CHAPTER 3

Application of Power Series Solution to Branching Processes

3.1 Introduction and Summary

Branching process is a special class of Markov chain. Following biological terminology, let us consider a situation where each organism of our generation produces a random number of offsprings to form the next generation. If the probability distribution of the number of offsprings produced by an organism is given, one becomes interested in many characteristics like the distribution of the size of the population for different generations and the probability of extinction.

Some of the natural phenomena can be modeled as a branching process. This relate to the survival of family names, verbal flow of information, electron multipliers, which amplify a weak current using a series of plates that generate new electrons when hit by an electron. But we prefer to restrict our attention to biological terminology of organism and offspring.

Simple mathematical model for branching processes was formulated by Galton and Watson. But we shall concentrate on Bellman-Harris process which is age dependant branching process.

Description of the Bellman-Harris (BH) process: Let us suppose that an ancestor at time $t = 0$ initiates the process and at the end of its life time
it produces a random number of offsprings having a distribution and the 
process continues as long as objects are present. Here assumption is that 
the offsprings act independent of each other. Also the lifetime of objects are 
i.i.d. random variables with distribution function $G$.

Under the above assumption let $\{X(t), \ t \geq 0\}$ be the number of objects 
available at time $t$. Then the stochastic process $\{X(t), \ t \geq 0\}$ is called 
an age-dependent branching process. It is important to note that an age­
dependent process is Non-Markovian.

In section (3.2) we consider BH process. The probability generating func­
tion for BH process is available in literature. Then we have applied power 
series method for solving integral equation for which the tools have been de­
developed. Applying this technique we have obtained expression for extinction 
probability at any time $t$. This has been done in (3.2.1). Also an example 
with two offsprings have been worked out. This is important for present day 
scenario, because in our society we do not prefer more that two children and 
it is important to know, whether a family name will be extinct or not.

In section (3.2.2) we have worked out the integral equation for mean pop­
ulation size at time $t$ for BH process. Taking the relevant integral equation 
we have demonstrated its solution by the same power series technique and 
also worked out the case with exponential life in terms of the form obtained 
earlier.

In section (3.3) we have considered the problem of offsprings with Bernoulli 
distribution, i.e., either there will be offspring or not.

In section (3.4), a different branching process of BH type is considered 
next. Here we have assumed that the life distributions are different for dif­
different generations. This assumption is quite realistic as with the passage of time, culture, socio-economic, environment, etc change and for these the age distributions vary. Of course, we have considered the case where there is a constant shift from generation to generation. We have obtained the integral equation describing its p.g.f at time $t$ and also have investigated the nature of extinction probability with a suitable example.

3.2 Application of power series solution of integral equation for Bellman-Harris process

3.2.1 General expression for probability of extinction at time $t$

Let $\{X(t), t \geq 0\}$ be the number of objects alive at time $t$ and if the number of offspring obtained from a unit has p.g.f $P(s)$ then as in (Karlin and Taylor) we see that the probability generating function $F(t, s)$ of $X(t)$ follows

$$F(t, s) = [1 - G(t)]s + \int_0^t P[F(t - u, s)]dG(u)$$

Then extinction probability at time $t$ can be obtained as $F(t, 0)$ i.e., putting $s = 0$

$$F(t, 0) = \int_0^t P[F(t - u, 0)]dG(u)$$

Putting $t - u = v$, then

$$F(t, 0) = \int_0^t P[F(v, 0)]g(t - v)dv$$
where \( g \) is density of \( G \). Putting

\[
E(v) = F(v, 0)
\]

we have

\[
E(t) = \int_0^t P[E(v)]g(t - v)dv
\]

Now we assume that the number of offsprings follows the distribution

- **Values**: \( 0, 1, \cdots, k \)
- **Probability**: \( p_0, p_1, \cdots, p_k \)

So \( P(s) = \sum_0^k s^k p_i \)

So in this case the integral equation (3.2.1) changes to

\[
E(t) = \int_0^t \sum_0^k p_i (E(v))^i g(t - v)dv
\]

\[
= \int_0^t g(t - v) \left( \sum_{i=0}^k p_i E(v)^i \right) dv
\]

(3.2.2) is a polynomial nonlinear equation which is considered in Chapter 7.

Thus we see that explicit analytic solution is important. So our problem is to find out the solution \( E(t) \). Let us see this in simpler cases.

**Example**

Let us give an application when \( p_0 = \frac{1}{4}, p_1 = \frac{1}{2}, p_2 = \frac{1}{4} \) which has much practical significance, then

\[
P(s) = \frac{1}{4} + \frac{1}{2} s + \frac{1}{4} s^2 = \frac{1}{4} (1 + s)^2
\]

\[
\Rightarrow E(t) = \int_0^t g(t - v) \frac{1}{4}(1 + E(v))^2 dv \quad \text{from (3.2.1)}
\]
& put \( f(t) = 1 + E(t) \), then above changes to
\[
f(t) = 1 + \int_0^t g(t-v) \frac{1}{4} f^2(v) dv
\]
(3.2.3)

Now according to our previous nonlinear integral equation in chapter 2

\[
F(t) = 1, \ \lambda = \frac{1}{4}, \text{ and } k(t,v) = g(t-v)
\]

\[\Rightarrow \phi_0(t) = 1\]

\[
\phi_1(t) = \int_0^t g(t-v) dv = [-G(t-v)]_0^t = G(t)
\]

\[
\phi_2(t) = \int_0^t 2G'(t-v)(G(v) - G(0)) dv
\]

\[
= 2 \int_0^t G'(t-v)G(v) dv - 2 \int_0^t G'(t-v)G(0) dv
\]

\[
= 2 \int_0^t G'(t-v)G(v) dv - 2G(0) \int_0^t G'(t-v) dv
\]

\[
= 2(G' \ast G)(t) - 2G(0)(G(t) - G(0))
\]

\[
= 2((G \ast G)(t) - G(0)(G(t) - G(0)))
\]

\[
= 2(G' \ast G)(t)
\]

\[
= 2(G \ast g)(t)
\]

\[
\phi_3(t) = \int_0^t \phi_1^2(v) g(t-v) dv + \int_0^t 2\phi_0(v)\phi_2(v) g(t-v) dv
\]

\[
= \int_0^t (G(v))^2 g(t-v) dv + 2 \int_0^t (G \ast g)(v) g(t-v) dv
\]

\[
= (G^2 \ast g)(t) + 2(G \ast g \ast g)(t)
\]

\[
= (G^* \ast g)(t) + 2G \ast (g^2)(t)
\]
Thus successively others functions can be obtained.

So the solution for
\[ f(t) = \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)^k \phi_k(t) \]

\[ \Rightarrow 1 + E(t) = \phi_0(t) + \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \phi_k(t) \quad \text{from (3.2.1) and (3.2.3)} \]

\[ \Rightarrow E(t) = \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \phi_k(t) \]

3.2 General expression for mean population size at time \( t \)

In this case the expectation function satisfies the following integral equation

\[ M(t) = [1 - G(t)] + m \int_0^t M(t - u)dG(u) \]

where \( m = P'(1) = \) mean of the offspring distribution

\[ M(t) = [1 - G(t)] + m \int_0^t M(t - u)g(u)du \]

where \( g \) is density function of \( G \).

Now putting \( t - u = v \)

\[ M(t) = [1 - G(t)] + m \int_0^t M(v)g(t - v)dv \]

We shall try to solve the above integral equation which is a usual Volterra equation of second kind.

From the form of the solution for linear equation we see

\[ M(t) = g_0(t) + mg_1(t) + m^2g_2(t) + \cdots \]
\[ g_0(t) + m g_1(t) + m^2 g_2(t) + \cdots \]

\[ = [1 - G(t)] + m \int_0^t g(t-v) \left[ g_0(v) + m g_1(v) + m^2 g_2(v) + \cdots \right] dv \]

\[ = [1-G(t)]+m \int_0^t g(t-v)g_0(v)dv+m^2 \int_0^t g(t-v)g_1(v)dv+m^3 \int_0^t g(t-v)g_2(v)dv+\cdots \]

\[ \Rightarrow g_0(t) = 1 - G(t) \]

\[ g_1(t) = \int_0^t g(t-v)g_0(v)dv = (g_0 * g)(t) \]

\[ g_2(t) = \int_0^t g(t-v)g_1(v)dv = (g_0 * g^2)(t) \]

\[ g_n(t) = (g_0 * g^n)(t) \]

Example: Now let us consider the exponential life distribution i.e.,

\[ G(t) = 1 - e^{-t} \]

\[ \Rightarrow g_0(t) = e^{-t} \]

Then

\[ g_1(t) = \int_0^t g(t-v)e^{-v}dv \]

\[ = \int_0^t e^{-(t-v)}e^{-v}dv \]

\[ = \int_0^t e^t dv = te^{-t} \]

\[ g_2(t) = \int_0^t e^{-(t-v)}ve^{-v}dv \]

\[ = \int_0^t ve^{-t}dv = e^{-t}t^2/2 \]

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\[ g_3(t) = \int_0^t e^{-(t-u)}e^{-\frac{u^2}{2}}du \]
\[ = \int_0^t \frac{v^2}{2}e^{-t}dv = e^{-t}t^3 \]
\[ \vdots \]
\[ g_n(t) = e^{-t}\frac{t^n}{n!} \]
\[ \Rightarrow M(t) = \sum_{k=0}^{\infty} e^{-t}\frac{t^k}{k!} \]
\[ = e^{-t}e^{mt} = e^{(m-1)t} \]

This is the mean size of population at time \( t \).

### 3.3 Expression for probability of extinction in Bellman-Harris process with one offspring

In this section we consider another application though it is not an application of our solution of the nonlinear integral equation but we borrow the solution of linear integral equation.

Suppose the time when a unit gives birth to one offspring that follows some distribution whose life distribution function is given by \( G(t), t \geq 0 \) and suppose the probability of single offspring = \( p \) and no offspring = \( 1-p = q \). This resembles to the present day family welfare management.

So it is clear that always at any time \( t \) either there will be an offspring from that generation or vanish at that time. So let us see the situation at any time \( t \).
In this case also we are to solve the similar type of integral equation

\[ F(t, s) = [1 - G(t)]s + \int_0^t P[F(t-u, s)]dG(u) \]

with probability generating function \( P(s) = q + ps \) i.e.,

\[ F(t, s) = [1 - G(t)]s + \int_0^t [q + pF(t-u, s)]g(u)du \]
\[ = [1 - G(t)]s + qG(t) + p\int_0^t F(t-v, s)g(u)du \]
\[ = s + (q-s)G(t) + p\int_0^t F(v, s)g(t-v)dv \]

and in this case in order to find out extinction probability at time \( t \) we are to find out \( F(t, 0) \) where

\[ F(t, 0) = qG(t) + p\int_0^t F(v, 0)g(t-v)dv \]

Now putting \( f(t) = F(t, 0) \)

\[ f(t) = qG(t) + p\int_0^t f(v)g(t-v)dv \]

This is Volterra integral equation of second kind and solution is given by

\[ f(t) = \phi_0(t) + p\phi_1(t) + p^2\phi_2(t) + \cdots \]

where

\[ \phi_0(t) = qG(t) \]
\[ \phi_1(t) = \int_0^t g(t-v)qG(v)dv = q(G * g)(t) \]
\[ \phi_2(t) = \int_0^t g(t-v)q(G * g)(v)dv = q(G * g^2)(t) \]
\[ \vdots \]

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\[
\phi_n(t) = \int_0^t g(t - v) q(G * g^{(n-1)}(v)) dv = q(G * g^n)(t)
\]

So

\[
f(t) = q \sum_{k=0}^{\infty} p^k (G * g^k)(t)
\]

### 3.4 Bellman-Harris type process with varying life distributions over generations

In this section we shall develop equation describing the branching process having age dependent life and it is true that there must not be age equal in all generations. It varies from generation to generation. With the advancement of time there is also change in distribution. So we assume that there is a constant shift in age distribution. Under this assumption we calculate probability generating function at first. In fact the probability generating function carries all information of a probability distribution. Thus we generalise Bellman-Harris process.

The generating function of Bellman-Harris process is given by

\[
F(t, s) = \sum_{k=0}^{\infty} Pr\{X(t) = k\} s^k of\{X(t), t \geq 0\}
\]

\(X_0 = 1\). Where

\[
F(t, s) = [1 - G(t)]s + \int_0^t P[F(t - u, s)]dG(u)
\]

holds. Hence \(P(s)\) is p.g.f of the offspring obtained from a unit having life distribution \(G(.)\) and this is same in all units and in all generations. We shall assume that it varies from generation to generation i.e., 0th generation age
distribution is $G(u)$ and in 1st generation it is $G_1(u) = G(u - \Delta)$ and second it is $G_2(u) = G(u - 2\Delta)$ and so on.

Let $F(t, s)$ be the generating function using age distribution $G(u), G(u - \Delta), G(u - 2\Delta)$ etc.

$F_1(t, s)$ be the generating function using age distribution $G(u - \Delta), G(u - 2\Delta)$ etc.

Similarly $F_2(t, s)$ be the generating function using age distribution $G(u - 2\Delta), G(u - 3\Delta)$ etc.

$$F(t, s) = [1 - G(t)]s + \int_0^t P[F_1(t - u, s)] dG(u)$$

and similarly

$$F_1(t, s) = (1 - G(t - \Delta))s + \int_0^t P[F_2(t - u, s)] dG_1(u)$$

$$= (1 - G(t - \Delta))s + \int_0^{t-\Delta} P[F_2(t - u - v, s)] dG_1(u - \Delta)$$

$$= (1 - G(t - \Delta))s + \int_0^{t-\Delta} P[F_2(t - \Delta - v, s)] dG(u)$$

$$F_2(t, s) = (1 - G(t - 2\Delta))s + \int_0^{t-2\Delta} P[F_3(t - 2\Delta - v, s)] dG(v)$$

Putting $F_1$ in equation we have

$$F(t, s) = (1 - G(t))s + \int_0^t P[(1 - G(t - \Delta - u))s$$

$$+ \int_0^{t-\Delta-u} P[F_2(t - \Delta - u - v, s)] dG(v)] dG(u)$$

$$= (1 - G(t))s + \int_0^{t-\Delta} P[(1 - G(t - \Delta - u))s$$

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+ \int_0^{t-\Delta-u} P[F_2(t - \Delta - u - v, s)] dG(v) \right] dG(u)

Repeating the same thing we have

= (1-G(t))s + \int_0^{t-\Delta} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - 3\Delta - u - v - w, s)] dG(w) \right] dG(v) \right] dG(u), v \leq t-\Delta-u

= (1-G(t))s + \int_0^{t-\Delta} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - 3\Delta - u - v - w, s)] dG(w) \right] dG(v) \right] dG(u)

= (1-G(t))s + \int_0^{t-3\Delta} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - 3\Delta - u - v - w, s)] dG(w) \right] dG(v) \right] dG(u)

= (1-G(t))s + \int_0^{t-3\Delta-u} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - 3\Delta - u - v - w, s)] dG(w) \right] dG(v) \right] dG(u)

= (1-G(t))s + \int_0^{t-3\Delta-u} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - a_2\Delta - u_1 - u_2, s)] dG(u_2) \right] dG(u_1)

Repeating this process we have

= (1-G(t))s + \int_0^{t-3\Delta-u} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-3\Delta-u-v))s \\
+ \int_0^{t-3\Delta-u-v} P[F_3(t - a_2\Delta - u_1 - u_2, s)] dG(u_2) \right] dG(u_1)

= (1-G(t))s + \int_0^{t-3\Delta-u} P \left[ (1-G(t-\Delta-u))s + \int_0^{t-3\Delta-u} P [(1-G(t-a_2\Delta-u_1-u_2))s \\
+ \int_0^{t-a_2\Delta-u_1-u_2} P[F_3(t - a_2\Delta - u_1 - u_2, s)] dG(u_3) \right] dG(u_2) \right] dG(u_1)

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\[ + \int_0^{t-a_n \Delta - u_1} \left[ P \left( 1 - G(t - a_2 \Delta - u_1 - u_2) \right) s \right. \\
+ \int_0^{t-a_n \Delta - u_1 - u_2} \left. P \left( 1 - G(t - a_n \Delta - u_1 - u_2 - u_3) s \right) \right] + \cdots \\
\cdots + \int_0^{t-a_n \Delta - u_1 - u_2 - \cdots - u_{n-1}} \left[ \left. P \left( 1 - G(t - a_n \Delta - u_1 - u_2 - \cdots - u_n) s \right) \right] \right] \right. \\
\left. + \int_0^{t-a_n \Delta - u_1 - u_2 - \cdots - u_n} \left. P \left( F_{n+1}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n+1}, s) \right) \right] \right] \right\} \\
\text{\(dG(u_{n+1})dG(u_n)\cdots dG(u_1)\)}

It is to be noted that during the time \([t-a_n \Delta, t]\) there will be no branch.

So

\[ F_{n+1}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n+1}, s) = 1 \]

So

\[ P[F_{n+1}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n+1}, s)] = 1, \ \forall s \]

So the above expression changes to

\[ F(t, s) = (1 - G(t)) s + \int_0^{t-a_n \Delta} P \left[ 1 - G(t - a_1 \Delta - u_1) s \right] \\
+ \int_0^{t-a_n \Delta - u_1} P \left[ 1 - G(t - a_2 \Delta - u_1 - u_2) s \right] \\
+ \int_0^{t-a_n \Delta - u_1 - u_2} P \left[ 1 - G(t - a_n \Delta - u_1 - u_2 - u_3) s \right] + \cdots \\
\cdots + \int_0^{t-a_n \Delta - u_1 - u_2 - \cdots - u_{n-1}} P \left[ 1 - G(t - a_n \Delta - u_1 - u_2 - \cdots - u_n) s \right] \\
+ \int_0^{t-a_n \Delta - u_1 - u_2 - \cdots - u_n} P (G(t - a_n \Delta - u_1 - u_2 - \cdots - u_n)) \left] \right] \right\} \right. \\
\text{\(n_{n-1} \cdots n_1\)}

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\[ dG(u_n) dG(u_{n-1}) \cdots dG(u_1) \]

So for each fixed \( t \) there exists \( n \), such that the p.g.f. turns out to be the above type. Now we can find out the expression for each life distribution.

The extinction probability in this case will be

\[
F(t,0) = \int_0^{t-a_n \Delta} P \left[ \int_0^{t-a_n \Delta-u_1} P \left[ \int_0^{t-a_n \Delta-u_1-u_2} \cdots P \left[ \int_0^{t-a_n \Delta-u_1-u_2-\cdots-u_n} \right] \right] \right] \cdots \\
\int_0^{t-a_n \Delta-u_1-u_2-\cdots-u_n} \right] dG(u_n) dG(u_{n-1}) \cdots dG(u_1)
\]

Note: It is to be noted that

\[
a_0 = 0, \ a_1 = 1, \ a_2 = 3, \ a_3 = 3 + a_2, \ a_4 = 4 + a_3, \ \text{i.e.,} \ a_n = n + a_{n-1} \ (3.4.1)
\]

3.4.1 Example with one offspring and varying life distribution

Assume \( G \sim \text{c.d.f. of any life distribution} \) and \( P(s) = q + ps \) i.e., p.g.f. of Bernoulli distribution

\[
\Rightarrow \int_0^{t-a_n \Delta-u_1-u_2-\cdots-u_n} (q + p(G(t-a_n \Delta-u_1-u_2-\cdots-u_n))dG(u_n) \\
= \int_0^x (q + pG(x-y))dG(y) \\
= qG(t-a_n \Delta-u_1-u_2-\cdots-u_{n-1}) + p \int_0^G(x-y)dG(y) \\
= qG(t-a_n \Delta-u_1-u_2-\cdots-u_{n-1}) + pG^2(x)
\]

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where \( x = t - a_n \Delta - u_1 - \cdots - u_{n-1} \)
\[
= qG(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-1}) + pG^{*2}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-1})
\]

So after 2nd step this becomes
\[
\int_0^{t-a_n \Delta - u_1 - u_2 - \cdots - u_{n-2}} q + p \{ qG(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-1}) \\
+ pG^{*2}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-1}) \} dG(u_{n-1})
\]
\[
= qG(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-2}) + pqG^{*2}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-2})
\]
\[
+ p^2 G^{*3}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-2})
\]

After 3rd step this becomes
\[
qG(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-3}) + pqG^{*2}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-3})
\]
\[
+ p^2 qG^{*3}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-3}) + p^3 G^{*4}(t - a_n \Delta - u_1 - u_2 - \cdots - u_{n-3})
\]

So after nth step it is given by
\[
qG(t - a_n \Delta) + pqG^{*2}(t - a_n \Delta) + p^2 qG^{*3}(t - a_n \Delta) + \cdots + p^n G^{*(n+1)}(t = a_n \Delta)
\]

(3.4.2)

So if we are given \( \Delta \) and \( t \) then we can find out \( n \) such that (3.4.1) will hold and then p.g.f. will be given by (3.4.2).

**Fate of the population:**

Taking Laplace transformation we have
\[
\mathcal{L}(F(t,0))(s) = q\mathcal{L}(G(s)) + pq(\mathcal{L}(G(s)))^2 + p^2 q(\mathcal{L}(G(s)))^3 + \cdots + p^n(\mathcal{L}(G(s)))^{n+1}
\]
\[
= \frac{q}{p} \left[ p\mathcal{L}(G(s)) + (p\mathcal{L}(G(s)))^2 + \cdots + (p\mathcal{L}(G(s)))^n \right] + p^n \left( \mathcal{L}(G(s)) \right)^{n+1}
\]

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\[
\frac{q}{p} \left[ p\mathcal{L}(G(s)) \frac{1 - (p\mathcal{L}(G(s)))^n}{1 - p\mathcal{L}(G(s))} \right] + p^n \mathcal{L}(G(s))^{n+1}
\]

Put \( H(t) = G^{*k}(t - a_n\Delta) \) Then

\[
\tilde{H}(s) = \int H(t)e^{-st}dt
\]

\[
= \int G^{*k}(t - a_n\Delta)e^{-st}dt
\]

\[
= \int G^{*k}(u)e^{-s(u+a_n\Delta)}du, \quad u = t - a_n\Delta
\]

\[
= \int_0^\infty G^{*k}(u)e^{-su}e^{-sa_n\Delta}du
\]

\[
= e^{-sa_n\Delta}\int_0^\infty G^{*k}(u)e^{-su}du
\]

\[
= e^{-sa_n\Delta}G^{*k}(s)
\]

So laplace transformation of above is given by

\[
qe^{-sa_n\Delta} \left[ \tilde{G}(s) + p\tilde{G}(s)^2 + p^2\tilde{G}(s)^3 + \cdots + p^{n-1}\tilde{G}(s)^n \right] + p^n e^{-sa_n\Delta}G(s)^{n+1}
\]

\[
\rightarrow \frac{\tilde{G}(s)}{1 - p\tilde{G}(s)} \lim_{s \to} e^{-sa_n\Delta} \rightarrow 0
\]

Laplace{sequence function} \( \rightarrow 0 \)

\( \Rightarrow \) sequence function \( \rightarrow 0 \)

So extinction probability \( \rightarrow 0 \) as \( t \to \infty \)

### 3.5 Concluding Remarks

In this Chapter we tackled known processes and their equations, but way of solving them is different. The main tool used here is the solution of non-linear integral equations (Chapter 2).
Of course, we have considered a new process here with changing life distribution. We have not worked out the case of scale shift. But we hope that this also can be tackled with suitable transformation.

The main problem is left regarding inference procedures for extinction probability e.g. their estimation, testing. We have explored this in short but there are some works left to be investigated.

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