - \lambda) f_i\). Hence \(f\) is a convex combination of \(f_i\) and \(f_4\).

(iii) \(B_f = \{v_2, v_3\}\).

Proceeding as in (ii), we can prove that \(f\) is a convex combination of \(f_2\) and \(f_i\).

(iv) \(B_f = \{v_1, v_2\}\).

In this case \(f\) is a convex combination of \(f_3\) and \(f_4\).

Clearly if \(f\) is any MDF of \(D\) then the convex combination of \(f_4\) and \(f\) is an MDF. Hence \(f_4\) is a UMDF of \(D\).

**Theorem 4.15.** Any indegree regular directed cycle \(C_n (n > 3)\) has a UMDF.

**Proof:** Let \(V(C_n) = \{v_1, v_2, \ldots, v_n\}\) and \(A(C_n) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)\}\). Define \(g : V \rightarrow [0,1]\) by \(g(v_i) = \frac{1}{2}\) for all \(v_i \in V\). Clearly \(g\) is a dominating function and \(B_g = P_x = V\) and \(B_{g^*} = P_{x^*}\). Hence \(g\) is an MDF of \(C_n\). Now let \(f\) be an arbitrary MDF of \(C_n\). Since \(I[v_1] = \{v_{n-1}\}\), it follows that if \(f(v_1) = 0\), then
\[ f[v_i] = 1 \text{ and } v_i \in B_f. \] If \( f(v_i) > 0 \) then either \( v_i \in B_f \) or \( v_{i+1} \in B_f \). Hence \( B_f \leftarrow v_i \}. \) Thus \( B_f \leftarrow V \). Also, since \( B_k = P_k = V \), we have \( B_f \cap B_k = B_f \) and \( P_f \cup P_k = V \). Hence \( B_f \cap B_k \leftarrow P_f \cup P_k = V \). Hence the convex combination of \( f \) and \( g \) is an MDF, so that \( g \) is a universal MDF.

**Definition 4.16.** Let \( D = (V, A) \) be a digraph. The *fractional domination number* \( \gamma_f(D) \) and the *upper fractional domination number* \( \Gamma_f(D) \) are defined by,

\[
\gamma_f(D) = \min \{ |f| / f \text{ is an MDF of } D \},
\]

\[
\Gamma_f(D) = \max \{ |f| / f \text{ is an MDF of } D \}.
\]

**Example 4.17.** Consider the digraph \( D \) given in Example 4.14. It is proved that any MDF of \( D \) is a convex combination of \( f_4 \) and one of the MDFs \( f_1, f_2, \) and \( f_3, f_4 \). If \( g \) is the convex combination of \( g_1 \) and \( g_2 \), then

\[
|g| \leq \max \{ |g_1|, |g_2| \} \text{ and } |g| \geq \min \{ |g_1|, |g_2| \}.
\]

Let \( f \) be any MDF of \( D \) then

\[
|f| \leq \max \{ |f_1|, |f_2|, |f_3|, |f_4| \} \quad \text{and} \quad |f| \geq \min \{ |f_1|, |f_2|, |f_3|, |f_4| \}.
\]

Hence

\[
\gamma_f(D) = \frac{1}{2} \quad \text{and} \quad \Gamma_f(D) = 2.
\]

**Remark 4.18** For any digraph \( D = (V, A) \), we have

\[
\gamma_f(D) \leq \gamma(D) \leq \Gamma(D) \leq \Gamma_f(D).
\]

Further for any digraph \( D \) having a unique MDF, \( \gamma_f(D) = \gamma(D) = \Gamma(D) = \Gamma_f(D) \).
Definition 4.19. Let $D = (V, A)$ be a directed graph. A function $f : V \rightarrow [0,1]$ is called an independent function of $D$ if for any vertex $v$ with $f(v) > 0$, we have

$$\sum_{x \in I[v] \cup O[v]} f(x) = 1.$$  

An independent function $f$ is called a maximal independent function, if $\sum_{x \in I[v] \cup O[v]} f(x) \geq 1$ for all $v \in V$.

The fractional independent domination number $i_f(D)$ and the fractional independence number $\beta_{0f}(D)$ are defined by,

$$i_f(D) = \min \{ |f| \mid f \text{ is an MIF of } D \} \quad \text{and} \quad \beta_{0f}(D) = \max \{ |f| \mid f \text{ is an MIF of } D \}$$

Remark 4.20. Let $G = (V, E)$ be the underlying graph of the digraph $D$. Since $I[v] \cup O[v] = N[v]$, it follows that the set of all independent functions of $D$ and the set of all independent functions of $G$ are same. Hence $i_f(D) = i_f(G)$ and $\beta_{0f}(D) = \beta_{0f}(G)$.

Lemma 4.21. If an independent function $f$ of a directed graph $D = (V, A)$ is also a dominating function, then $f$ is a maximal independent function.

Proof: Let $f : V \rightarrow [0,1]$ be a function which is both independent and dominating. Since $f$ is dominating, $\sum_{x \in I[v]} f(x) \geq 1$ for all $v \in V$. So $\sum_{x \in I[v] \cup O[v]} f(x) \geq 1$ for all $v \in V$. Hence $f$ is an MIF.
**Definition 4.22.** Let \( D = (V, A) \) be a directed graph. A function \( g : V \to [0,1] \) is called an *irredundant function*(IRF) if for every vertex \( v \in V \) with \( g(v) > 0 \), there exists a vertex \( w \in O[v] \) such that \( g(I[w]) = 1 \). An irredundant function \( g \) is called a *maximal irredundant function*(MIRF) if there does not exist an irredundant function \( h \neq g \), with \( h(v) \geq g(v) \) for all \( v \in V \) and \( h(v) > g(v) \) for some \( v \in V \).

The *fractional irredundance number* \( ir_f \) and the *upper functional irredundance number* \( IR_f \) are defined by,

\[
ir_f = \min \left\{ |f|/f \text{ is an MIRF of } D \right\} \text{ and }
\]

\[
IR_f = \max \left\{ |f|/f \text{ is an MIRF of } D \right\}.
\]

**Lemma 4.23.** Every minimal dominating function of a directed graph \( D = (V, A) \) is a maximal irredundant function.

**Proof:** Let \( f \) be a minimal dominating function. Since \( B_i \leftarrow P_i \), for every vertex \( v \) such that \( f(v) > 0 \) there exists \( w \in O[v] \) such that \( f(I[w]) = 1 \). Hence \( f \) is an irredundant function. Now suppose there exists an irredundant function \( g \) such that \( g \neq f \) and \( g \geq f \). Then there exists \( v \in V \) such that \( g(v) > f(v) \). Since \( g \) is irredundant, there exists \( x \in O[v] \) such that \( g(I[x]) = \sum_{y \in I[x]} g(y) = 1 \).
Hence \( \sum_{v \in [x,y]} g(y) + g(v) = 1 \). Now since \( f(v) < g(v) \), it follows that \( f([v]) < 1 \), which is a contradiction. \( \square \)

**Corollary 4.24.** For any digraph \( D = (V, A) \), \( ir, (D) \leq \gamma, (D) \leq \Gamma, (D) \leq IR, (D) \).

**Conclusion and scope.** Although domination and related topics have been extensively studied, the respective analogues on digraphs have not received much attention. A survey of results on domination in directed graphs is given in Chapter 15 of [24]. In this chapter we have extended the concept of dominating functions to directed graphs and have developed some basic properties of these functions. Obviously there is much scope for carrying out this study further. The following are some fundamental problems for further investigation.

(i) Define and develop the theory of total dominating functions in digraphs.

(ii) Find more properties of universal minimal dominating functions of directed graphs.

(iii) Cockayne et al. [9] have obtained a characterization of universal minimal dominating functions of a graph (Theorem 1.29). Obtain a similar characterization for UMDFs of a directed graph.

(iv) Examine the equality of parameters given in the inequality chain
Yu [31] have defined a splitting operation in a graph (Definition 1.38) and used this to prove the existence of UMDFs in some classes of graphs. Is it possible to define a similar operation in directed graphs and prove the existence of UMDFs in similar classes of digraphs?