Chapter 3

FUZZY LOCAL CONVEXITY*

Introduction

In this chapter we introduce the concept of fuzzy local convexity in an ftfcs and define a locally ftfcs. Also it is proved that an FCC, F-continuous, F-open image of a locally ftfcs is a locally ftfcs. We also study the subspace, product and quotient of a locally ftfcs.

3.1 Locally Fuzzy Topology Fuzzy Convexity Spaces

Definition 3.1.1.

Let $a_\lambda$ be a fuzzy point in an ftfcs $(X, \mathcal{L}, T)$. Then a fuzzy set $N$ is called a fuzzy neighbourhood of $a_\lambda$ if there exists $A \in \mathcal{T}$ such that $a_\lambda \in A \subseteq N$.

Definition 3.1.2.

An ftfcs $(X, \mathcal{L}, T)$ is said to be locally fuzzy convex at a fuzzy point $a_\lambda$ if for every fuzzy neighbourhood $U$ of $a_\lambda$ there is some convex fuzzy neighbourhood $C$ of $a_\lambda$ which is contained in $U$.

* Almost all the results of this chapter appeared as a research paper in Fuzzy Sets and Systems 1994 [28].
\[(X, \mathcal{L}, T)\] is locally fuzzy convex if it is locally fuzzy convex at each of its fuzzy points.

Example:

The Euclidean space \(\mathbb{R}^n\) is locally convex under usual convexity and considering crisp as a special case of 'fuzzy', this means \(\mathbb{R}^n\) is locally fuzzy convex.

Example:

Let \(X = \{a, b, c\}\)
\[
T = \{\emptyset, X\} \cup \{\{a\} \mid 0 < a < \frac{1}{2}\}
\]
\[
\mathcal{L} = \{\emptyset, X, \{a\}, \{a, b\}\} \cup \{\{a\} \mid 0 < a < \frac{1}{2}\}
\]

\((X, \mathcal{L}, T)\) is locally fuzzy convex.

Proposition 3.1.3.

An FCC, F-open, F-continuous image of a locally ftfcs is a locally ftfcs.

Proof:

Let \(f: (X, \mathcal{T}_1, T_1) \rightarrow (Y, \mathcal{T}_2, T_2)\) be an FCC, F-open, F-continuous onto map. Let \(a_\lambda\) be a fuzzy point in \(Y\). Then we can find a point \(b\) in \(X\) such that \(f(b) = a_\lambda\). Then
clearly \( f(b_\lambda) = a_\lambda \). Let \( U \) be a fuzzy neighbourhood of \( a_\lambda \) in \( Y \). Then \( f^{-1}(U) \) is a fuzzy neighbourhood of \( b_\lambda \) in \( X \). Since \( X \) is a locally ftfc, there exists an \( \mathcal{L}_1 \)-convex fuzzy neighbourhood \( C \) of \( b_\lambda \) in \( X \) such that

\[ b_\lambda \in C \subseteq f^{-1}(U) \]

\[ \therefore f(b_\lambda) \subseteq f(C) \subseteq U \]

i.e. \( a_\lambda \in f(C) \subseteq U \)

Since \( f \) is an FCC, \( F \)-open onto map, \( f(C) \) is an \( \mathcal{L}_2 \)-convex fuzzy neighbourhood of \( a_\lambda \) in \( Y \). Hence \( Y \) is a locally ftfc.

Proposition 3.1.4.

Any subspace of a locally ftfc is a locally ftfc.

Proof:

Let \( (X, \mathcal{L}, T) \) be a locally ftfc. Let \( M \subset X \) and \( (M, \mathcal{L}_M, T_M) \) be the corresponding subspace of \( (X, \mathcal{L}, T) \). Let \( a_\lambda \) be a fuzzy point in \( M \) and let \( U \) be an \( F \)-open neighbourhood of \( a_\lambda \) in \( M \). i.e. \( a_\lambda \in U \subseteq T_M \). Since \( U \) is \( F \)-open in \( M \), we have \( U = V \cap M \) where \( V \in T \). Since \( X \) is locally fuzzy convex, there exists a convex fuzzy neighbourhood \( C \) of \( a_\lambda \) such that \( a_\lambda \in C \subseteq V \). Then
\[ a \in \mathcal{C} \cap M \subseteq \mathcal{V} \cap M. \] Now \( \mathcal{C} \cap M \) is a convex fuzzy neighbourhood of \( a \) in \((M, \mathcal{T}_M, \mathcal{T}_M)\) and so \( M \) is locally fuzzy convex.

Remark: **Fuzzy Topology Fuzzy Convexity Fuzzy Subspace**

Let \((X, \mathcal{L}, \mathcal{T})\) be an ftfcs and \( M \) a fuzzy subset of \( X \).

Then define

\[
\mathcal{L}_M = \left\{ L \cap M \mid L \in \mathcal{L} \right\}
\]

and

\[
\mathcal{T}_M = \left\{ A \cap M \mid A \in \mathcal{T} \right\}.
\]

We can say that \((M, \mathcal{L}_M, \mathcal{T}_M)\) is a fuzzy topology fuzzy convexity fuzzy subspace of \((X, \mathcal{L}, \mathcal{T})\) in the following sense:

1. \( \emptyset, M \in \mathcal{L}_M \)
2. If \( A_i \in \mathcal{L}_M \) for each \( i \in I \), then \( \bigcap A_i \in \mathcal{L}_M \).
3. If \( A_i \in \mathcal{L}_M \) for each \( i \in I \) and if \( A_i \)'s are totally ordered by inclusion, then \( \bigcup A_i \in \mathcal{L}_M \).

Again

1. \( \emptyset, M \in \mathcal{T}_M \)
2. If \( A, B \in \mathcal{T}_M \) then \( A \cap B \in \mathcal{T}_M \)
3. If \( A_i \in \mathcal{T}_M \), then \( \bigcup A_i \in \mathcal{T}_M \).
Imitating the proof of the above Proposition 3.1.4, we can show that any such fuzzy subspace of a locally ftfcs is a locally ftfcs with obvious definition for locally ftfcs in the case of the subspace.

In the following chapters also wherever we prove results for crisp subspaces, we could obtain analogous results for fuzzy subspaces; however we would be restricting ourselves to crisp subspaces only.

Remark:

We proved in Proposition 1.2.5 that a map \( f : (X, \mathcal{L}_1) \rightarrow (Y, \mathcal{L}_2) \) is an FCC function if and only if \( \mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S)) \), for each fuzzy subset \( S \) of \( X \). However, as in the crisp case (cf. Van de Vel [33]) we can prove the following:

Lemma 3.1.5.

In a product space, the polytopes (i.e. fuzzy convex hulls of finite fuzzy sets) are of the product type and hence for each finite subset \( S \) of the product,

\[
\mathcal{L}(S) = \bigcap_{\alpha \in I} \mathcal{L}(\pi_\alpha(S))
\]
Lemma 3.1.6.

\[ f : (X, \mathcal{L}_1) \longrightarrow (Y, \mathcal{L}_2) \] is FCC if and only if

\[ \mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S)) \]

for each finite fuzzy set \( S \) of \( X \).

Proposition 3.1.7.

The projection map \( \pi_\alpha \) of a product to its factors is both FCP and FCC.

Proof:

In a product space \( X = \prod_{\alpha \in I} X_\alpha \), a fuzzy alignment can be generated by the convex fuzzy sets of the form

\[ \left\{ \pi_\alpha^{-1}(C_\alpha) \mid C_\alpha \text{ is a convex fuzzy set in } X_\alpha, \alpha \in I \right\}, \]

it follows that each \( \pi_\alpha \) is FCP.

That each \( \pi_\alpha \) is FCC is a consequence of the two lemmas and the remark above.

Proposition 3.1.8.

A nonempty product space \( \prod_{\alpha \in I} (X_\alpha, \mathcal{L}_\alpha, \Gamma_\alpha) \) is locally fuzzy convex if and only if each factor is locally fuzzy convex.
Proof:

Suppose each $X_\alpha$ is locally fuzzy convex. Let $\lambda$ be a fuzzy point in $X = \overline{\bigcup X_\alpha}$ and consider a basic fuzzy neighbourhood

$$\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_n)$$

of $\lambda$ in $X$ where $\pi_\alpha$ is the projection map from $X$ to $X_\alpha$. Now $U_i$ is a fuzzy neighbourhood of $(a_{\alpha_i})_\lambda$ in $X_\alpha$ for $i = 1, 2, 3, \ldots, n$, and since each $X_\alpha$ is locally fuzzy convex, $U_i$ contains a fuzzy convex neighbourhood $C_i$ of $(a_{\alpha_i})_\lambda$. i.e. $(a_{\alpha_i})_\lambda \in C_i \subset U_i$. Then

$$\pi_{\alpha_1}^{-1}(C_1) \cap \pi_{\alpha_2}^{-1}(C_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(C_n)$$

is a convex fuzzy neighbourhood of $\lambda$ contained in

$$\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_n).$$

Thus every fuzzy neighbourhood of $\lambda$ contains a convex fuzzy neighbourhood and hence $X$ is locally fuzzy convex.
Conversely let \((a_\lambda)_\lambda\) be a fuzzy point in \(X_\alpha\).
Then we can choose a fuzzy point \(a_\lambda\) in \(X\) such that 
\(\pi_\alpha(a_\lambda) = (a_\lambda)_\lambda\). Let \(U_\alpha\) be a fuzzy neighbourhood of 
\((a_\lambda)_\lambda\) in \(X_\alpha\). Then \(\pi_\alpha^{-1}(U_\alpha)\) is a fuzzy neighbourhood of 
\(a_\lambda\) in \(X\). Since \(X\) is locally fuzzy convex, there exists 
a convex fuzzy neighbourhood \(C\) of \(a_\lambda\) contained in 
\(\pi_\alpha^{-1}(U_\alpha)\). Since \(\pi_\alpha\) is FCC, \(\pi_\alpha(C)\) is a convex fuzzy 
neighbourhood of \((a_\lambda)_\lambda\) in \(X_\alpha\) contained in \(U_\alpha\). Hence 
\(X_\alpha\) is locally fuzzy convex.

Proposition 3.1.9.

Quotient of a locally ftfcs is a locally ftfcs 
if the quotient map is an FCC, F-open map.

Proof:

Let \((X, \mathcal{L}, T)\) be a locally ftfcs. Let \(f\) be the 
quotient map from \(X\) to \(Y\). Let \(G\) be an open fuzzy 
neighbourhood of a fuzzy point \(a_\lambda\), \(0 < \lambda < 1\) in \(Y\). Then 
we can find a point \(b\) in \(X\) such that \(f(b) = a\). Then 
clearly \(f(b_\lambda) = a_\lambda\). Therefore \(f^{-1}(G)\) is an open fuzzy 
neighbourhood of \(b_\lambda\) in \(X\). Since \(X\) is locally fuzzy 
convex, there exists a convex fuzzy neighbourhood \(C\) of \(b_\lambda\)
such that \( b_\lambda \in C \subseteq f^{-1}(G) \).

Then \( f(b_\lambda) \in f(C) \subseteq G \)

i.e. \( a_\lambda \in f(C) \subseteq G \).

Since \( f \) is an FCC, F-open map, \( f(C) \) is a convex fuzzy neighbourhood of \( a_\lambda \) in \( Y \) and hence \( Y \) is a locally ftfcs.

Remark:

If \( X \) is a fuzzy topological vector space with fuzzy convexity as defined in \( \text{l.l} \), then the quotients are locally ftfcs, if the quotient map is F-open. This is because, if \( X \) is a fuzzy topological vector space, then a quotient map \( f \) is a linear map and under a linear map the image of a convex fuzzy set is a convex fuzzy set [11].