CHAPTER 1
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Introduction

1.1 Stochastic Processes and its Applications

The theory of stochastic process was developed to meet dynamic part of the statistical theory. "Currently in the period of dynamic indeterminism in science, there is hardly a serious piece of research, which if treated realistically, does not involve operations on stochastic processes" (Neyman, 1960).

The theory of stochastic processes was developed a lot for last five decades. Its field of applications is growing. Some of the important areas are population growth, operations research, time-series analysis, biological sciences, management sciences etc. Also stochastic processes give models for physical phenomena as thermal noise in electric circuits and Brownian motion of a particle in a fluid.

Though these are some of the important areas, let us state briefly the important tools and type of processes which are mainly used in these applications.

Let us denote $X_n$ the value of a stock at $n$th unit of time, one may represent its evolution by a countable number of random variable \{$X_0, X_1, \cdots$\} indexed by the discrete-time parameter $n \in T$. The number $X_t$ of telephone calls in a telephone exchange during the time interval $[0,t]$ gives rise to a collection of random variables \{$X_t : t \geq 0$\} indexed by the continuous time
parameter $t$. The velocity vector $X_u$ at a point $u$ is a turbulent flow field, gives a family of random variables \( \{X_u : u \in \mathbb{R}^3 \} \) indexed by a multidimensional spatial parameter $u$. To be specific the following definition is given in literature.

**Definition 1.1** For a given index set $I$, a stochastic process indexed by $I$ is a collection of random variables \( \{X_i : i \in I \} \) on a probability space \( (\Omega, \Gamma, P) \) taking values in a set $S$. The set $S$ is called the state space of the process.

If the index set has natural ordering, then this ordering coincides with the sense of evolution of the process. This order is lost for the process where index is multidimensional parameter as $\mathbb{R}^3$ has no natural ordering. Such processes are random fields.

For a process the values of the random variable corresponding to the occurrence of a sample point $\omega \in \Omega$ gives a sample realisation of the process. In applied statistics after collection of empirical data, theoretical distribution is fitted in order to extract more information from the data. If fitting is good, then the properties of the data set can be approximated by the properties of the distribution. In a similar situation, let us suppose that a real life process has been observed to have the characteristic of a stochastic process. Then the idea of the behaviour of the stochastic process is then highly desirable in understanding the real life situation. This is true when the system deals with complex phenomenon. Some of the applications in the real world phenomena are given in the following paragraphs.

In the study of social structure the influence of parents on the children is a major consideration. Consider a society consisting of $m$ distinct social
classes, such as professional, managerial, skilled, unskilled, etc. Also let us assume that male head decides the social class of the family. Under this assumption each person can be considered to be occupying one of the $m$ states and social states of the family in a generation can be modeled as a stochastic process which can assume one of the $m$ values. A survey conducted in England and Wales (Glass 1954) justifies the use of such model.

It is important to consider the growth of a population as stochastic rather than deterministic. The external influences such as weather conditions, disease and availability of food, are too varying and uncertain for being deterministic. After these factor are identified and accounted for, the population size at any time can be considered as a stochastic process. In problems of this nature, it is not only important to know the behaviour of the process but also using such information, control of growth or decline of the population are done. Otherwise the situation could result in the species being extinct.

In case of recovery, relapse and death from some major diseases such as cancer is governed by several random causes, and therefore stochastic models have been found useful in the study of hospital data related to such diseases. For example, four different states of the patient can be identified: (a) the initial state of being under treatment, (b) the state of being dead immediately following treatment (c) the state of recovering and (d) the state of being lost after test. From such a model, problems related to treatment can be studied.

Many jobs of varied lengths come to a service center from various sources. The number of jobs arriving as well as their lengths can be said to follow
certain distributions. Under such conditions the number of jobs waiting at any time and the time a job has to spend can be represented by stochastic models. Under a strictly first come first served policy there is a good chance of a long job delaying a much more important shorter job over a long period of time. For an efficient operator of the system, in addition to minimizing the number of jobs waiting and total delay it may be necessary to adopt a different service policy.

The reliability of a system or a component defined as the probability that it performs its assigned target. If the target is to accomplish a task with no reference of time, we can consider it as the probability of success at a time epoch. When the success has to be measured over a period of time, with several parts to the system, one has to consider the life distributions of the different parts and the system structure in determining its reliability.

Let us consider a traffic light system with red, green and another lights mounted at a north-south, east, west intersection. The system is run by a clock. Assuming that the electrical connections and switching system are perfect, then all 16 bulbs and the time clock to be considered. Life distributions of all these items are now relevant. The light system are such that north-south and east-west lights are separately synchronized. The mission of the light system can be defined in many ways: for instance, the entire system is trouble free, or when a specific number of light bulbs are out, the system is not hazardous. Depending on the mission, let us suppose one is interested in determining the probability of accomplishing that mission during a specified lengths of time. Because of the probabilistic nature of the life lengths of the components, the underlying process is a stochastic process and the reliability
of the system can be determined from the properties of the process.

In time series problems, observations are sequential in time and one would like to identify the factors regulating them.

A study of such system has three essential aspects. These can be referred to as behavioral, statistical and operational. In order to obtain the maximum benefit, the study is complete only if all three aspects of the system are investigated. A behavioral study is aimed at understanding the general as well as specific details of the behavior of the system. This can be carried out in many ways. Our way is to built an exact model and deduce the behavior of the system from the model. A miniature of an intended migration system, the simulation models of complex systems are examples of such exact models.

Another method of study is to built a model for a complex situation and deduce some of the general properties from this model. An idealised model helps us to weed out the unnecessary details and look into only the essential features of the system. Once the general properties are known, the specific can be deduced by other means.

Even if the behavior of the wide variety of stochastic process is known, selection of the right model carefully be made by statistical means. Needed data have to be collected and analysed, and the essential characteristics of the system have to be estimated. Hypotheses about the features of the system need to be tested through reliable means. Identification of the correct model does not complete the analysis. For the study to be useful, the results of the analysis should be used for improving the system operation.

If one is interested in the number of people why buy a certain brand of an item, a survey conducted over a period of time, estimates are obtained
for the consumer brand-switching. Then the number could be represented as a stochastic process. The behaviour of the consumer can also be considered as a stochastic process. Some of the questions arise are: What the expected number of months that a consumer stays with a specific brand? Which is the product preferred most by the customers in the long run?

The theory of heredity, originated by Mendel (1822-1884), provides instructive illustrations for applicability of simple stochastic models. Heritable characters depend on special carriers, called genes. All cell of the body, except the reproductive cells or gamets, carry exact replicas of the same gene structure. The reproductive cells, or gamets are formed by a splitting process and receive one gene only. New organism are derived from two parental gamets in equal numbers. The genotype of offspring depends on a chance process. At every occasion, each parental gene has probability \( \frac{1}{2} \) to be transmitted and the successive trials are independent. There are three genotypes. There are stationary distributions of these three. In practice, deviations will be observed, but for large populations we say whatever the composition of the parent population may be, random mating within one generation produce an approximately stationary genotype distribution with unchanged gene frequencies. From the second generation on, there is no tendency toward a systematic change; a steady state reached with the first filial.

Certain special classes of stochastic processes have undergone extensive mathematical development. The Brownian motion process is the most renowned. As a physical phenomenon the Brownian motion was discovered by botanist Brown in 1827. A mathematical description of this phenomenon was first derived from the laws of physics by Einstein in 1905. Today Brownian motion
process and its many generalisation and extensions arise in numerous and
diverse areas of pure and applied science such as economics, communication
theory, biology, management science and mathematical statistics.

For studying the dynamics of financial markets, the questions of most
interest are (i) nature of short and long term trends in the market, (ii) nature
of fluctuations around the trend or mean behavior, (iii) correlation between
the variables studied and (iv) short term forecasts. Since the behavior of
the market is reflected in the corresponding time series, it forms the basis
upon which we formulate the study of dynamics of the market. Intuitively
we expect that the dynamics of the average behaviour has a deterministic
character. We also believe that the fluctuations are random in nature and
hence its dynamics would be governed by stochastic laws. It is important to
have a very reliable separation of mean behaviour and fluctuations. This is
because our interest is primarily in fluctuations and an improper separation
may give rise to spurious effects.

Branching process known as Galton-Watson process dates back to 1874
when Galton and Watson formulated the problem of extinction families.
Model at that time did not attract but during last 40 years much atten-
tion has been devoted to it. Let us suppose that we are observing a physical
system consisting of a finite number of particles either of the same type or of
several different types. With the passage of time each particle can disappear
or turn into a group of new particles, independently of other particles. Phe-
nomena described by such a scheme are frequently encountered in natural
science and technology, sociology and demography, e.g. showers of cosmic
rays, growth of large organic cells, the development of biological populations
and the spread of epidemics. All of these processes are characterised by the same property, their development has a branching form.

Congestion is a natural phenomenon in real system. A service facility get congested if there are more people than the server can handle. Queueing for milk, railway tickets, etc are examples of queueing system. In such situations the uncertainties are related to the system characteristics, such as arrival of customers, time needed for service. The processes occurring in these systems are represented by stochastic processes.

Besides these there many other areas like geology, epidemiology, etc where stochastic models are used.

1.2 The problems and perspective

From the above discussion one can have idea regarding the numerous applications of stochastic processes and in all these applications we have seen that many integral equations come out when we want to know about the processes or some of their characteristics. Some of these equations are highly non-linear. So one has to solve these equations analytically. But the non-linear equations are really difficult to solve in full generality. In the existing literature some existence theorems are available. For example one can look at (Davis Harold T). In that way, necessities for solving such equations come from some useful stochastic processes. Also these generate interest to solve equations from other general setup.

While dealing with branching process, the present author has noticed many non-linear equations for probability generating functions (p.g.f.). So
main idea is to solve such equations with full generality. A particular application made with this is to find probability of extinction of a branching process. From these processes, other processes have been built up considering some more realistic assumptions though some of them are linear integral equations. The theory and method of solving linear integral equations are standard. For example one can look at the book by Lovitt William Vernon (1950).

Then the present author is interested with solutions of integral equations both non-linear or linear, some of them are found in the context of stochastic differential equations, Bucy-Kalman filter. Then it is necessary to examine whether the method developed can be applied in modified form to tackle randomness involved.

Even similar things come while looking at kernel density estimation. Here during proper smoothing a sequence of density estimates ultimately gives integral equations. Obviously, the next step is to study the existence of limit or choice of kernel for such existence, etc.

In this study, we are concerned with the following specific problems:

Before stating the problems, we introduce the following definition which will be used from time to time.

**Definition:** If $t$ is a real valued function, then the integral equation is called $t$ nonlinear if

$$
\phi(x) = F(x) + \lambda \int_0^x K(x, \xi) t(\phi(\xi)) d\xi
$$

Hence it is square non-linear if $t(x) = x^2$, cubic non-linear if $t(x) = x^3$, $k$th power non-linear if $t(x) = x^k$, polynomial non-linear if $t(x)$ is a polynomial
and so on

i) In case of square non-linear integral equation we are concerned with
   a) analytic solution of the equation,
   b) verify uniform convergence of the above series solution,
   c) to write down the compact form as far as possible.

ii) To apply the previous method in solving the equations obtained from
    Bellmall-Harris process, so that
    a) expression for probability of extinction may be obtained,
    b) to give important applications,
    c) to obtain mean population size,
    d) to study the effect of family name through extinction probability, in
      case of only one offspring which has been found now-a-days.

    Even, to generalise and study Bellman-Harris process by relaxing the con­
    dition of age distribution which is assumed to be same for all generations. 
    But from practical point of view it changes as many factors in nature change
    from time to time.

iii) To generalise and study usual branching process by assuming \( k \) offsprings
    in the life of a unit and assuming dependency among the time of births of
    offsprings, because it seems more realistic.

iv) Also we are concerned with the estimation of probability of extinction
    from sample data, because from of the distribution or probability we have
    general expression, but data will speak the final or create urge for modifica­
    tions.

iv) We are also concerned with the modifications of power series method
    for solving stochastic differential equation, though there are other methods
available.

vi) To see whether power series method works for estimating trend function from some integral equations related with Kalman-Bucy filter, with the desirable property of literature.

vii) To study non-parametric density estimate, as primary estimate is the integration of samples by empirical distribution function or averaging with proper kernel. By iterating integration we can generate a stochastic process and to see the process in ultimate i.e., limiting density of this sequence of estimates. Because in this case, we have seen that we are to deal with approximate integral equation.

viii) For an addition in mathematical analysis, to generalise the power series method from square non linearity to

a) cubic non-linearity,

b) polynomial non-linearity,

c) 'smooth' non-linearity when \( t(x) \) can be represented by a sequence of polynomial as in Bernstein theory.

ix) Also we are concerned with the generalisation of the solutions in case where integrations are

a) Riemann-Stieltjes instead of Riemann integration.

b) With respect to some finite measure over a measurable space.

1.3 A brief review of the literature

The existing literature on applications of stochastic processes is already so vast that it will not be possible to give a comprehensive resume within
a short span. Instead of attempting that we make here a selective review concentrating on those works which have direct relevance to our problems.

The actual development of the theory of integral equations began only at the end of nineteenth century due to the works of the Italian mathematician V. Volterra (1896) and principally to the year 1900 in which the Sweedish mathematician Fredholm published his work on the method of population for the Dirichlet problem.

In spite of difficulties of the general problem, there exists a need for a systematic treatment of nonlinear equation. Mechanics, relying as it does upon the Calculus of Variation, Euler’s equation, and Hamilton’s principle provides a wealth of examples. "The present status of a theory which must await the future efforts of mathematicians for a more satisfactory formulation" (Davis H. T. (1962)). Volterra attempted to incorporate in his problem of growth of populations the influence of heredity. There is existence theorem for nonlinear integral equation of Voterra type. T. Lalesco, E. Cotton, M. Picone, G. C. Evans extended their proofs to a functional equation sufficiently general. Also Lalesco has given an existence proof under general condition on kernel function. G. Bratu has studied the special cases of square nonlinearity and exponential nonlinearity.

Though Galton and Watson formulated the problem of extinction of families in 1874 but in last 40 years a vast amount of research work has been done. Bellman & Harris introduced an age-dependent branching process. Sevasymov, introduced the possibility of dependence between mother’s life distribution and offspring life distribution. He considered the generalisation that an object produced offsprings not necessarily at the end of its life time.
but at randomly chosen instants during its life time.

Continuous time branching processes in varying and random environments, processes with immigration (Pakes 1971, 1974, 1979) and processes with disasters (Kaplan, 1979) and others have received considerable attention. Mode (1971), Jagers (1975), Keyfitz (1968) discussed many interesting models in several fields like biology, epidemiology, demography and also in nuclear physics and chemistry.

In some recent works a special stochastic process on the lattice to model mass extinctions has been discussed (Schinazi, 2005). Another generalisation is logistic branching process (Lambert, 2005). It can be seen as a fragmentation process combined with constant coagulation rate. Cohn and Wang (2003) studied convergence of multi-type branching process in varying environment.

Ito, K., and McKean (1965) developed techniques of stochastic integrals and stochastic calculus in the Ito sense. Gikman and Skorohod (1969, 1974) present extensive treatments of the theory of stochastic differential equations covering one or higher dimensions. Friedman (1975-76) elaborates the theory of stochastic differential equations. A far ranging advanced exposition by several authors integrates the subjects of stochastic integration with respect to generalised martingales instead of Brownian motion, the theory of stochastic differential equations with special attention to diffusion process on manifolds, featuring theory on comparisons and approximations of different processes. Arnold (1974) has provided a presentation of stochastic differential equations and applications, Ewens (1979) exhibits the pervasive in featuring a number of important population genetics models.

Rao (1973), Silverman (1986) and Titterington (1985) considered the problem of non-parametric density estimation. It is useful for different purposes. For example, the efficiency of liquid-liquid extraction columns depend on probability distribution of the drops. In image processing the probability density function of the brightness is commonly used.

Kernel estimators were first proposed by Nadaraya and Watson (1964). The methods related to the estimation of densities are closely related to these estimators. Nadaraya and Watson proposed an interpolation procedure. Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables with density \( f(x), x \in \mathbb{R}^d \). The Rosenblatt and Parzen estimator of density \( f(x) \) is a suitably smoothed histogram. It is given by

\[
\hat{f}_n(x_1, \ldots, x_n; x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left( \frac{x - x_i}{h_n} \right)
\]

where \( K(.) : \mathbb{R} \rightarrow \mathbb{R}_+ \) is the so called kernel function which represents a fixed bounded density function, \( \{h_n\} \) is a sequence of positive numbers, \( h_n \rightarrow 0 \), as \( N \rightarrow \infty \). There exists many variants of Rosenblatt-Parzen
method. There is attempt of smoothing like

\[
\hat{f}_1(x_1; x) = \frac{1}{h_1} K\left(\frac{x - x_1}{h_1}\right)
\]

\[
\hat{f}_2(x_1, x_2; x) = (1 - a_2)\hat{f}_1(x_1; x) + a_2 \frac{1}{h_2} K\left(\frac{x - x_1}{h_2}\right)
\]

Such recursive variant has been considered by Wolverton and Wagner and Yamato. Zhang (1997) considered estimation which is limited to wavelets of small dimension.

1.4 Summary of the works done

(1) In many situations of stochastic processes we are to deal with integral equations and some of them are non-linear and even some problems of stochastic processes can be formalised in terms of integral equations.

The theory of integral equations is one of the most important areas like mathematical analysis, boundary value problems, etc. Integral equations occur in many fields of mechanics and mathematical physics. It has relation with mechanical vibrations, theory of analytic functions, orthogonal systems, quadratic forms of infinitely many variables.

It is known that integral equations come out in many cases of stochastic process viz renewal equation, age-dependent branching process, etc. For standard reference one can look at any standard text book on stochastic process. Also the theory of linear integral equations is standard. But the non-linear integral equations are complicated to deal with. In this study we deal with square non-linear integral equation and solve the equation by simple power series method and sufficient argument is given regarding the
assumptions for the solution and also check its uniform convergence and uniqueness. To show this we have used the solution of a suitable differential equation to dominate the terms of the solution of the integral equation.

(2) Branching process is a special class of Markov chain. Following biological terminology, let us consider a situation where each organism of our generation produces a random number of offsprings to form the next generation. If the probability distribution of the number of offsprings produced by an organism is given, one becomes interested in many characteristics like the distribution of the size of the population for different generations and the probability of extinction.

Some of the natural phenomena can be modeled as a branching process. This relate to the survival of family names, verbal flow of information, electron multipliers, which amplify a weak current using a series of plates that generate new electrons when hit by an electron. But we prefer to restrict our attention to biological terminology of organism and offspring.

Simple mathematical model for branching processes was formulated by Galton and Watson. But we shall concentrate on Bellman-Harris process which is age dependent branching process.

Description of the Bellman-Harris (BH) process: Let us suppose that an ancestor at time $t = 0$ initiates the process and at the end of its life time it produces a random number of offsprings having a distribution and the process continues as long as objects are present. Here assumption is that the offsprings act independent of each other. Also the lifetime of objects are i.i.d. random variables with distribution function $G$.

Under the above assumption let $\{X(t), \ t \geq 0\}$ be the number of objects
available at time $t$. Then the stochastic process $\{X(t), \ t \geq 0\}$ is called an age-dependent branching process. It is important to note that an age-dependent process is Non-Markovian.

The probability generating function for BH process is available in literature. Then we have applied power series method for solving integral equation for which the tools have been developed. Applying this technique we have obtained expression for extinction probability at any time $t$. Also an example with two offsprings have been worked out. This is important for present day scenario, because in our society we do not prefer more that two children and it is important to know, whether a family name will be extinct or not.

Next we have worked out the integral equation for mean population size at time $t$ for BH process. Taking the relevant integral equation we have demonstrated its solution by the same power series technique and also worked out the case with exponential life in terms of the form obtained earlier.

Also we have considered the problem of offsprings with Bernoulli distribution, i.e., either there will be offspring or not.

A different branching process of BH type is considered next. Here we have assumed that the life distributions are different for different generations. This assumption is quite realistic as with the passage of time, culture, socio-economic, environment, etc change and for these the age distributions vary. Of course, we have considered the case where there is a constant shift from generation to generation. We have obtained the integral equation describing its p.g.f at time $t$ and also have investigated the nature of extinction probability with a suitable example.

(3) Throughout the application, we have tried to use the solution of
integral equations. Some of them are known like BH processes. But we have presented different way of solving. We have generalised the process and have examined the consequences.

We have considered another generalisation i.e., branching occurs at random time which is different from the process in Chapter 2.

We consider the age-dependent branching processes. Strictly speaking we start with a unit and assume that first branch occurs after a random time \( \tau_1 \) and after occurrence of first branch it requires \( \tau_2 \) random time for second branching and so on upto \( k \)th branching. Also we assume that the unit may have life \( T \), random time. This holds for all units and they behave independently after birth.

In section 4.2 we have described the process where \( X(t) \) = total number of units available at time \( t \). Here ancestors and offsprings are included and the integral equation for \( EX(t) \) have been obtained, and also its solution is provided.

In section 4.2.1 we give an example where \( \tau_i - \tau_{i-1} \) are independent i.e., spacings are independently exponentially distributed and the mean population size is obtained.

In section 4.2.2 fate of such population is discussed and it is the consequence that such population never vanishes with probability 1.

In section 4.3 we have obtained the expression for p.g.f. function under general \( F(\tau_1, \tau_2, \cdots, \tau_k) \). Here spacings may not be independent.

In section 4.4 we have considered a method for estimation of probability extinction.

(4) Next we have discussed two areas. First one is regarding first order
stochastic differential equation and second one is regarding Kalman-Bucy filter. For this we need to add something in literature for prerequisites.

For the first one:

We consider the processes which satisfy the first order stochastic differential equation.

\[ dX(t) + K(t)X(t) = dB(t) \]

where \( K(t) \) is a non-random function of \( t \), others \( X(t) \) and \( B(t) \) depend on sample points. Here \( B(t) \) is a standard one dimensional Brownian motion. This comes in the following way.

The increment of the process from \( t - \Delta \) to \( t \) time is

\[ X(t) - X(t - \Delta) \propto X(t - \Delta), \text{ state at time } t - \Delta \]

and also \( \propto K(t - \Delta) \)

i.e., rate of increment depends on the present state and on some constraints which are time dependent.

So combining these two we must have

\[ X(t) - X(t - \Delta) = K(t - \Delta)X(t - \Delta)\Delta + \text{ normal error} \]

where \( K(t - \Delta)X(t - \Delta) \) is velocity at \( t - \Delta \).

Then

\[ X(t) - X(t - \Delta) = K(t - \Delta)X(t - \Delta)\Delta + (B_t - B_{t-\Delta}) \]

Such things are available in literature (Karlin, S. and Taylor, H. (1975, 1981)).
So the above equation gives the following integral equation

\[ X_t = X_0 + \int_0^t K(t - s)X_{t-s}ds + \int_0^t dB_s \]

It is to be noted that it is not defined through usual differentiation because Brownian paths are non-differentiable with probability one. It is in the sense of Itô.

There is estimator of diffusion parameter which is consistent for large samples and thus one knows the function \( K(t) \). So for the time being we assume that \( K(.) \) is known. Here the solutions are available in literature (Rao (1999)). But our target is to look this solution by the solution obtained from the power series method for solving integral equation. It is really a new way to look sample paths in terms of our solution as done in Chapter 2. But here we are to modify this because of the Brownian part.

Basic tools used here are Markov inequality, Borel Cantelli lemma and Cauchy-Schwartz inequality.

For second one:

In section (5.3) we consider state and measurement processes of a Kalman-Bucy linear system.

Here p-dimensional 'state' process \( X \) and a q-dimentional 'measurement' process \( Z \) are assumed to be related by the stochastic differential system

\[
\begin{align*}
dX(t) &= A(t)X(t)dt + B(t)u(t)dt + dW(t), \\
dZ(t) &= C(t)X(t) + dV(t)
\end{align*}
\]

where \( W \) and \( V \) are independent p-dimentional and q-dimentional Wiener processes for \( 0 \leq t \leq T \), \( u \) is known input function and \( A, B, C \) are known
non-random time dependent matrices of suitable dimensions and \( X(0) \) is independent of \( W \) and \( V \). Also the processes \( \{Z(s), \ 0 \leq s \leq t\} \) is observable whereas \( \{X(s), \ 0 \leq s \leq t\} \) is not. Such models are used in control system.

In real applied problem, often the additive trends are present in the state and measurement processes. So the state space is given with extra additional additive trend terms.

\[
dX(t) = f(t)dt + A(t)X(t)dt + B(t)u(t)dt + dW(t)
\]

and the observed process instead of \( Z \) is given by

\[
dY(t) = g(t)dt + dZ(t)
\]

For first one, a new method for obtaining solution to stochastic differential equation has been suggested. The power series method has been altered in order to accommodate Brownian part.

For the second, an estimate of trend in Kalman filter is indicated using power series solution and that function estimate belong to \( L^2[0, T] \).

(5) Here let us consider the estimation of density function. If the form of the density is known except some parameters, then we estimate those parameters but in non-parametric case we cannot assume the form of the density function. And we estimate density function using unstructured approach. As in non-parametric approaches, the density function is estimated locally by a small number of neighbouring samples, this estimate is less reliable than the parametric form.

Two kinds of estimation technique in nonparametric estimation are available. They are Parzen density estimate and the other is K-nearest neigh-
bouring density estimate. Here some issues of Parzen density estimate have been considered.

The simplest estimate for density function is

\[ \hat{p}(X) = \hat{p}_s(X) * K(X) = \int \hat{p}_s(Y) K(X - Y) dY \]

where \( \hat{p}_s(X) * K(X) \) is convolution of \( \hat{p}_s(.) \) and \( K(.) \), \( \hat{p}_s(.) \) is an impulsive density function with impulses at the location of existing \( N \) samples,

\[ \hat{p}_s(Y) = \frac{1}{N} \sum_{i=1}^{N} \delta(Y - X_i) \]

where

\[ \delta(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0, & \text{if } x \neq 0 
\end{cases} \]

With this background we give a technique for smoothing density in section (6.2) and show that the iteration procedure gives a convergent sequence of densities. So proper choice of kernel gives the true unknown density.

From idea of section (6.2) a non-parametric test called k-test has been suggested to test goodness of fit.

Then as kernel plays an important role, we try to give sparingly new properties of kernel function. Remaining properties like symmetric, concentrated to origin are known in literature.

(6) In Chapter 2 it is mentioned that integral equations are important in science and technology. The Theory of linear integral equations is a standard, but the non-linear equations are really difficult to deal with. In Chapter 2 we have dealt with the simplest square nonlinearity but in many branches at least in probability theory where probability generating functions are polynomials
of higher degree, we need to obtain the solution of higher degree nonlinear equations.

We have used the same technique of Chapter 2 to obtain solution of three degree but the nature of solution requires cumbersome combinatorial calculations. Then solution of degree $k$ where $k$ is greater than 3 is obtained. The above solutions have been generalised to nonlinear integral equations of polynomial nonlinearity. This idea is used to solve the equations with any continuous 'smooth' nonlinearity.

(7) As linear integral equations have been studied extensively by others, we have not discussed those things, but we have generalised some of the results where the set up of the equations are different. To begin with we consider equations which resemble Fredholm integral equation of second kind. But the set up are different. Here we have generalised in three different cases: (1) Integration is Riemann-Stieltjes integration with respect to an integrator of bounded variation, (2) integration w.r.t. step function integrator and (3) integration on any probability space or measure space.

The solutions in the above mentioned three cases and the necessary and sufficient conditions for existence of solution in each case also have been obtained.

The Fredholm integral of second kind equation is

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi) \phi(\xi) d\xi$$

Now consider the following equation as in (1)

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi) \phi(\xi) dg(\xi)$$
where \( g(.) \) is any function of bounded variation

**Definition 8.1.1** (Apostol) A function \( g(\cdot) \) is called a bounded variation on \([a, b]\) if for any partition \( P: a = x_0 < x_1 < \cdots < x_n = b \) we have

\[
\sup_P \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \text{ is bounded.}
\]

\[\phi(x) = F(x) + \lambda \int_{a}^{b} K(x, \xi) \phi(\xi) dG(\xi) \text{ as in (2)}\]

where \( G(.) \) is a step function and discontinuity at \( c_1, c_2, \cdots, c_s \in (a, b) \)

\[\phi(x) = F(x) + \lambda \int_{X} K(x, \xi) \phi(\xi) d\mu(\xi) \text{ as in (3)}\]

where \( \mu \) is a finite measure on the sample space \((X, \Gamma)\).

where

\[K : X \times X \rightarrow \mathbb{R}\]

\[F : X \rightarrow \mathbb{R}\]

We have considered these in sections (8.2), (8.3) and (8.4). Then we obtain solution in each case. Some examples are worked out.

### 1.5 Dimensions of the work

In this study we have considered various problems related to stochastic processes and their applications. We have tried to solve these problems at length but still there is a scope for further study.

(1) We have obtained solution for square non-linear integral equations by power series method. This idea will be used in more combinatorial way to
obtain solution of higher order non-linear integral equations. More theoretical investigations should be carried out regarding the nature of solution i.e., to find all possible $\lambda$ for which solution exists. Even numerical algorithms to develop software are yet to be developed.

Here for some $\lambda$ we have obtained solution and have shown that for some restricted choice of kernel function and $F(x)$ the solution is meaningful, as the series is uniformly convergent. The trick is used by connecting the solution of suitable differential equation to that of integral equation.

(2) Next we tackled known processes and their equations, but way of solving them is different. The main tool used here is the solution of nonlinear integral equations (Chapter 2).

Of course, we have considered a new process here with changing life distribution. We have not worked out the case of scale shift. But we hope that this also can be tackled with suitable transformation.

The main problem is left regarding inference procedures for extinction probability e.g. their estimation, testing. We have explored this in short but there are some works left to be investigated.

(3) There are enough scope for elaborating the inference part of section 4.4, like observing samples with more generations. The testing part in this regard should be carefully handled, as the family distribution here is a curved exponential family.

(4) We have considered two areas related stochastic differential equations.

Firstly, we have used the power series solution of ordinary integral equation, then have accomodated Brownian part iteratively and finally the limiting paths thus obtained are solution to SDE.
In future non-linear stochastic integral equation are to be investigated in the same line.

Secondly, we have suggested that the power series solution gives the estimates of trend function of Kalman Bucy system in dimension 1. But this is yet to be generalised in higher dimension.

(5) We should choose kernel function carefully. In this section we could not explore the properties of kernel function, as we feel that it heavily depends on operator theory, generalisation of matrix.

One way of choosing the kernel is indicated in the following:

Suppose \( f(x) \) is true density then for right choice of kernel we must have

\[
\int K(x-y)f(y)dy = f(x)
\]

\[ \Rightarrow \text{ The operator } J(f)(x) = \int K(x-y)f(y)dy = f(x) \]

\[ \Rightarrow (J - I)f \equiv 0 \text{ for a wide class of densities } f \]

\[ \Rightarrow \|J - I\| < \epsilon \text{ for small number } \epsilon \]

Perhaps, that is why we choose \( k \) such that \( k \) receives maximum weight around 0.

(6) We have solved many types of non-linear integral equations. These equations look like Volterra's integral equations. But our equations are not of that form. Volterra's equation is linear. We have defined non-linearity of equation and have solved in most general form. These are, of course, important in solving many problems of stochastic processes.

In Chapters 2 and 7 we have tackled the nonlinear integral equation where integration is in Riemann sense, but for full generality this method
can be applied in general setup like (1) when the integration is Riemann-Stieltjes integration, (2) when integrators are step functions, and (3) when the integration done over any measure space or probability space.

Some theoretical investigations regarding λ and kernel function are left for further study and we have not developed numerical algorithm for software development.

(7) We have also extended the idea of power series technique to linear equations but with different setup. For non-linear case there is scope for further study.