CHAPTER-3

THEORETICAL AND MATHEMATICAL ANALYSIS
## CHAPTER 3

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### CHAPTER 3: THEORETICAL AND MATHEMATICAL ANALYSIS

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3.1. UNSTEADY MHD THREE DIMENSIONAL FLOW OF MAXWELL FLUID THROUGH A POROUS MEDIUM IN A PARALLEL PLATE CHANNEL UNDER THE INFLUENCE OF INCLINED MAGNETIC FIELD

Let us consider the unsteady hydro magnetic flow of Maxwell fluid which is electrically conducting in a parallel plate channel under the influence of a uniform transverse magnetic field of strength $H_0$ at an angle of inclination $\alpha$ normal to the channel walls. In $xyz$ plane, let us assume that boundary plates are parallel to $xy$-plane and to the $z$-axis the magnetic field in the transverse $xz$-plane. The $z$-direction component induces a secondary flow in the same direction while its $x$-components change perturbation to axial flow. The fluid is driven by prescribed pressure gradient at $t>0$ which is parallel to channel walls. When we take a Cartesian co-ordinate system $O(x, y, z)$ then the boundary walls will be at $z=0$ and $z=l$. Since the plates extend to infinity along side $x$ and $y$ directions, all physical quantities excepting the pressure depends only on $z$ and $t$. The hydro magnetic equations which are unsteady and which govern the electrically conducting Maxwell fluid by influence of transverse magnetic field in reference to a frame are

$$\rho \left[ \frac{\partial V}{\partial t} + (V \cdot \nabla)V \right] = -\nabla p + div \ S + J \times B + R \tag{3.1}$$

$$\nabla \cdot V = 0 \tag{3.2}$$

$$\nabla \cdot B = 0 \tag{3.3}$$

$$\nabla \times B = \mu_m J \tag{3.4}$$
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.5) \]

Where \( B \) is the total magnetic field, \( J \) is the present density, \( E \) is the total electric field, \( \mu_m \) is the magnetic permeability, \( V = (u, v, w) \) is the velocity field, \( T \) is the Cauchy stress tensor, \( B \) is the total magnetic field so that \( B = B_0 \sin \alpha + b \), where \( B_0 \) is the applied magnetic field which is parallel to the \( z \)-axis and considering \( b \) as the induced magnetic field. This \( b \) is negligible, as a result \( B = (0, 0, B_0 \sin \alpha) \), the Lorentz force \( \mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 \sin^2 \alpha \mathbf{V} \). Where \( \rho \) is density of fluid, \( \sigma \) is the electrical conductivity of fluid, \( \frac{D}{Dt} \) is the material derivative and \( R \) is the Darcy resistance. The extra tensor \( S \) for a Maxwell fluid is

\[ T = -\rho \mathbf{I} + S \quad (3.6) \]

\[ S + \lambda \left( \frac{DS}{Dt} - L S - S L^T \right) = \mu A \quad (3.7) \]

here \( -\rho \mathbf{I} \) is the stress because of constraint of the impermeability in which \( p \) is static fluid pressure, identity tensor is \( \mathbf{I} \), \( \mu \) the viscosity of the fluid, \( \lambda \) the material time constants considered as relaxation time, assuming \( \lambda \geq 0 \). \( A_1 \) the first Rivlin-Erickson tensor can be written as

\[ A_1 = (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \quad (3.8) \]

Considering the viscous Navier-Stokes fluid as a special case for this model when \( \lambda = 0 \). Stress tensor and the velocity component can be indicated as

\[ \mathbf{V}(z, t) = (u, 0, w) \quad (3.9) \]
As said by Tan and Masuoka [5] Darcy’s resistance in an Oldroyd-B fluid satisfies the following expression:

\[
\left( 1 + \lambda \frac{\partial}{\partial t} \right) R = -\frac{\mu \phi}{k} \left( 1 + \lambda \frac{\partial}{\partial t} \right) V
\]

(3.10)

Where \( \lambda_r \) is the retardation time, \( \phi \) is the porosity (0<\( \phi \)<1), and \( k \) is the permeability of the porous medium. For Maxwell fluid \( \lambda_r = 0 \) and hence,

\[
\left( 1 + \lambda \frac{\partial}{\partial t} \right) R = -\frac{\mu \phi}{k} V
\]

(3.11)

When we take these equations (3.6), (3.7) and (3.8), the equation (3.1) is reduced to

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xz}}{\partial z} - \sigma B_o^2 \sin^2 \alpha u + R_x
\]

(3.12)

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial S_{yz}}{\partial z} - \sigma B_o^2 \sin^2 \alpha w + R_z
\]

(3.13)

Where \( R_x \) and \( R_z \) are \( x \) and \( z \)-components of Darcy’s resistance \( R \);

\[
\left( 1 + \lambda \frac{\partial}{\partial t} \right) S_{xz} = \mu \frac{\partial u}{\partial z} \quad \text{and} \quad \left( 1 + \lambda \frac{\partial}{\partial t} \right) S_{yz} = \mu \frac{\partial w}{\partial z}
\]

(3.14)

The equations (3.12) and (3.13) reduces to

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xz}}{\partial z} - \sigma B_o^2 \sin^2 \alpha u - \frac{\mu \phi}{k} u
\]

(3.15)

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial S_{yz}}{\partial z} - \sigma B_o^2 \sin^2 \alpha v - \frac{\mu \phi}{k} w
\]

(3.16)

Let \( q = u + iw \) merging two equations (3.15) and (3.16) we get
\[ \rho \frac{\partial q}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z} (S_{x} + iS_{y}) - \sigma B_{0}^{2} \sin^{2} \alpha q - \frac{\mu \phi}{k} q \]  

(3.17)

Since

\[ \left(1 + \lambda \frac{\partial}{\partial t}\right) (S_{x} + iS_{y}) = \mu \frac{\partial q}{\partial z} \]  

(3.18)

(3.18) is substituted in the equation (3.17), the governing flow equation is obtained through porous medium then the rotating frame takes the form of

\[ \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial q}{\partial t} + \left(\frac{\sigma B_{0}^{2} \sin^{2} \alpha}{\rho} + \frac{v \phi}{k}\right) \left(1 + \lambda \frac{\partial}{\partial t}\right) q = -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial x} + v \frac{\partial^{2} q}{\partial z^{2}} \]  

(3.19)

Initial and boundary conditions follows:

\[ q = 0 \quad t > 0, \quad z = 0 \]  

(3.20)

\[ q = 0, \quad t \neq 0, \quad z = l \]  

(3.21)

\[ q(z, t) = 0, \quad \frac{dq(z, t)}{dt} = 0, \quad t \leq 0, \quad \text{for all } z \]  

(3.22)

The following are the variables of non dimension:

\[ z' = \frac{z}{l}, \quad q' = \frac{q l}{v}, \quad t' = \frac{t v}{l}, \quad \omega' = \frac{\omega l^{2}}{v}, \quad \phi' = \frac{\phi l}{v}, \quad P' = \frac{Pl^{2}}{\rho v} \]

The resultant governing equations by using these non dimensional variables given below

\[ \left(1 + \beta_{i} \frac{\partial}{\partial t}\right) \frac{\partial q}{\partial t} + (M^{2} \sin^{2} \alpha + D^{2}) \left(1 + \beta_{i} \frac{\partial}{\partial t}\right) q = \frac{1}{\rho} \left(1 + \beta_{i} \frac{\partial}{\partial t}\right) P + \frac{\partial^{2} q}{\partial z^{2}} \]  

(3.23)
where, Hartmann number is \( M^2 = \frac{\sigma \mu^2 H_0^2 l^2}{\rho v} \), inverse Darcy Parameter is \( D^{-1} = \frac{l^2}{k} \), \( \beta_i = \frac{\lambda v}{l^2} \) is the material parameter related to relaxation time and \( P = -\frac{\partial p}{\partial x} \) is the pressure gradient.

The resultant initial with boundary conditions are

\[
q = 0 \quad t > 0, \quad z = 0 \quad (3.24)
\]
\[
q = 0, \quad t \neq 0, \quad z = 1 \quad (3.25)
\]
\[
q(z,t) = 0, \quad \frac{dq(z,t)}{dt} = 0, \quad t \leq 0, \quad \text{forall } z \quad (3.26)
\]

The pressure is given by (assumption)

\[
P = \begin{cases} 
  P_0 + P e^{i\omega t}, & t > 0 \quad \forall \ z, \\
  0, & t < 0 \quad \forall \ z
\end{cases} \quad (3.27)
\]

For equations (3.23) and (3.27) we apply Laplace transforms using initial conditions (3.26) in terms of the transformed variable the governing equations reduces to

\[
\frac{d^2 \tilde{q}}{dz^2} - \left[ \beta_i s^2 + (1 + \beta_i (M^2 \sin^2 \alpha + D^i \phi))s + (M^2 \sin^2 \alpha + D^i \phi) \right]\tilde{q} =
\]
\[
- (I + i\beta_i \omega_i) \frac{P_i}{s - i\omega_i} - \frac{P_0}{s}
\]

(3.28)

Solving equation (3.28), by using conditions (3.24) and (3.25) we get
\[ q = - \frac{P_i (1 + i \beta \omega_j) \cosh(\lambda_i z)}{\lambda_i^2 (s - i \omega_j)} - \frac{P_o \cosh(\lambda_i z)}{\lambda_i s} \]

\[ + \frac{P_i (1 + i \beta \omega_j) \cosh(\lambda_i z) \sinh(\lambda_i z)}{\lambda_i^2 (s - i \omega_j) \sinh(\lambda_i)} + \frac{P_o \cosh(\lambda_i z) \sinh(\lambda_i z)}{\lambda_i s \sinh(\lambda_i)} - \frac{P_i (1 + i \beta \omega_j) \sinh(\lambda_i z)}{\lambda_i^2 (s - i \omega_j) \sinh(\lambda_i)} \]

\[ \text{Where} \]

\[ \lambda_i^2 = \beta_j s^2 + (1 + \beta \omega_j) s + (M^2 \sin \alpha + D^2 \varphi) \]

Inverse Laplace transforms is applied for equations (3.29) on either sides, we get

\[ q = - \frac{P_o \cosh(b_0 z)}{b_0^2} + \frac{P_o \cosh(b_0 z) \sinh(b_0 z)}{b_0^2 \sinh(b_0)} - \frac{P_o \sinh(b_0 z)}{b_0^2 \sinh(b_0)} + \frac{P_o}{b_0^2} + \]

\[ + \frac{P_i (1 + i \beta \omega_j)}{(s - jx_i)(s - jx_j)} \left\{ - \cosh(b_i z) + \frac{\cosh(b_i z) \sinh(b_i z)}{\sinh(b_i)} \right\} + \]

\[ - \frac{\sinh(b_i z)}{\sinh(b_i)} + 1 \right\} e^{i \omega_j t} + \left\{ - \frac{P_i (1 + i \beta \omega_j) \cosh(b_j z)}{(s_j - x_j)(s_j - i \omega_j)} \right\} + \]

\[ + \frac{P_i (1 + i \beta \omega_j) \cosh(b_j z)}{(s_j - x_j)(s_j - i \omega_j)} - \frac{P_i (1 + i \beta \omega_j) \sinh(b_j z)}{(s_j - x_j)(s_j - i \omega_j) \sinh(b_j)} \]

\[ + \frac{P_i (1 + i \beta \omega_j)}{(s_j - x_j)(s_j - i \omega_j)} - \frac{P_i \cosh(b_j z)}{(s_j - x_j)(s_j)} + \frac{P_o \cosh(b_j z)}{(s_j - x_j)(s_j) \sinh(b_j)} - \frac{P_o \sinh(b_j z)}{(s_j - x_j)(s_j)} \right\} e^{\omega_j t} + \]

\[ + \left\{ - \frac{P_i (1 + i \beta \omega_j) \cosh(b_j z)}{(s_j - x_j)(s_j - i \omega_j)} + \right\]
\[ + P_1 \frac{(1 + \beta z \omega_s \omega_i) \cosh (b_z) \sinh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - P_1 \frac{(1 + \beta z \omega_s \omega_i) \sinh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \]

\[ + \frac{P_z \cosh (b_z) z}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - \frac{P_z \cosh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \] + \left\{ \begin{array}{c} P_0 \frac{\cosh (b_z) z}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - \frac{P_0 \cosh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \\ \frac{P_0 \sinh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - \frac{P_0 \sinh (b_z z)}{(s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \end{array} \right\} e^{\varepsilon t} + \]

\[ + \sum_{n=0}^\infty \left\{ \frac{P_n (1 + \beta z \omega_s \omega_i) \cosh (b_z) \sinh (b_z z)}{b_z^2 (s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - \frac{P_n (1 + \beta z \omega_s \omega_i) \sinh (b_z z)}{b_z^2 (s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \right\} e^{\varepsilon t} \]

\[ + \frac{P_0 \cosh (b_z) \sinh (b_z z)}{b_z^2 (s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} - \frac{P_0 \sinh (b_z z)}{b_z^2 (s_z - s_i)(s_z - i\omega_i) \sinh (b_z)} \right\} e^{\varepsilon t} \]

(3.30)

(Where the constants are mentioned in the appendix)

The shear stresses for both upper and lower plates are:

\[ \tau_u = \left( \frac{dq}{dz} \right)_{\varepsilon = 1} \text{ and } \tau_l = \left( \frac{dq}{dz} \right)_{\varepsilon = 0} \]

(3.31)
3.2. HALL EFFECTS ON UNSTEADY MHD THREE DIMENSIONAL FLOW OF ELECTRICALLY CONDUCTING MAXWELL FLUID THROUGH A POROUS MEDIUM IN A PARALLEL PLATE CHANNEL WITH EFFECT OF INCLINED MAGNETIC FIELD

We consider the unsteady flow of an electrically conducting Maxwell fluid through porous medium in a channel of parallel plate subjected to an even transverse magnetic field, with strength $H_0$ leaning at an angle of inclination $\alpha$ normal to the channel walls taking hall current into account. To the $xy$-plane the boundary plates are assumed to be parallel and to the $z$-axis the magnetic field which is in the transverse $xz$-plane. The secondary flow of the component is induced along $z$-direction and at the same time its $x$-components changes perturbation to axial flow. When the condition is at $t > 0$, the fluid is driven by the prescribed pressure gradient is parallel to the walls of the channel. Considering Cartesian system $O(x, y, z)$ in order that the distance of the boundary walls are at $z=0$ and $z=l$ along $x$ and $y$ directions the plates extends to infinity, except for the pressure all the physical quantities depend on $z$ and $t$ alone. The unsteady hydro magnetic equations ruling the electrically conducting Maxwell fluid under the influence of transverse magnetic field with reference to a frame, the equation of motion, continuity equation and the Maxwell equations are shown below:

$$\rho \left[ \frac{\partial V}{\partial t} + (V \cdot V) \right] = -\nabla p + \text{div} S + J \times B + R \quad (3.32)$$

$$\nabla . V = 0 \quad (3.33)$$
\[ \nabla \cdot B = 0 \quad (3.34) \]

\[ \nabla \times B = \mu_m J \quad (3.35) \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad (3.36) \]

Where \( B \) is the total magnetic field, \( J \) is the current density, \( E \) is the total electric field, \( V \) is velocity field \((u, v, w)\), Cauchy stress tensor is \( T \), magnetic permeability is \( \mu_m \). So that \( B = B_0 \sin \alpha + b \), in this equation \( B_0 \) is the applied magnetic field which is parallel to the \( z \)-axis and the induced magnetic field is \( b \). This \( b \) is negligible so that \( B = (0, 0, B_0 \sin \alpha) \), the Lorentz force \( J \times B = -\sigma B_0 \sin \alpha v \), \( \rho \) is the density of the fluid, \( \sigma \) is the electrical conductivity of the fluid and \( \frac{D}{Dt} \) is the material derivative and \( R \) is the Darcy resistance. The extra tensor \( S \) for a Maxwell fluid is

\[ T = -\rho J + S \quad (3.37) \]

\[ S + \dot{\lambda} \left( \frac{DS}{Dt} - L S - S L^T \right) = \mu A \quad (3.38) \]

In the above equation \(-\rho J\) is the stress due to constraint of the impermeability, the static fluid pressure is \( \rho \), the identity tensor is \( I \), the viscosity of the fluid is \( \mu \), the material time constants referred to as relaxation time is \( \lambda \), it is assumed that \( \lambda \geq 0 \), then the first Rivlin-Ericksen tensor \( A_1 \) is defined as

\[ A_1 = (\nabla V) + (\nabla V)^T \quad (3.39) \]
It is observed that the above model is a special case which includes the viscous Navier-Stokes fluid for $\lambda = 0$. Indicating the stress tensor as and the velocity components as:

$$V(z, t) = (u, 0, w).$$

(3.40)

According to Tan and Masuoka [5] Darcy’s resistance in an Oldroyd-B fluid satisfies the following expression:

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) R = -\frac{\mu \phi}{k} \left(1 + \lambda_r \frac{\partial}{\partial t}\right) V$$

(3.41)

Where $\lambda_r$ is the retardation time, $\phi$ is the porosity ($0 < \phi < 1$) and $k$ is the permeability of the porous medium. For a Maxwell fluid $\lambda_r = 0$ so

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) R = -\frac{\mu \phi}{k} V$$

(3.42)

Making use of the equations (3.37), (3.38) and (3.39) the equation (3.32) reduces to

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{\gamma \gamma}}{\partial z} - B_0 J_z \sin \alpha + R_x$$

(3.43)

$$\rho \frac{\partial w}{\partial t} = \frac{\partial S_{\gamma \gamma}}{\partial z} + B_0 J_x \sin \alpha + R_z$$

(3.44)

Where $R_x$ and $R_z$ are $x$ and $z$-components of Darcy’s resistance $R$;

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) S_{\gamma \gamma} = \mu \frac{\partial u}{\partial z} \quad \text{and} \quad \left(1 + \lambda \frac{\partial}{\partial t}\right) S_{\gamma \gamma} = \mu \frac{\partial w}{\partial z}$$

(3.45)

The equations (3.43) and (3.44) reduces to
\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S}{\partial z} - B_0 J_z \sin \alpha - \frac{\mu \varphi}{k} u \tag{3.46}
\]

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial S}{\partial z} + B_0 J_\alpha \sin \alpha - \frac{\mu \varphi}{k} w \tag{3.47}
\]

When the strength of the magnetic field is extremely large, the generalized Ohm’s law is modified to take account of the hall current, therefore

\[
J + \frac{\omega_e \tau_e}{H_0} J \times H = \sigma (E + \mu_e q \times H) \tag{3.48}
\]

Where \( J \) is the current density vector, \( \omega_e \) is cyclotron frequency, \( \tau_e \) is electron collision time, \( H \) is the magnetic field intensity vector, \( \sigma \) is fluid conductivity, \( E \) is the electric field, \( \mu_e \) is magnetic permeability, \( q \) is the velocity vector. In the equation (3.48) electron pressure gradient, thermo-electric effects and ion-slip are ignored. Lets suppose the electric field \( E=0 \), it reduces to

\[
J_x - m J_z \sin \alpha = -\sigma \mu_e H_0 w \sin \alpha \tag{3.49}
\]

\[
J_z + m J_x \sin \alpha = -\mu_e \sigma H_0 \mu \sin \alpha \tag{3.50}
\]

In the above equation \( m = \tau_e \omega_e \) is the Hall parameter. Solving the equations (3.49) and (3.50) we get

\[
J_x = \frac{\sigma \mu_e H_0 \sin \alpha}{1 + \frac{m^2}{2} \sin^2 \alpha} (um \sin \alpha - w) \tag{3.51}
\]

\[
J_z = \frac{\sigma \mu_e H_0 \sin \alpha}{1 + \frac{m^2}{2} \sin^2 \alpha} \left( u + m \sin \alpha \right) \tag{3.52}
\]
Substituting the equations (3.51) and (3.52) in the equations (3.46) and (3.47) respectively. We obtain the equations of the motion which governs the flow through a porous medium with reference to a rotating frame can be given as

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{\omega}}{\partial z} - \frac{\sigma \mu_e H_o^2 \sin \alpha}{\rho (1 + m^2 \sin^2 \alpha)} (u + m \sin \alpha) - \frac{\mu \phi}{k} u \quad (3.53)
\]

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial S_{\omega}}{\partial z} + \frac{\sigma \mu_e H_o^2 \sin \alpha}{\rho (1 + m^2 \sin^2 \alpha)} (m \sin \alpha - w) - \frac{\mu \phi}{k} w \quad (3.54)
\]

Let \( q = u + iw \) combining equations (3.53) and (3.54) we obtain

\[
\rho \frac{\partial q}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z} (S_{\omega} + iS_{\eta}) - \frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} q - \frac{\mu \phi}{k} q \quad (3.55)
\]

Since

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) (S_{\omega} + iS_{\eta}) = \mu \frac{\partial q}{\partial z} \quad (3.56)
\]

Substituting the equation (3.56) in the equation (3.55), we obtain the equation of the motion which governs the flow through a porous medium with reference to a rotating frame can be given as

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial q}{\partial t} + \left(\frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + \frac{v \phi}{k}\right) \left(1 + \lambda \frac{\partial}{\partial t}\right) q = -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial z} + v \frac{\partial^2 q}{\partial z^2} \quad (3.57)
\]

Initial and boundary conditions:

\[
q = 0 \quad t > 0, \quad z = 0 \quad (3.58)
\]

\[
q = 0, \quad t \neq 0, \quad z = l \quad (3.59)
\]

\[
q(z,t) = 0, \quad \frac{dq(z,t)}{dt} = 0, \quad t \leq 0, \quad \text{for all } z \quad (3.60)
\]
Non dimensional variables:

\[
\zeta = \frac{z}{l}, \quad q' = \frac{q}{l}, \quad t' = \frac{t}{l^2}, \quad \omega' = \frac{\omega}{l}, \quad \xi' = \frac{\xi}{l}, \quad P' = \frac{P l^2}{\rho v^2}
\]

For non dimensional variables the governing equations are:

\[
\left(1 + \beta_i \frac{\partial}{\partial t}\right) \frac{\partial q'}{\partial t} + \left(\frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + D^2 \phi\right) \left(1 + \beta_i \frac{\partial}{\partial t}\right) q = \frac{1}{\rho} \left(1 + \beta_i \frac{\partial}{\partial t}\right) P + \frac{\partial^2 q'}{\partial z'^2}
\]

(3.61)

where, Hartmann number is \(M^2 = \frac{\sigma \mu_e^2 H_0^2 l^2}{\rho v}\),

the inverse Darcy Parameter is \(D^{-1} = \frac{l^2}{k}\),

hall parameter is \(m = \tau_w \omega_e\),

Material parameter related to relaxation time is \(\beta_i = \frac{\lambda \nu}{l^2}\),

and pressure gradient is \(P = -\frac{\partial p}{\partial x}\).

Corresponding initial and boundary conditions:

\[
q = 0, \quad t > 0, \quad z = 0
\]

(3.62)

\[
q = 0, \quad t = 0, \quad z = 1
\]

(3.63)

\[
q(z, t) = 0, \quad \frac{dq(z, t)}{dt} = 0, \quad t \leq 0, \quad \text{for all } z
\]

(3.64)

The pressure \(P\) is given by

\[
P = \begin{cases} 
  P_0 + P e^{\nu t}, & t > 0, \quad \forall z, \\
  0, & t < 0, \quad \forall z
\end{cases}
\]

(3.65)
Applying Laplace transforms for the equations (3.61) and (3.65), using initial conditions (3.64) the governing equations with respect to transformed variable is reduced to

\[
\frac{d^2 \tilde{q}}{dz^2} = \left[ \beta_s s^2 + \left( 1 + \beta_i \left( \frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + D^{-i \varphi} \right) \right) s + \left( \frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + D^{-i \varphi} \right) \right] \tilde{q} =
\]

\[
= -(1 + i \beta_i \omega_i) \frac{P_i}{s - i \omega_i} - \frac{P_o}{s}
\]  \hspace{1cm} (3.66)

Solving equation (3.66) subjected to the conditions (3.62) and (3.63), we obtain

\[
\tilde{q} = - \frac{P_o (1 + i \beta_i \omega_i) \cosh(\lambda_i z)}{\lambda_i^2 (s - i \omega_i)} - \frac{P_o \cosh(\lambda_i z)}{\lambda_i^2 s} + \frac{P_i (1 + i \beta_i \omega_i) \cosh(\lambda_i) \sinh(\lambda_i z)}{\lambda_i^2 (s - i \omega_i) \sinh(\lambda_i)} - \frac{P_o \sinh(\lambda_i z)}{\lambda_i^2 s \sinh(\lambda_i)} + \frac{P_o (1 + i \beta_i \omega_i)}{\lambda_i^2 (s - i \omega_i)} + \frac{P_o}{\lambda_i^2 s (1 + s \alpha)}
\]  \hspace{1cm} (3.67)

Where

\[
\lambda_i^2 = \beta_s s^2 + \left( 1 + \beta_i \left( \frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + D^{-i \varphi} \right) \right) s + \left( \frac{M^2 \sin^2 \alpha}{(1 - im \sin \alpha)} + D^{-i \varphi} \right)
\]

On applying inverse Laplace transforms to the equations (3.67) on both sides. We get the equation as shown below:

\[
q = - \frac{P_o \cosh(a_i z)}{a_o^i} + \frac{P_o \cosh(a_i) \sinh(a_i z)}{a_o^i \sinh(a_i)} - \frac{P_o \sinh(a_i z)}{a_o^i \sinh(a_i)} + \frac{P_o}{a_o^i} + \frac{P_i (1 + i \beta_i \omega_i)}{(i \omega_i - s_i)(i \omega_i - s)} \left[ - \cosh(a_i z) + \frac{\cosh(a_i) \sinh(a_i z)}{\sinh(a_i)} \right] - \frac{\sinh(a_i z)}{\sinh(a_i)} + 1 \right] e^{i \omega_i t} + \left[ - \frac{P_i (1 + i \beta_i \omega_i) \cosh(a_i z)}{(s_i - s)(s_i - i \omega_i)} +
\]
\[ + \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Cosh}(a_x) \cdot \text{Sinh}(a_x, z)}{(s_x - s_x)(s_x - i \omega_i) \cdot \text{Sinh}(s_x)} - \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Sinh}(a_x, z)}{(s_x - s_x)(s_x - i \omega_i) \cdot \text{Sinh}(s_x)} \]

\[ + \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Cosh}(s_x, z)}{(s_x - s_x)(s_x - i \omega_i)} - \frac{P_0 \cdot \text{Cosh}(s_x, z)}{(s_x - s_x) s_x} + \frac{P_0 \cdot \text{Cosh}(a_x, z)}{(s_x - s_x)(s_x - i \omega_i) \cdot \text{Sinh}(s_x)} - \]

\[ - \frac{P_0 \cdot \text{Sinh}(a_x, z)}{(s_x - s_x) s_x} + \frac{P_0}{(s_x - s_x) s_x} \cdot e^{i \alpha} + \]

\[ \sum_{n=0}^{\infty} \left\{ \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Cosh}(a_x, z)}{a^2_x(s_x - s_x)(s_x - i \omega_i)} - \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Sinh}(a_x, z)}{a^2_x(s_x - s_x)(s_x - i \omega_i)} + \right. \]

\[ + \frac{P_0 \cdot \text{Cosh}(a_x, z)}{a^2_x(s_x - s_x)(s_x)} - \frac{P_0 \cdot \text{Sinh}(a_x, z)}{a^2_x(s_x - s_x)(s_x)} \cdot e^{i \alpha} + \]

\[ \left. + \sum_{n=0}^{\infty} \left\{ \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Cosh}(a_x, z)}{a^2_x(s_x - s_x)(s_x - i \omega_i)} - \frac{P_1 (1 + i \beta \omega_i) \cdot \text{Sinh}(a_x, z)}{a^2_x(s_x - s_x)(s_x - i \omega_i)} + \right. \]

\[ + \frac{P_0 \cdot \text{Cosh}(a_x, z)}{a^2_x(s_x - s_x)(s_x)} - \frac{P_0 \cdot \text{Sinh}(a_x, z)}{a^2_x(s_x - s_x)(s_x)} \cdot e^{i \alpha} \right\} \]  

(3.68)

(Where the constants are mentioned in the appendix)

On the upper and lower plates the shear stresses is as shown below

\[ \tau_U = \left( \frac{dq}{dz} \right)_{z=L} \quad \text{and} \quad \tau_L = \left( \frac{dq}{dz} \right)_{z=0} \]
3.3. HALL EFFECTS ON UNSTEADY MHD FLOW OF A NON-NEWTONIAN FLUID THROUGH A POROUS MEDIUM WITH UNIFORM SUCTION AND INJECTION

We consider an incompressible viscous and electrically conducting non Newtonian Bingham fluid in a parallel plate channel bounded by a loosely packed porous medium. The fluid is driven by a uniform pressure gradient parallel to the channel plates and the entire flow field is subjected to a uniform magnetic field of strength $H_0$ with the normal to the boundaries in the transverse $xz$-plane. The geometrical figure of the problem is given in Fig.3.1. The fluid is supposed to be incompressible, laminar and obeys Bingham model and flows between two infinite horizontal plates located at the $y = \pm h$ planes and extends from $x = 0$ to $\infty$ and from $z = 0$ to $\infty$. The upper plate moves with uniform velocity $U_0$ while the lower plate is stationary. Both the upper and lower plates are kept at two different and constant temperatures $T_2$ and $T_1$ respectively with $T_2 > T_1$. The fluid is acted upon by a constant pressure gradient $\frac{dp}{dx}$ in $x$-direction, uniform suction from above and an injection from below which are applied at $t = 0$. A uniform magnetic field $B_0$ is applied in the positive $y$-direction which is assumed undisturbed as the induced magnetic field to be neglected by assuming a very small magnetic Reynolds number. Hall Effect is considered and consequently a $z$-component for the velocity is expected to rise. The uniform suction indicates that the $y$-component of the velocity is constant. Then, the fluid velocity vector is given by
\[ v(y,t) = v(y,t)i + v_0j + w(y,t)k \]. It is to be noted that the outcome of problem is a linear one. In hydrodynamic case without the suction and injection, the problem lessens to Poiseuille problem [75], the classical hydrodynamic linear problem. Without including suction–injection and also by neglecting the Hall current, it lessens to Hartmann-Poiseuille problem [78]; the classical MHD linear problem. The inclusion of a constant suction-injection as well as the Hall term [67] preserves linearity. Then obviously does change the Newtonian fluid to a non-Newtonian one in the present study. The classical problems (Poiseuille and Hartmann-Poiseuille) of channel flow and the related pipe flow of Newtonian fluid are known to be obtain in practice and to give results in excellent concord with experimentation. The fully developed profiles can be noticed away from the inlet and the side-walls of the channel. The usage of a non-Newtonian fluid is unlikely to cause a problem. The fluid motion starts from rest at \( t = 0 \) and the no-slip condition at the plates implies that the fluid velocity has neither a \( z \) nor an \( x \)-component at \( y = \pm h \). The incipient temperature of the fluid is supposed to be equal to \( T_1 \). Since the plates are unbounded in the \( x \) and \( z \)-directions, these physical quantities will remain in these directions.
The fluid flow is governed by the momentum equation

\[ \rho \frac{Dv}{Dt} = \nabla \cdot (\mu \nabla v) - \nabla p + J \times B_0 \]

(3.69)

Whereas \( \rho \) is density of the fluid, \( \mu \) is the apparent viscosity of the model and is given by

\[ \mu = K + \frac{\tau_0}{\sqrt{\left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2}} \]

(3.70)

Where \( K \) is the plastic viscosity of Bingham fluid, \( \tau_0 \) is the yield stress. If the Hall term is taken, the current density \( J \) is as follows

\[ J = \sigma \left[ V \times B_0 - \beta (J \times B_0) \right] \]

(3.71)

Where \( \sigma \) is the electric conductivity of the fluid and \( \beta \) is the Hall factor [78]. The equation (3.71) can be solved in \( J \) to give in

\[ J \times B_0 = -\frac{\sigma B_i}{l + m} \left[ (u + mw)i + (w - mu)k \right] \]

(3.72)
Where \( m \) is the Hall parameter and \( m = \sigma \beta B_0 \).

Thus, the two components of the momentum equation (3.69) becomes

\[
\rho \frac{\partial u}{\partial t} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) - \frac{\sigma B^2}{I + m^2} (u + mw) - \frac{\nu}{k} u \quad (3.73)
\]

\[
\rho \frac{\partial w}{\partial t} + \rho v \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) - \frac{\sigma B^2}{I + m^2} (w - mu) - \frac{\nu}{k} w \quad (3.74)
\]

The energy equation with Joule dissipation and viscous dissipation is given by

\[
\rho c_p \frac{\partial T}{\partial t} + \rho c_v v \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \left( \frac{\sigma B^2}{I + m^2} - \frac{\nu}{k} \right) (u^2 + w^2) \quad (3.75)
\]

Where \( c_p \) and \( k \) are respectively, the specific heat capacity and the thermal conductivity of the fluid. On the right-hand side, the second and the third terms represent the viscous and Joule dissipations respectively. Here each term will have two components. This is due to Hall effect that brings about a velocity \( w \) in the \( z \)-direction. The initial condition and boundary condition of the problem are given by

\[
u = w = 0 \text{ at } t \leq 0 \quad (3.76)
\]

\[
w = 0 \text{ at } y = -h \text{ and } y = h \text{ for } t > 0 \quad (3.77)
\]

\[
u = 0 \text{ at } y = -h \text{ for } t > 0, \quad u = U_0 \text{ at } y = h \text{ for } t > 0 \quad (3.78)
\]

\[
T = T_1 \text{ at } t \leq 0, \quad (3.79)
\]

\[
T = T_1 \text{ at } y = h, \quad T = T_2 \text{ at } y = -h \text{ for } t > 0 \quad (3.80)
\]

that the boundary conditions do not show dependence on \( x \) suggests that the problem has a fully developed solution of the form, \( u = u(y, t) \).
\( v = v_0, \ p = P + G \ x \) where \( P \) is the pressure at \( x = 0 \) (constant), \( G \) is the constant pressure gradient (negative). Under these conditions the continuity equation \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) is automatically satisfied. It is useful to write the above equations in the non-dimensional form. For this, let us introduce the non-dimensional variables

\[
\begin{align*}
x^* &= \frac{x}{h}, \quad y^* = \frac{y}{h}, \quad z^* = \frac{z}{h}, \quad t^* = \frac{t U_0}{h}, \quad w^* = \frac{w}{U_0}, \quad p^* = \frac{p}{\rho U_0^2}, \quad \theta = \frac{T - T_1}{T_2 - T_1}, \quad \mu^* = \frac{\mu}{K}
\end{align*}
\]

Making use of the above non-dimensional variables, the equations from (3.73 to 3.80) and (3.70) are respectively written as (where the asterisks are dropped for convenience)

\[
\begin{align*}
&\frac{\partial u}{\partial t} + \frac{S}{Re} \frac{\partial u}{\partial y} = - \frac{dp}{dx} + \frac{1}{Re} \left[ \frac{\partial}{\partial y} \left( \frac{\mu}{\partial y} \right) - \frac{M^2}{1 + m^2} (u + mw) - \frac{v}{k} u \right], \quad (3.81) \\
&\frac{\partial w}{\partial t} + \frac{S}{Re} \frac{\partial w}{\partial y} = - \frac{dp}{dx} + \frac{1}{Re} \left[ \frac{\partial}{\partial y} \left( \frac{\mu}{\partial y} \right) - \frac{M^2}{1 + m^2} (w - mu) - \frac{v}{k} w \right], \quad (3.82) \\
&\frac{\partial \theta}{\partial t} + \frac{S}{Pr} \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2} + Ec \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \left( \frac{\sigma B_n^2}{1 + m^2} - \frac{v}{k} \right) (u^2 + w^2) \quad (3.83)
\end{align*}
\]

Equivalent initial, boundary conditions:

\[
\begin{align*}
u = w = 0 \text{ at } \ t \leq 0 , \quad (3.84) \\
u = w = 0 \text{ at } y = -1 \quad (3.85) \\
u = 1, \ w = 0 \text{ at } y = 1 \text{ for } t > 0 , \quad (3.86) \\
\theta = 0 \text{ for } t \leq 0 , \text{ and } \theta = 0 \text{ at } y = -1 , \ \theta = 1 \text{ at } y = 1 \text{ for } t > 0 \quad (3.87)
\end{align*}
\]
\[ \mu = I + \frac{\tau_D}{\sqrt{(\frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y})^2}}, \]  

(3.88)

Where

\[ \tau_D = \frac{\tau_0 h}{K U_0} \] is the Bingham number (dimensionless yield stress);

\[ Re = \frac{\rho U_0 h}{K} \] is the Reynolds number;

\[ S = \frac{\rho_v h}{K} \] is the suction parameter;

\[ Pr = \frac{\rho_c h U_0}{K} \] is the Prandtl number;

\[ Ec = \frac{U_0 K}{\rho c_v h (T_h - T_i)} \] is the Eckert number;

\[ M^2 = \frac{\sigma B_0^2 h^2}{K} \] is the Hartmann number squared:

\[ D^{-1} = \frac{h^2}{kK} \] is the inverse Darcy parameter

Equations (3.81), (3.82) and (3.88) represent coupled system of non-linear partial differential equations which are solved numerically under the initial and boundary conditions (3.84, 3.85 and 3.86) using the finite difference approximations. The implicit method of Crank–Nicolson is used [60]. Finite Difference equations relating to the variables are obtained by writing the equations at the midpoint of the computational cell and then replacing the different terms by their second order central difference approximations in y-direction. These
diffusion terms are replaced by taking the average of the central differences at two successive time levels. In equations (3.81 and 3.82) the non-linear terms are first linearized and then an iterative scheme is used at every time step to solve the linearized system of difference equations using Thomas algorithm to determine the velocity distributions. By substituting values of the velocity components in the right-hand side of equation (3.83) which is solved numerically under the initial and boundary conditions (3.87). The computational domain is divided into meshes each is of dimension $\Delta t$ and $\Delta y$ in time and space respectively as shown in Fig. 3.2.

![Mesh diagram](image)

**Fig. 3.2.** Mesh diagram

We define the variables $v = \frac{\partial u}{\partial y}$, $B = \frac{\partial w}{\partial y}$, $H = \frac{\partial \theta}{\partial y}$ and $\mu = \frac{\partial \mu}{\partial y}$ to reduce the second order differential equations to first order differential equations. Finite difference representations for equations (3.81) and (3.82) take the form
\( \left( \frac{u_{i+1,j+1} - u_{i,j+1} + u_{i+1,j} - u_{i,j}}{2\Delta t} \right) + \frac{S}{Re} \left( \frac{v_{i+1,j+1} + v_{i,j+1} + v_{i+1,j} + v_{i,j}}{4} \right) = \)

\[- \frac{dp}{dx} + \left( \frac{\overline{\mu}_{i+1,j} + \overline{\mu}_{i,j}}{2Re} \right) \left( \frac{v_{i+1,j+1} + v_{i,j+1} - v_{i+1,j} - v_{i,j}}{2\Delta y} \right) + \]

\[- \left( \frac{M^2}{1 + m^2} + D^4 \right) \left( \frac{u_{i+1,j+1} + u_{i,j+1} + u_{i+1,j} + u_{i,j}}{2\Delta t} + m \frac{w_{i+1,j+1} + w_{i,j+1} + w_{i+1,j} + w_{i,j}}{2\Delta t} \right) \]

\(- \left( \frac{\overline{\mu}_{i+1,j} + \overline{\mu}_{i,j}}{2Re} \right) \left( \frac{B_{i+1,j+1} + B_{i,j+1} + B_{i+1,j} + B_{i,j}}{4} \right) \)

\(- \left( \frac{M^2}{1 + m^2} + D^4 \right) \left( \frac{w_{i+1,j+1} + w_{i,j+1} + w_{i+1,j} + w_{i,j}}{4Re} - m \frac{u_{i+1,j+1} + u_{i,j+1} + u_{i+1,j} + u_{i,j}}{4Re} \right) \] (3.89)

\[- \left( \frac{\overline{\mu}_{i+1,j} + \overline{\mu}_{i,j}}{2Re} \right) \left( \frac{B_{i+1,j+1} + B_{i,j+1} + B_{i+1,j} + B_{i,j}}{4} \right) \]

\(- \left( \frac{M^2}{1 + m^2} + D^4 \right) \left( \frac{\theta_{i+1,j+1} - \theta_{i,j+1} + \theta_{i+1,j} - \theta_{i,j}}{2\Delta t} + \frac{S}{Re} \left( \frac{H_{i+1,j+1} + H_{i,j+1} + H_{i+1,j} + H_{i,j}}{4} \right) = \right) \]

\( = \frac{1}{Pr} \left( \frac{H_{i+1,j+1} + H_{i,j+1} - H_{i+1,j} - H_{i,j}}{2\Delta t} \right) + DISP \) (3.91)

The variables with bars are given initial guesses from the previous time steps and an iterative scheme is used at every time to solve the linearized system of difference equations. The finite difference form given for the energy equation (3.83) can be written as
Where, DISP exhibit the Joule and viscous dissipation terms which are known from the solution of the momentum equations and can be evaluated at the midpoint \((i, j)\) of the computational cell. Computations have been made for \((dp/dx)=5\), \(Pr=1\), \(Re=1\), \(\alpha = \pi/3\) and \(Ec=0.2\). Step sides \(\Delta t =0.0001\) and \(\Delta y =0.005\) for time and space respectively are chosen and then the scheme converges in nearly 7 iterations at each time step. Small step sizes don’t show any significant change in results. Convergence of this scheme is assumed when each of \(u, v, w, B, \theta\) and \(H\) for the last two approximations disagree from unity by less than \(10^{-6}\) for all values of \(y\) in \(-1 < y< 1\) at each time step. Fewer than seven iterations are required to satisfy this intersection criteria for all ranges of the parameters studied here. So as to examine the correctness and accuracy of the solutions, the result of the time development of velocity components \(u\) and \(w\) at the centre of the channel for the Newtonian case is compared and shown as depicted in Table 3.1, to have a complete agreement with those reported by Attia [67]. This ensures the satisfaction of all the governing equations like energy and momentum equations as well as mass continuity.
Table 3.1: When comparing the current results and the well known results of Attia [65] and Attia [67] for Newtonian fluid $\tau_D = 0$ for $M=2$, $\alpha = \pi / 2$, $n=1$, $S=1$ and $y=0$.

<table>
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<tr>
<th>$t$</th>
<th>The values of $u$</th>
<th>The values of $w$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Present Results</td>
<td>Attia [65]</td>
</tr>
<tr>
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<td>0.4673</td>
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3.4. HALL CURRENT EFFECTS ON MHD FREE AND FORCED CONVECTION FLOW THROUGH A POROUS MEDIUM IN AN INFINITE VERTICAL PLATE IN PRESENCE OF INCLINED MAGNETIC FIELD

Lets take a two dimensional unsteady flow of MHD which is viscous, incompressible and electrically conducting fluid occupying a semi infinite region of space bounded by porous medium through an infinite vertical plate moving with the constant velocity $U$, under the influence of a uniform inclined magnetic field of strength $H_0$ inclined at an angle of inclination $\alpha$ with the normal to the boundaries. The surface temperature of the plate oscillates with small amplitude about a non-uniform mean temperature. The co-ordinate system is such that the $x$-axis is taken along the plate and $z$-axis is normal to the plate. A uniform transverse magnetic field $B_0$ is imposed parallel to $z$-direction. All the fluid properties are considered constant except the influence of the density variation in the buoyancy term, according to the classical Boussinesq approximation. The radiation heat flux in the $x$-direction is considered negligible in comparison to the $z$-direction. The unsteady MHD equation takes the vectorial form when it is governed by the fluid through the porous medium with the influence of transverse magnetic field with buoyancy force is as shown below.

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = \nu \nabla^2 \mathbf{q} + \frac{1}{\rho} \mathbf{J} \times \mathbf{B} + g \beta(T - T_\infty)$$

(3.92)

The equation of continuity is

$$\nabla \cdot \mathbf{q} = 0$$

(3.93)
The Ohm’s law for moving conductor states that

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{q} \times \mathbf{B}) \]  

(3.94)

Maxwell’s electromagnetic field equations are

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} \] (Ampere’s Law)  

(3.95)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \] (Faraday’s Law)  

(3.96)

\[ \nabla \cdot \mathbf{B} = 0 \] (Solenoidal relation i.e., magnetic field continuity)  

(3.97)

\[ \nabla \cdot \mathbf{J} = 0 \] (Gauss’s Law i.e., Conservation of electric charge)  

(3.98)

In which \( \mathbf{q}, \mathbf{B}, \mathbf{E} \) and \( \mathbf{J} \) are, respectively, the velocity vector, magnetic field vector, electric field vector and current density vector. \( T \) is the temperature of the fluid, \( T_\infty \) is the temperature far away the plate, \( g \) is the gravitational acceleration, \( \beta \) is the coefficient of volume expansion, \( \rho \) is the density of fluid, \( \sigma \) is the electrical conductivity, \( \mu \) is the magnetic permeability of the fluid, \( t \) is time, \( \nu \) is dynamic viscosity and \( Bo \) is the magnetic flux density component normal to the plate surface. According to Shercliff [101] and Hughes and young [94], the following assumptions are compatible with the fundamental equations (3.92) to (3.98) of magneto hydro dynamics.

\[ \mathbf{q} = (u, 0, w), \mathbf{B} = (B_x, B_0 \sin \alpha, B_z) \]  

(3.99)

\[ \mathbf{E} = (E_x, E_y, E_z), \mathbf{J} = (J_x, 0, J_z) \]  

(3.100)

Where, \( u \) and \( w \) are the velocity components along the \( x \)-direction and \( z \)-direction respectively. Since magnetic Reynolds number is very small
for metallic liquid or partially ionized fluid the induced magnetic field produced by the electrically conducting fluid is negligible. Also as no external electric field is applied, the polarization voltage is negligible so that following Meyer [97], $E=0$. Ohmic and viscous heating effects are also neglected. The appropriate boundary conditions to be satisfied by equations (3.92) and (3.94) are

$$u'=U, \ w'=0, \ \Phi = T - T_e = \theta_v(x)(1 + \zeta e^{i\omega z'}), \quad \text{at} \quad z'=0;$$

$$u' \rightarrow 0, \ w' \rightarrow 0, \ \Phi \rightarrow 0 \quad \text{at} \quad z' \rightarrow \infty; \quad (3.101)$$

Where $\Phi$ designated wall-free stream temperature difference, $\zeta = \frac{U}{v}$ i.e dimensionless velocity ratio and $\omega$ is the frequency of oscillation in the surface temperature of the plate. The conditions (3.101) suggest solutions to equations (3.92) to (3.94) for the variables $u', v'$and $\Phi$ of the form,

$$u' = u_0' + \varepsilon e^{i\omega t} u_1', \quad (3.102)$$

$$w' = w_0' + \varepsilon e^{i\omega t} w_1', \quad (3.103)$$

$$\Phi = \theta_v(x)(\theta_0' + \zeta e^{i\omega t} \theta_1') \quad (3.104)$$

When the strength of the magnetic field is very large, generalized Ohm’s law modified to include the Hall current, such that

$$J + \frac{\omega_e \tau_e}{H_0} J \times H = \sigma (E + \mu_e q \times H) \quad (3.105)$$

Where, $J$ is the current density vector, $\omega_e$ is the cyclotron frequency, $\tau_e$ is the electron collision time, $H$ is the magnetic field intensity vector, $\sigma$ is the fluid conductivity, $E$ is the electric field, $\mu_e$ is the magnetic
permeability, \( q \) is the velocity vector. In the above equation the ion-slip, the electron pressure gradient and effects of thermo-electric are ignored. We also suppose that the electric field \( E = 0 \) then these assumptions are reduced to

\[
J_y - mJ_z = 0
\]

(3.106)

\[
J_z - mJ_y = \sigma B_0 \sin \alpha u
\]

(3.107)

Solving the above equations (3.106) and (3.107), we get

\[
J_y = \frac{\sigma B_0 \sin \alpha mu}{1 + m^2}, \quad J_z = 0, \quad J_z = \frac{\sigma B_0 \sin \alpha u}{1 + m^2}
\]

(3.108)

In the above equation, \( m = \omega \tau \) is the hall parameter.

Consider the case of an optically-thin gray gas, thermal radiation flux gradient can be expressed as follows (Siegel and Howell [102])

\[
-\frac{\partial q_r}{\partial y'} = 4a \sigma^* (T_{\infty}^{\prime 4} - T^{\prime 4})
\]

(3.109)

And \( q_r \) is the radiative heat flux, \( a \) is absorption coefficient of the fluid and \( \sigma^* \) is the Stefan-Boltzmann constant. We assume that the temperature differences within the flow are sufficiently small such that \( T^{\prime 4} \) may be expressed as a linear function of the temperature. This is accomplished by expanding \( T^{\prime 4} \) in a Taylor series about \( T_{\infty} \) and neglecting higher order terms, leading to:

\[
T^{\prime 4} = 4T_{\infty}^{3} T' - 3T_{\infty}^{4}
\]

(3.110)
Making use of the equation (3.108) the components $u_0$, $w_0$ and $\theta_0$ represent the steady mean flow and temperature fields and satisfy the following equations:

$$\frac{\partial u_0}{\partial x'} + \frac{\partial w_0}{\partial z'} = 0$$

(3.111)

$$0 = \nu \frac{\partial^2 u_0}{\partial z'^2} + \frac{\sigma B_0^2 \sin^2 \alpha}{\rho(1 + m^2)} u_0 \frac{\partial}{\partial x'} + \frac{\nu}{k} u_0$$

(3.112)

$$0 = \nu \frac{\partial^2 w_0}{\partial z'^2} - \frac{\sigma B_0^2 \sin^2 \alpha}{\rho(1 + m^2)} w_0 \frac{\partial}{\partial z'} + \frac{\nu}{k} w_0$$

(3.113)

$$0 = \frac{K}{\rho c_p} \frac{\partial^2 \theta_0}{\partial z'^2} - \frac{1}{\rho c_p} \frac{\partial q_z}{\partial z}$$

(3.114)

Where $K$ designates thermal conductivity and $c_p$ is the specific heat capacity under constant pressure. The corresponding boundary conditions are

$$u_0 = U, w_0 = 0, T = T_w \quad \text{for} \quad z' = 0$$

(3.115)

$$u_0 \to 0, w_0 \to 0, T \to T_w \quad \text{for} \quad z' \to \infty$$

(3.116)

Again making use of the equation (3.108), the components $u_j$, $w_j$ and $\theta_j$ represent the steady mean flow and temperature fields which satisfy the following equations:

$$\frac{\partial u_j}{\partial x'} + \frac{\partial w_j}{\partial z'} = 0$$

(3.117)
\[ i\omega' u_i = v \frac{\partial^2 u_i}{\partial z'^2} + g \beta \theta_w'(x) \theta_i - \frac{\sigma B^2 \sin^2 \alpha}{\rho(1 + m^2)} u_i - \frac{v}{k} u_i \]  
(3.118)

\[ i\omega' w_i = v \frac{\partial^2 w_i}{\partial z'^2} - \frac{\sigma B^2 \sin^2 \alpha}{\rho(1 + m^2)} w_i - \frac{v}{k} w_i \]  
(3.119)

\[ \frac{\partial \Phi}{\partial t} = \frac{K}{\rho c_p} \frac{\partial^2 \theta_i}{\partial z'^2} - \frac{1}{\rho c_p} \frac{\partial q_i}{\partial z} \]  
(3.120)

Corresponding the boundary conditions:

\[ u_i = U, w_i = 0, T = T_w \quad \text{for} \quad z' = 0 \]  
(3.121)

\[ u_i \to 0, w_i \to 0, T \to T_\infty \quad \text{for} \quad z' \to \infty \]  
(3.122)

Proceeding with the analysis we introduce dimensionless quantities to normalize the flow model:

\[ u_0 = \frac{u_i}{U}, w_0 = \frac{w_i}{U}, u_i = \frac{w_i}{U}, w_i = \frac{z'}{v}, \theta_0 = \frac{\theta'_0}{UL}, \theta_i = \frac{\theta'_i}{U Le}, \omega = \frac{\omega'}{U^2}, \quad Gr = \frac{g \beta v^2 \theta_w'(x)}{U^4 L}, \quad M^2 = \frac{\sigma B^2 v}{\rho U^2}, \quad D^{-1} = \frac{\varphi^2}{k u^2}, \quad K_i = \frac{16 a \sigma v^2 T_\infty^3}{K U^2} \]

Where \(Gr\) is Grashof number,

\(M^2\) is the Hartmann (magneto hydro dynamic number),

\(K_i\) is the thermal radiation-conduction number,

\(K\) is thermal conductivity and \(\theta_i\) is dimensionless temperature,

inverse Darcy parameter is \(D^{-1}\).
Using equation (3.120) together with the equations (3.109) and (3.110) the dimensionless form of equation (3.114) becomes:

\[
\frac{d^2 \theta_0}{dz^2} - K \frac{\partial}{\partial z} \theta_0 = 0
\]  

(3.123)

Making use of non-dimensional variables, together with equations (3.109) and (3.110) the dimensionless form of equation (3.120) becomes

\[
\frac{d^2 \theta}{dz^2} - (K_i + i\omega Pr) \theta = 0
\]  

(3.124)

We are introducing complex variables

\[
u_o + i\omega u_o = F,
\]

(3.125)

\[
u_i + i\omega u_i = H
\]

(3.126)

Where \(i = \sqrt{-1}\).

Combining equations (3.112) and (3.113) with the help of (3.125), the differential equation for steady mean flow in dimensionless form becomes:

\[
\frac{d^2 F}{dz^2} - \left( \frac{M^2 \sin \alpha}{1 + m^2} + D^4 \right) F = -Gr \theta_o
\]

(3.127)

Combining equations (3.118) and (3.119) with the help of (3.126), the differential equation for unsteady mean flow in dimensionless form reduces to

\[
\frac{\partial^2 H}{\partial z^2} - \left( \frac{M^2 \sin \alpha}{1 + m^2} + D^4 + i\omega \right) H = -Gr \theta_i
\]

(3.128)
Corresponding boundary conditions for the steady mean flow (non-dimensional) are

\[ u_o = 1, \ w_o = 0, \ \theta_o = 1 \quad \text{for} \quad z = 0 \]  \hspace{1cm} (3.129)
\[ u_o = 0, \ w_o = 0, \ \theta_o = 0 \quad \text{for} \quad z \to \infty \]  \hspace{1cm} (3.130)

The corresponding boundary conditions for unsteady mean flow (non-dimensional) are

\[ u_1 = 1, \ w_1 = 0, \ \theta_1 = 1 \quad \text{at} \quad z = 0 \]  \hspace{1cm} (3.131)
\[ u_1 = 0, \ w_1 = 0, \ \theta_1 = 0 \quad \text{at} \quad z \to \infty \]  \hspace{1cm} (3.132)

The boundary conditions (3.129), (3.130), (3.131) and (3.132) can be written subject to equation (3.125 and 3.126) as follows:

\[ F = 1, \ \theta_o = 1 \quad \text{at} \quad z = 0 \]  \hspace{1cm} (3.133)
\[ F = 0, \ \theta_o = 0 \quad \text{at} \quad z \to \infty \]  \hspace{1cm} (3.134)

and

\[ H = 1, \ \theta_1 = 1 \quad \text{at} \quad z = 0 \]  \hspace{1cm} (3.135)
\[ H = 0, \ \theta_1 = 0 \quad \text{at} \quad z \to \infty \]  \hspace{1cm} (3.136)

Equations (3.127) and (3.123) subjects to the boundary conditions (3.133) and (3.134) can be solved and the solution for the steady mean flow can be expressed as
\[ F = u_0(z) + iw_0(z) = e^{-\left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right)z} + \frac{Gr}{K_1 - \left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right)} e^{-\left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right)z} - e^{-\sqrt{K_1}z} \]

(3.137)

in which \(-e^{-\sqrt{K_1}z} = \theta_0(z)\).

Equations (3.127) and (3.124) subjects to the boundary conditions (3.135) and (3.136) may also be solved yielding the following solution for unsteady mean flow

\[ H = u_1(z,t) + iw_1(z,t) = e^{-\left(C_1 \cdot iD_1\right)z} + \frac{\alpha - i\beta}{\alpha^2 + \beta^2} Gr\left[e^{-\left(C_1 \cdot iD_1\right)z} - e^{-\left(C_2 \cdot iD_2\right)z}\right] \]

(3.138)

And \(\theta_1(z,t) = e^{-\sqrt{K_1 \cdot x\cdot t}}\).

Where, the functions \(\theta_0\) and \(\theta_1\) denote the temperature fields due to the main flow and cross flows respectively. Of interest in practical MHD plasma energy generator design are the dimensionless shear stresses at the plate which may be defined for steady and unsteady mean flow respectively as follows:

\[ \frac{dF}{dy}\big|_{y=0} = \left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right) + \frac{Gr}{K_1 - \left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right)} \left[-\left(\frac{M^2 \sin^2\alpha}{1 + m^2} + D^4\right) + \sqrt{K_1}\right] \]

(3.139)

\[ \frac{\partial H}{\partial y}\big|_{y=0} = -(C_1 + iD_1) + \frac{\alpha - i\beta}{\alpha^2 + \beta^2} Gr\left[-(C_1 + iD_1) + (C_2 + iD_2)\right] \]

(3.140)
It is evident from equations (3.139) and (3.140) that the shear stress component due to the main flow for the steady mean flow equations (3.139) and the shear stress components due to main and cross flows given by equation (3.140) do not vanish at the plate.

Inspection of these expressions also reveals that the shear stress component as defined by equation (3.139) due to a steady mean flow is subjected to a non-periodic oscillation that depends on Hartmann number, inverse Darcy parameter and radiation-conduction parameter.

In contrast to this, the shear stress components as computed in equation (3.140) due to the main and cross flows for an unsteady mean flow are subjected to periodic oscillation which is a function of not only Hartmann number and radiation-conduction parameter, but also the Prandtl number and the frequency of oscillation. The shear stress for equation (3.139) will vanish at the plate (z=0) at a critical value of the free convection parameter i.e. Grashof number, defined by the condition:

\[ Gr_{crit} = \left( \frac{M^2 \sin^2 \alpha}{1 + m^2} + D^{-1} \right) \left( \sqrt{K_1 + \left( \frac{M^2 \sin^2 \alpha}{1 + m^2} + D^{-1} \right)} \right) \]  

(3.141)

The shear stress for equation (3.4.48) will vanish at the plate (z=0) when

\[ Gr_{crit} = (C_1 + i D_1)[(C_1 + C_2) + i(D_1 + D_2)] \]  

(3.142)

Also of interest in plasma MHD generator design is the dimensionless temperature gradient at the plate. This can be shown to take the form for the unsteady main flow as follows:

\[ \left. \frac{d\theta}{dz} \right|_{z=0} = -\sqrt{K_1} \]  

(3.143)
For the unsteady cross flow the dimensionless temperature gradient at the plate (y=0) is

\[
\frac{d\theta_i}{dz} \bigg|_{z=0} = -\sqrt{K_r + i\omega Pr}
\]

(3.144)

Comparing equations (3.142) and (3.143) it is immediately deduced that in the absence of an oscillating surface i.e., for \( \omega = 0 \), the dimensionless temperature gradient due to a steady and unsteady mean flows are identical.

### 3.5. STEADY MHD COUETTE FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID THROUGH A POROUS MEDIUM BETWEEN TWO INFINITE PARALLEL PLATES UNDER EFFECT OF INCLINED MAGNETIC FIELD

Under the effect of inclined magnetic field at an inclination \( \beta \), we reflect on the steady hydro magnetic three dimensional couette flow which is viscous and incompressible through a porous medium between two infinite parallel plates. The stationary plate is on xz-plane and the other plate at a distance \( h \), moving with uniform velocity \( U \) along the x-direction. The motionless plate subjected to sinusoidal surface temperature and sinusoidal suction velocity together, varying in the z-direction as well as moving plate is assumed to be isothermal with a uniform injection velocity and taking viscous dissipation into account. The steady hydro magnetic equations through a porous medium with inclined magnetic field are governed by the following equations.
\[ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3.145) \]

\[ v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\sigma \mu^2 H^2 \sin \beta}{\rho} u - \frac{\nu}{k} u \quad (3.146) \]

\[ \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\sigma \mu^2 H^2 \sin \beta}{\rho} v - \frac{\nu}{k} v \quad (3.147) \]

\[ \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\sigma \mu^2 H^2 \sin \beta}{\rho} w - \frac{\nu}{k} w \quad (3.148) \]

\[ \rho C_p \left( v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = K_1 \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \phi \quad (3.149) \]

Where \( \phi \) is the viscous dissipation function given by

\[ \phi = 2\mu \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2, \]

\( u, v \) and \( w \) are the velocity components in the directions \( x, y \) and \( z \) respectively. \( \mu \) is the coefficient of viscosity, \( k \) is the medium permeability, \( H_o \) is the applied magnetic field, \( \rho \) is the density of the fluid, \( \mu_r \) is the magnetic permeability, \( p \) is the pressure, \( \sigma \) is the conductivity of the medium, \( \nu \) is the kinematic viscosity, \( T \) is the temperature, \( K_1 \) is the thermal conductivity and \( C_p \) is the specific heat of the fluid at constant pressure.

The boundary conditions are

\[ u = 0, \quad v = -V \left( 1 + \psi \cos \left( \frac{\pi z}{h} \right) \right) \]

\[ w = 0, \quad T = T_1 \left( 1 + \psi \cos \left( \frac{\pi z}{h} \right) \right), \quad p = 0 \quad \text{at} \quad y = 0 \quad (3.150) \]
\[ u = U, \ v = -V, \ w = 0 \]
\[ T = T, \ T, T > T, \]
\[ p = V \mu h \]
\[ \text{at} \quad y = h \]  
(3.151)

Where \( \psi \ll 1 \). \( U \) and \( V \) are constants with dimension of velocity, \( h \)
and \( T \) are constants with dimension of length and temperature respectively. Introducing the non-dimensional parameters

\[ z^* = \frac{z}{h}, \ y^* = \frac{y}{h}, \ u^* = \frac{u}{U}, \ v^* = \frac{v}{U}, \ w^* = \frac{w}{U}, \ p^* = \frac{p}{pU^2}, \ \theta = \frac{T - T}{T_2 - T} \]

By using non-dimensional variables, the governing equation reduces to

\[
\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]
(3.152)

\[
\frac{v}{\partial y} + \frac{w}{\partial z} = \frac{1}{R} \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) u
\]
(3.153)

\[
\frac{v}{\partial y} + \frac{w}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) v
\]
(3.154)

\[
\frac{v}{\partial y} + \frac{w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{R} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) w
\]
(3.155)

\[
\frac{v}{\partial y} + \frac{w}{\partial z} = \frac{1}{PR} \left( \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) + \frac{E}{R} \phi
\]
(3.156)

The viscous dissipation function gives

\[
\phi = 2 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2
\]
(3.157)
Corresponding boundary conditions are

\[
\begin{align*}
u = 0, \quad v &= -\alpha (1 + \psi \cos(\pi z)), \\
w = 0, \quad \theta &= a \psi \cos(\pi z), \\
p &= 0
\end{align*}
\]

at \( y = 0 \) \hspace{1cm} (3.158)

and

\[
\begin{align*}
u = 1, \quad v &= -\alpha, \\
w = 0, \quad \theta &= 1, \quad p = d
\end{align*}
\]

at \( y = 1 \) \hspace{1cm} (3.159)

Non-dimensional parameters are:

\[
P = \frac{\mu C_p}{K} \text{ is the Prandtl number,}
\]

\[
R = \frac{U h}{v} \text{ is the Reynolds number,}
\]

\[
M^2 = \frac{\frac{\sigma \mu z H_0^2 H^2}{\rho \nu}} \text{ is the Hartmann number,}
\]

\[
E = \frac{U^2}{C_p(T_2 - T_1)} \text{ is the Eckert number,}
\]

\[
D = \frac{k}{h^2} \text{ is the Darcy parameter,}
\]

\[
\alpha = \frac{V}{U} \text{ is the Suction parameter and}
\]

\[
d = \frac{\alpha}{R D}
\]

In order to solve the above partial differential equations, we can take
\[u(y,z) = u_o(y) + \psi u_i(y,z)\]
\[v(y,z) = v_o(y) + \psi v_i(y,z)\]
\[w(y,z) = \psi w_i(y,z)\]
\[p(y,z) = p_o(y) + \psi p_i(y,z)\]
\[\theta(y,z) = \theta_o(y) + \psi \theta_i(y,z)\]  \hspace{1cm} (3.160)

This is valid, since it is assumed that amplitude \(\psi \ll 1\) of sinusoidal suction velocity is small compared to its wavelength.

Substituting equation (3.160) in (3.152 to 3.159) and taking \(\psi = 0\). The unperturbed quantities satisfy the following equations

\[\frac{dy_0}{dy} = 0\]  \hspace{1cm} (3.161)

\[D \frac{d^2u_o}{dy^2} + RD\alpha \frac{du_o}{dy} - (M^2 \sin^2 \beta D + 1)u_o = 0\]  \hspace{1cm} (3.162)

\[\frac{dp_o}{dy} = d\]  \hspace{1cm} (3.163)

\[\frac{d^2\theta_o}{dy^2} + PR\alpha \frac{du_o}{dy} + EP \left(\frac{du_o}{dy}\right)^2 = 0\]  \hspace{1cm} (3.164)

With the boundary conditions

\[\begin{align*}
&u_0 = 0, v_0 = -\alpha \hspace{1cm} \text{at} \hspace{1cm} y = 0 \\
&\theta_0 = 0, p_0 = 0
\end{align*}\]  \hspace{1cm} (3.165)

\[\begin{align*}
&u_0 = 1, v_0 = -\alpha \hspace{1cm} \text{at} \hspace{1cm} y = 1 \\
&\theta_0 = 1, p_0 = d
\end{align*}\]  \hspace{1cm} (3.166)

The solutions of the equations (3.161 to 3.164) by using boundary conditions (3.165) and (3.166) are

\[v_o(y) = -\alpha\]  \hspace{1cm} (3.167)
\[ u_0(y) = \frac{e^{m y} - e^{m y}}{e^m - e^m} \]  
\[ p_0(y) = d \ y \]  
\[ \theta_0(y) = C_1 + C_2 e^{-(PR\alpha) y} - \frac{EP}{(e^m - e^m)^2} \left\{ \frac{m e^{2 m y}}{2(2m_1 + PR\alpha)} + \frac{m e^{2 m y}}{2(2m_2 + PR\alpha)} - \frac{2m_1 e^{(m+m_2)y}}{(m_1 + m_2)^2 + (PR\alpha)(m_1 + m_2)} \right\} \] 
\[ (3.168) \]
\[ (3.169) \]
\[ (3.170) \]

Likewise the perturbed quantities in equations (3.161 to 3.164) satisfies the following equations

\[ \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \]  
\[ (3.171) \]

\[ v_1 \frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial y} = \frac{1}{R} \left( \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) u_1 \]
\[ (3.172) \]

\[ -\alpha \frac{\partial v_1}{\partial y} = \frac{\partial p_1}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) v_1 \]
\[ (3.173) \]

\[ -\alpha \frac{\partial w_1}{\partial y} = \frac{\partial p_1}{\partial z} + \frac{1}{R} \left( \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right) - \left( \frac{M^2 \sin^2 \beta D + 1}{RD} \right) w_1 \]
\[ (3.174) \]

\[ -\alpha v \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_1}{\partial y} = \frac{1}{PR} \left( \frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right) + \frac{2E}{R} \frac{\partial u_0}{\partial y} \frac{\partial u_1}{\partial y} \]
\[ (3.175) \]

Corresponding the boundary conditions

\[ \begin{align*}
  u_1 &= 0, v_1 = -\alpha \cos(\pi z) \\
  w_1 &= 0, \theta_1 = -a \cos(\pi z) \\
  p_1 &= 0
\end{align*} \]
\[ \text{at } y = 0 \]  
\[ (3.176) \]

and
\[ \begin{align*}
  u_i &= 0, v_i = -\alpha \\
  w_i &= 0, \theta_i = 0 \\
  p_i &= 0 \\
\end{align*} \]

at \( y = 1 \) \hspace{1cm} (3.177)

The equations (3.171 to 3.175) are linear partial differential equations which describes the perturbed three dimensional flow due to variation of the suction velocity stationary surface temperature along three directions. The form of the suction velocity stationary surface temperature suggests the following forms

\[ u_i = u_2(y)\cos(\pi z) \]  \hspace{1cm} (3.178)

\[ v_i = v_2(y)\cos(\pi z) \]  \hspace{1cm} (3.179)

\[ w_i = -\frac{1}{\pi}v_2(y)\cos(\pi z) \]  \hspace{1cm} (3.180)

\[ p_i = p_2(y)\cos(\pi z) \]  \hspace{1cm} (3.181)

\[ \theta_i = \theta_2(y)\cos(\pi z) \]  \hspace{1cm} (3.182)

The expressions for \( v_i \) and \( w_i \) have been chosen so that the equation of continuity (3.171) is trivially satisfied. The equations (3.173) and (3.174) being independent of the main flow and temperature field can be solved initially. Therefore substituting equations (3.179), (3.180) and (3.181) in equations (3.173) and (3.174). Hence the following simultaneous ordinary differential equations are obtained.

\[ \begin{align*}
  D \frac{d^3 v_2}{dy^3} + RD\alpha \frac{d^2 v_2}{dy^2} - (D\pi^2 + M^2 \sin^2 \beta + 1)v_2 &= \pi^2 RD p_2 \\
\end{align*} \]  \hspace{1cm} (3.183)
\[ D \frac{d^2 v}{dy^2} + RD \frac{dv}{dy} - (D \pi^2 + M^2 \sin^2 \beta + l) v = RD \frac{dp}{dy} \quad (3.184) \]

Corresponding the boundary conditions

\[ v = -\alpha \quad \text{and} \quad \frac{dv}{dy} = 0 \quad \text{at} \quad y = 0 \quad (3.185) \]

\[ v = 0 \quad \text{and} \quad \frac{dv}{dy} = 0 \quad \text{at} \quad y = 1 \quad (3.186) \]

Solving the equations (3.183) and (3.184) using the equations (3.185) and (3.186), which is straight-forward and is substituted in equations (3.179) to (3.181). Therefore we obtain the following expressions.

\[ v_1 = (C_1 e^{\pi \nu} + C_4 e^{m \nu} + C_5 e^{x \nu} + C_6 e^{-x \nu}) \cos (\pi z) \quad (3.187) \]

\[ w_1 = -\frac{1}{\pi} (C_1 e^{\pi \nu} + C_4 e^{m \nu} + C_5 e^{x \nu} + C_6 e^{-x \nu}) \cos (\pi z) \quad (3.188) \]

\[ p_1 = \frac{1}{\pi^2 RD} (C_1 e^{\pi \nu} + C_4 e^{m \nu} + C_5 e^{x \nu} + C_6 e^{-x \nu}) \cos (\pi z) \quad (3.189) \]

Now for the main flow and the temperature field, the expressions (3.178, 3.182) are substituted in equations (3.172 and 3.175) to give the following ordinary differential equations

\[ D \frac{d^2 u}{dy^2} + RD \frac{du}{dy} - (D \pi^2 + M^2 \sin^2 \beta + l) u = RD \frac{d^2 u}{dy^2} \quad (3.190) \]

\[ \frac{d^2 \theta}{dy^2} + PR \frac{d \theta}{dy} - \pi^2 \theta = v \frac{PR \frac{d \theta}{dy} - 2PE \frac{du}{dy}}{dy} \quad (3.191) \]
Corresponding the boundary conditions

\[ u_2 = 0 \quad \text{and} \quad \theta_2 = 0 \quad \text{at} \quad y = 0 \quad \text{(3.192)} \]

and

\[ u_2 = 0 \quad \text{and} \quad \theta_2 = 0 \quad \text{at} \quad y = 1 \quad \text{(3.193)} \]

Solving equations (3.190) and (3.191) with boundary conditions (3.192) and (3.193) and substituting the solutions in the equations (3.178) and (3.179), the expressions for \( u_i \) and \( \theta_i \) can be written as

\[
u_i = \left\{ \frac{C_1}{m_1} e^{m_1 y} + C_2 e^{m_2 y} + \frac{R}{e^{m_1} - e^{m_2}} \left( B_1 e^{(m_1 + m_2) y} + B_2 e^{(m_1 - m_2) y} + B_3 e^{(m_1 + m_2) y} + B_4 e^{(m_1 - m_2) y} \right) \right\} \cos (\pi z) \quad \text{(3.194)}
\]

\[
\theta_i = \left\{ \frac{G - e^{m_1} (F - a)}{e^{m_2} - e^{m_3}} \right\} e^{m_2 y} + \left\{ \frac{G - e^{m_3} (F - a)}{e^{m_1} - e^{m_2}} \right\} e^{m_1 y} + f(y) \cos (\pi z) \quad \text{(3.195)}
\]

The skin friction components \( \tau_x \) and \( \tau_z \) along main flow and transverse direction and the non dimensional rate of surface heat transfer (Nusselt number) at both the plates can be calculated as

\[
\tau_x = \frac{1}{\mu U} \left( \frac{du_0}{dy} \right)_{y=0} + \psi \left( \frac{du_z}{dy} \right)_{y=0} \cos (\pi z)
\]

\[
= \left( \frac{m_i - m_0}{e^{m_0} - e^{m_i}} \right) + \psi \left\{ \frac{R m_0 (B - Ae^{m_0})}{(e^{m_0} - e^{m_i})(e^{m_0} - e^{m_2})} + \frac{R m_0 (B - Ae^{m_2})}{(e^{m_2} - e^{m_0})(e^{m_0} - e^{m_2})} \right\}
\]
\begin{align*}
&+ \frac{R}{(e^{m_1} - e^{m_2})} + (B_1(m_1 + m_3) + B_2(m_1 + m_4) + B_3(m_1 + \pi) + B_4(m_1 - \pi) - \\
&- B_1(m_1 + m_3) - B_2(m_1 + m_4) - B_3(m_1 + \pi) - B_4(m_1 - \pi) \bigg) \cos(\pi z) \\
&\tau_z = \frac{d\tau_z}{\mu U} = \wp \left( \frac{d\psi}{dy} \right)_{y=0} \\
&= -\frac{\wp}{\pi} \left\{ C_1 m_1^2 + C_2 m_2^2 + C_3 \pi^2 + C_4 \pi^2 \right\} \sin(\pi z)
\end{align*}

The Nusselt number at the motionless plate:

\begin{align*}
\text{Nu} &= -\frac{h}{T_2 - T_1} \left( \frac{\partial T}{\partial y} \right)_{y=0} \\
&= -\left\{ -PR\alpha C_2 - \frac{EP}{(e^{m_1} - e^{m_2})^2} \left[ \frac{m_1^2}{2m_1 + PR\alpha} + \frac{m_2^2}{2m_2 + PR\alpha} + \frac{2m_1 m_2}{m_1 + m_2 + PR\alpha} \right] + \\
&+ \wp \left[ \left( \frac{G - e^{m_1} (F - a)}{(e^{m_1} - e^{m_2})} \right) a_1 + \left( \frac{G - e^{m_2} (F - a)}{(e^{m_1} - e^{m_2})} \right) a_2 + f'(0) \right] \cos(\pi z) \right\}
\end{align*}

(3.196)

The Nusselt number at the moving plate:

\begin{align*}
\text{Nu} &= -\frac{h}{T_2 - T_1} \left( \frac{\partial T}{\partial y} \right)_{y=1} \\
&= -\left\{ -PR\alpha C_2 e^{PR\alpha} - \frac{EP}{(e^{m_1} - e^{m_2})^2} \left[ \frac{m_1 e^{2m_1}}{2m_1 + PR\alpha} + \frac{m_2 e^{2m_2}}{2m_2 + PR\alpha} + \frac{2m_1 m_2 e^{m_1+m_2}}{m_1 + m_2 + PR\alpha} \right] + \\
&+ \wp \left[ \left( \frac{G - e^{m_1} (F - a)}{(e^{m_1} - e^{m_2})} \right) a_1 e^{m_1} + \left( \frac{G - e^{m_2} (F - a)}{(e^{m_1} - e^{m_2})} \right) a_2 e^{m_2} + f'(1) \right] \cos(\pi z) \right\}
\end{align*}

(3.197)

(3.198)