Chapter 4

On $*$-bimultipliers, generalized $*$-biderivations in rings with involution

4.1 Introduction

An additive mapping $x \mapsto x^*$ on a ring $R$ is said to be an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x^*$ holds for all $x, y \in R$. If $R$ is an algebra we assume additionally that $(\alpha x)^* = \overline{\alpha} x^*$ for all $x \in R$ and $\alpha$ is in some field $F$. A ring (algebra) which is endowed with an involution is called a ring (algebra) with involution or $*$-ring ($*$-algebra). A biadditive (i.e., additive in both arguments) mapping $B : R \times R \to R$ is called a biderivation on $R$ if it is a derivation in each argument i.e., for every $x \in R$, the maps

$y \mapsto B(x, y)$ and $y \mapsto B(y, x)$ are derivations of $R$ into $R$ (see [165], where biderivations satisfying some special properties are studied). We call a symmetric biadditive mapping $B : R \times R \to R$ a symmetric biderivation on $R$ if $B(xy, z) = B(x, z)y + xB(y, z)$ holds for all $x, y, z \in R$. Typical examples are mappings of the forms $(x, y) \mapsto c[x, y]$, where $c$ is an element of the center of $R$. The notion of symmetric biderivation arises naturally in the study of additive commuting mappings, since every commuting additive mapping $f : R \to R$ gives rise to a symmetric biderivation of $R$. Namely, the linearization of the relation $[f(x), x] = 0$ for all $x \in R$ yields that $[f(x), y] = [x, f(y)]$ for all $x, y \in R$. Therefore, we note that the mapping $(x, y) \mapsto [f(x), y]$ is a symmetric biderivation. The concept of symmetric biderivation was introduced by G. Maksa [133] (see also [134], where an example can be found). It was shown in [134] and [166] that symmetric biderivations are related to general solutions of some functional equations.

Further, Brešar et al. [76] established that every biderivation $B$ of a noncommutative prime ring $R$ is of the form $B(x, y) = \lambda [x, y]$ for all $x, y \in R$ and for some $\lambda \in C$. the

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extended centroid of $R$. Some more results related to symmetric biadditive mappings on prime and semiprime rings can be looked in [61], [66], [143], [144] and [165].

The purpose of this chapter is to study symmetric biadditive mappings when the ring $R$ is equipped with an involution. Section 4.2 is devoted to the study of left (resp. right) $*$-bimultipliers in the setting of semi(prime) $*$-rings. In this section, we establish that every left (right) $*$-bimultiplier on a semiprime $*$-ring $R$ maps $R \times R$ into $Z(R)$. Also, we proved that if a prime $*$-ring admits a nonzero left (resp. right) $*$-bimultiplier, then $R$ is commutative.

Section 4.3 deals with the study of symmetric generalized $*$-biderivations (resp. symmetric generalized reverse $*$-biderivations) on semi(prime) $*$-rings and prove that if a semiprime $*$-ring admits a symmetric generalized $*$-biderivation (resp. symmetric generalized reverse $*$-biderivation) $G$ with associated a nonzero symmetric $*$-biderivation (resp. symmetric reverse $*$-biderivation) $B$, then $G$ maps $R \times R$ into $Z(R)$. The prime version of these results have also been given.

In the last section, we establish corresponding results in the setting of $C^*$-algebras.

### 4.2 Left (resp. right) $*$-bimultipliers

Following [178], an additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer (or multiplier) if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. According [143, 144], a biadditive mapping $B : R \times R \to R$ is called a left (resp. right) bimultiplier if $B(xy, z) = B(x, z)y$ (resp. $B(xy, z) = xB(y, z)$) holds for all $x, y, z \in R$. Inspired by the above mentioned concepts in rings, the notion of $*$-bimultiplier can be introduce as follows:

**Definition 4.2.1.** Let $R$ be a $*$-ring. A symmetric biadditive mapping $M : R \times R \to R$ is said to be a symmetric left $*$-bimultiplier if $M(xy, z) = M(x, z)y^*$ holds for all $x, y, z \in R$.

**Definition 4.2.2.** Let $R$ be a $*$-ring. A symmetric biadditive mapping $M : R \times R \to R$ is said to be a symmetric right $*$-bimultiplier if $M(xy, z) = x^*M(y, z)$ holds for all $x, y, z \in R$.

If $M$ is both symmetric left as well as right $*$-bimultiplier, then $M$ is a symmetric $*$-bimultiplier. In the present section, we prove some results related to the left
*-bimultipliers in the setting of prime and semiprime rings with involution. We begin our discussion with the following theorem:

**Theorem 4.2.1.** Let $R$ be a semiprime $*$-ring. If $M : R \times R \to R$ is a biaadditive mapping such that $M(xy, z) = M(x, z)y^*$ for all $x, y, z \in R$, then $M$ maps $R \times R$ into $\mathbb{Z}(R)$.

**Proof.** By the hypothesis, we have

$$M(xy, z) = M(x, z)y^* \text{ for all } x, y, z \in R. \quad (4.2.1)$$

Replacing $y$ by $yw$ in (4.2.1), one hand we obtain

$$M(xyw, z) = M(x(yw), z) = M(x, z)y^*w^* \text{ for all } w, x, y, z \in R. \quad (4.2.2)$$

On the other hand, we have

$$M(xyw, z) = M((xy)w, z) = M(x, z)y^*w^* \text{ for all } w, x, y, z \in R. \quad (4.2.3)$$

Subtracting (4.2.2) from (4.2.3), we obtain

$$M(x, z)[y^*, w^*] = 0 \text{ for all } w, x, y, z \in R. \quad (4.2.4)$$

Substituting $y^*$ for $y$ and $w^*$ for $w$ in (4.2.4), we arrive at

$$M(x, z)[y, w] = 0 \text{ for all } w, x, y, z \in R. \quad (4.2.5)$$

Replacing $y$ by $yM(x, z)$ in the above expression we find that

$$M(x, z)[y, w]M(x, z) + M(x, z)y[M(x, z), w] = 0 \text{ for all } w, x, y, z \in R.$$ \hspace{1cm} \quad (4.2.6)$$

Application of relation (4.2.5) forces that

$$M(x, z)y[M(x, z), w] = 0 \text{ for all } w, x, y, z \in R. \quad (4.2.6)$$

Multiplying by $w$ to (4.2.6) from left yields that

$$wM(x, z)y[M(x, z), w] = 0 \text{ for all } w, x, y, z \in R. \quad (4.2.7)$$
Now putting \( wy \) for \( y \) in (4.2.6), we get

\[
M(x, z)wy[M(x, z), w] = 0 \text{ for all } w, x, y, z \in R. \tag{4.2.8}
\]

Combining (4.2.7) with (4.2.8) we arrive at

\[
[M(x, z), w][M(x, z), w] = 0 \text{ for all } w, x, y, z \in R. \tag{4.2.9}
\]

This implies that \([M(x, z), w]R[M(x, z), w] = (0)\) for all \( w, x, z \in R \). Thus, we obtain, \([M(x, z), w] = 0\) for all \( w, x, z \in R \) by the semiprimeness of \( R \). Hence, \( M \) maps \( R \times R \) into \( Z(R) \). This completes the proof of our first theorem. \( \square \)

We now prove another theorem in this vein that is,

**Theorem 4.2.2.** Let \( R \) be a semiprime \(*\)-ring. If \( M : R \times R \rightarrow R \) is a biadditive mapping such that \( M(xy, z) = x^*M(y, z) \) for all \( x, y, z \in R \), then \( M \) maps \( R \times R \) into \( Z(R) \).

**Proof.** We compute \( M(xy, z) \) in two different ways. Then, we have

\[
M(xyw, z) = x^*y^*M(w, z) \text{ for all } w, x, y, z \in R, \tag{4.2.10}
\]

and

\[
M((xy)w, z) = y^*x^*M(w, z) \text{ for all } w, x, y, z \in R. \tag{4.2.11}
\]

Comparing the expressions so obtained for \( M(xy, z) \), we arrive at

\[
[x^*, y^*]M(w, z) = 0 \text{ for all } w, x, y, z \in R. \tag{4.2.12}
\]

Henceforth, using similar approach as we have used after (4.2.4) in the proof of the last paragraph of Theorem 4.2.1 with necessary variations, we find that \([M(w, z), y] = 0\) for all \( w, y, z \in R \). Hence, \( M \) maps \( R \times R \) into \( Z(R) \). \( \square \)

**Corollary 4.2.1.** Let \( R \) be a semisimple \(*\)-ring. If \( M : R \times R \rightarrow R \) is a biadditive mapping such that \( M(xy, z) = M(x, z)y^* \) for all \( x, y, z \in R \) or \( M(xy, z) = x^*M(y, z) \) for all \( x, y, z \in R \), then \( M \) maps \( R \times R \) into \( Z(R) \).

**Proof.** As a consequence of Theorems 4.2.1 & 4.2.2 and of the fact that every semisimple \(*\)-ring is semiprime \(*\)-ring. \( \square \)

Next, let us consider the prime versions of Theorem 4.2.1 and Theorem 4.2.2.
Theorem 4.2.3. Let $R$ be a prime *-ring. If $M : R \times R \to R$ is a nonzero biadditive mapping such that $M(xy, z) = M(x, z)y^*$ for all $x, y, z \in R$, then $R$ is commutative.

Proof. In view of Theorem 4.2.1, we have $M(x, z)[y, w] = 0$ for all $w, x, y, z \in R$. Substituting $yx$ for $y$, we obtain $M(x, z)y[x, w] + M(x, z)[y, w]x = 0$ for all $w, x, y, z \in R$. This implies that $M(x, z)y[x, w] = 0$ for all $w, x, y, z \in R$, and hence $M(x, z)R[x, w] = (0)$ for all $w, x, z \in R$. Thus, the primeness of $R$ forces that for each $x \in R$ either $[x, w] = 0$ or $M(x, z) = 0$ for all $w, z \in R$. The set of all $x \in R$ for which these two properties hold are additive subgroups of $R$ whose union is $R$. But a group can not be the set-theoretic union of two of its proper subgroups, therefore $M(x, z) = 0$ or $[x, w] = 0$ for all $w, x, z \in R$. But $M(x, z) \neq 0$, we conclude that $[x, w] = 0$ for all $w, x \in R$ and hence $R$ is commutative. \hfill $\Box$

Similarly, we can prove the following:

Theorem 4.2.4. Let $R$ be a prime *-ring. If $M : R \times R \to R$ is a nonzero biadditive mapping such that $M(xy, z) = x^*M(y, z)$ for all $x, y, z \in R$, then $R$ is commutative.

4.3 Generalized *-biderivations

During the last few decades there has been a great deal of work concerning generalized derivation in context of algebras on certain normed spaces (for references see [110], where further references can be found). By a generalized derivation on an algebra $A$, one usually means a mapping of the form $x \mapsto ax + xb$, where $a$ and $b$ are fixed elements in $A$. We prefer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e., the mapping of the form $x \mapsto ax - xa$). Now in a ring $R$, let $F$ be a generalized inner derivation given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_b(y)$, where $I_b(y) = by - yb$ is the inner derivation defined by $b \in R$. In the year 1991, Brešar [57] introduced the concept of generalized derivation in rings. Recently, Hvala [110] initiated the algebraic study of generalized derivation, a function more general than derivation and extended some results concerning derivations to generalized derivations. In the present section, we continue the study in this direction. Let $R$ be a *-ring. An additive mapping $d : R \to R$ is called a *-derivation (resp. reverse *-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(xy) = d(y)x^* + yd(x)$) holds for all $x, y \in R$. Following [20], a symmetric biadditive mapping $B : R \times R \to R$ is called a symmetric *-biderivation if $B(xy, z) = B(x, z)y^* + xB(y, z)$ holds for all $x, y, z \in R$, and $B$ is called a symmetric reverse *-biderivation if $B(xy, z) = B(y, z)x^* + yB(x, z)$ holds for all $x, y, z \in R$. Motivated by
the definition of symmetric $*$-biderivation (resp. symmetric reverse $*$-biderivation), the concept of symmetric generalized $*$-biderivation (resp. symmetric generalized reverse $*$-biderivation) define as follows:

**Definition 4.3.1.** Let $R$ be a $*$-ring. A symmetric biadditive mapping $G : R \times R \rightarrow R$ is called a symmetric generalized $*$-biderivation if there exists a symmetric $*$-biderivation $B$ such that

$$G(xy, z) = G(x, z)y^* + xB(y, z)$$

holds for all $x, y, z \in R$.

**Definition 4.3.2.** Let $R$ be a $*$-ring. A symmetric biadditive mapping $G : R \times R \rightarrow R$ is called a symmetric generalized reverse $*$-biderivation if there exists a symmetric reverse $*$-biderivation $B$ such that

$$G(xy, z) = G(y, z)x^* + yB(x, z)$$

holds for all $x, y, z \in R$.

**Example 4.3.1.** Let $R$ be a $*$-ring. If $B$ is any symmetric $*$-biderivation of $R$ and $f : R \times R \rightarrow R$ is a biadditive mapping such that

$$f(xy, z) = f(x, z)y^* \text{ and } f(x, yz) = f(x, y)z^*$$

for all $x, y, z \in R$. Then $f + B$ is a symmetric generalized $*$-biderivation on $R$.

**Example 4.3.2.** Let $R$ be a $*$-ring. If $B$ is any symmetric reverse $*$-biderivation of $R$ and $f : R \times R \rightarrow R$ is a biadditive mapping such that

$$f(xy, z) = f(y, z)x^* \text{ and } f(x, yz) = f(x, z)y^*$$

for all $x, y, z \in R$. Then $f + B$ is a symmetric generalized reverse $*$-biderivation on $R$.

If $G : R \times R \rightarrow R$ is a symmetric generalized $*$-biderivation of $R$ related to a symmetric $*$-biderivation $B : R \times R \rightarrow R$, then it is easy to see that $G$ is a symmetric generalized $*$-biderivation of $R$ if and only if $G$ is of the form $G = B + M$, where $B$ is a symmetric $*$-biderivation and $M$ is symmetric left $*$-bimultiplier of $R$. Hence, we write $M = G - B$. In the proof of Theorem 4.3.2 below, we use this technique which can be regarded as a contribution to the theory of $*$-bimultipliers in $*$-rings. In [70], Brešar and Vukman proved that if a prime $*$-ring $R$ admits a $*$-derivation (resp. reverse
Then either \( d = 0 \) or \( R \) is commutative. Very recently, Ashraf and Shakir [20] extend above mentioned result for semiprime \(*\)-ring involving symmetric \(*\)-biderivation. In fact, the result which we want to refer states as follows:

**Theorem 4.3.1** ([20, Theorem 3.1]). Let \( R \) be a semiprime \(*\)-ring. Suppose that \( \alpha \) and \( \beta \) are endomorphisms of \( R \) such that \( \alpha \) is surjective. If \( R \) admits a symmetric \((\alpha, \beta)^*\)-biderivation \( B : R \times R \to R \), then \( B \) maps \( R \times R \) into \( Z(R) \).

Motivated by the above mentioned result, we prove the following theorem:

**Theorem 4.3.2.** Let \( R \) be a semiprime \(*\)-ring. If \( R \) admits a symmetric generalized \(*\)-biderivation \( G : R \times R \to R \) with associated a symmetric \(*\)-biderivation \( B : R \times R \to R \), then \( G \) maps \( R \times R \) into \( Z(R) \).

**Proof.** Let us give the proof of this theorem in the following two steps:

**Step 1.** We assume that \( G \) is a symmetric generalized \(*\)-biderivation with associated a symmetric \(*\)-biderivation \( B \). If \( B = 0 \), then \( G \) is a left \(*\)-bimultiplier on \( R \). Thus in view of Theorem 4.2.1, we get the required result.

**Step 2.** On the other hand, suppose that the associated \(*\)-biderivation \( B \neq 0 \). Then, we set \( G = B + M \) and hence \( M = G - B \) where \( M \), \( G \) and \( B \) are biadditive maps on \( R \). Therefore, we have

\[
M(xy, z) = G(xy, z) - B(xy, z)
= G(x, z)y^* + xB(y, z) - B(x, z)y^* - xB(y, z)
= (G(x, z) - B(x, z))y^*
= (G - B)(x, z)y^*
= M(x, z)y^* \quad \text{for all } x, y, z \in R.
\]

This implies that \( M(xy, z) = M(x, z)y^* \) for all \( x, y, z \in R \). That is, \( M \) is a left \(*\)-bimultiplier on \( R \). Therefore, we conclude that \( G \) can be written as \( G = B + M \), where \( B \) is a symmetric \(*\)-biderivation and \( M \) is a left \(*\)-bimultiplier on \( R \). Thus, in view of Theorem 4.2.1 and Theorem 4.3.1, (for \( \alpha = \beta = I_R \), the identity mapping on \( R \)), we conclude that \( G \) maps \( R \times R \) into \( Z(R) \). This proves the theorem completely.

Next, we turn to a corresponding result in the case of generalized reverse \(*\)-biderivation.
Theorem 4.3.3. Let $R$ be a semiprime $*$-ring. If $R$ admits a symmetric generalized reverse $*$-biderivation $G : R \times R \to R$ with associated a nonzero symmetric reverse $*$-biderivation $B : R \times R \to R$, then $[B(x,y),t] = 0$ for all $x, y, t \in R$.

Proof. We are given that $G$ is a symmetric generalized reverse $*$-biderivation with associated a nonzero symmetric reverse $*$-biderivation $B$, we have

$$G(x, yz) = G(x, z)y^* + zB(x, y) \quad \text{for all } x, y, z \in R. \quad (4.3.1)$$

Replacing $z$ by $zt$ in the above relation, we find that

$$G(x, y(zt)) = G(x, t)z^*y^* + tB(x, z)y^* + ztB(x, y) \quad \text{for all } x, y, z, t \in R. \quad (4.3.2)$$

Also, we have

$$G(x, (yz)t) = G(x, t)z^*y^* + tB(x, z)y^* - tzB(x, y) \quad \text{for all } x, y, z, t \in R. \quad (4.3.3)$$

Comparing (4.3.2) with (4.3.3), we obtain

$$[z, t]B(x, y) = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.4)$$

Substituting $B(x, y)z$ for $z$ in (4.3.4) we find that

$$B(x, y)[z, t]B(x, y) + [B(x, y), t]zB(x, y) = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.5)$$

In view of (4.3.4), the above expression reduces to

$$[B(x, y), t]zB(x, y) = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.6)$$

Taking $z = zt$ in (4.3.6), we get

$$[B(x, y), t]ztB(x, y) = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.7)$$

Right multiplication by $t$ to equation (4.3.6) forces that

$$[B(x, y), t]zB(x, y)t = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.8)$$

Subtracting (4.3.7) from (4.3.8), we arrive at

$$[B(x, y), t]z[B(x, y), t] = 0 \quad \text{for all } x, y, z, t \in R. \quad (4.3.9)$$
The last equation can be rewritten in the form \([B(x, y), t]R[B(x, y), t] = 0\) for all \(x, y, t \in R\). It follows from the semiprimeness of \(R\) that \([B(x, y), t] = 0\) for all \(x, y, t \in R\). This proves the theorem.

**Theorem 4.3.4.** Let \(R\) be a prime *-ring. If \(R\) admits a symmetric generalized reverse *-biderivation \(G\) with associated a nonzero symmetric reverse *-biderivation \(B\), then \(R\) is commutative.

**Proof.** By Theorem 4.3.3, we have \([B(x, y), t] = 0\) for all \(x, y, t \in R\). Replace \(y\) by \(yz\) in the last expression and using the fact that \(B\) is a reverse *-biderivation, we obtain \(B(x, z)[y^*, t] + [z, t]B(x, y) = 0\) for all \(t, x, y, z \in R\). This implies that \(B(x, z)[y^*, z] = 0\) for all \(x, y, z \in R\) by (4.3.4). Substituting \(y^*_1\) for \(y\) in the last relation, we get \(B(x, z)[y_1, z] = 0\) for all \(x, y_1, z \in R\). Now replace \(y_1\) by \(w\) to get \(B(x, z)[w, t; z] = 0\) for all \(w, x, z, t \in R\). That is, \(B(x, z)R[t, z] = 0\) for all \(x, z, t \in R\). The primeness of \(R\) yields that either \([t, z] = 0\) or \(B(x, z) = 0\) for all \(x, t \in R\). Now, we put \(A_1 = \{z \in R \mid [t, z] = 0\text{ for all } t \in R\}\) and \(A_2 = \{z \in R \mid B(x, z) = 0\text{ for all } x \in R\}\). Then, clearly \(A_1\) and \(A_2\) are additive subgroups of \(R\). Moreover, by the discussion given, \(R\) is the set-theoretic union of \(A_1\) and \(A_2\). But a group can not be the set-theoretic union of two of its proper subgroups, hence \(A_1 \neq R\) or \(A_2 \neq R\). If \(A_1 = R\), then \([t, z] = 0\) for all \(z, t \in R\) and hence \(R\) is commutative. On the other hand if \(A_2 = R\), then \(B(x, z) = 0\) for all \(x, z \in R\), a contradiction. With this the proof is complete. □

Similarly, we can prove the following:

**Theorem 4.3.5.** Let \(R\) be a prime *-ring. If \(R\) admits a symmetric generalized *-biderivation \(G\) with associated a symmetric *-biderivation \(B\), then \(R\) is commutative.

4.4 Applications to \(C^*\)-algebras

The objective of the present section is to discuss the applications of our previous results to \(C^*\)-algebras. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with the norm \(\| \cdot \|\) satisfying the multiplicative inequality: \(\|xy\| \leq \|x\|\|y\|\) for all \(x\) and \(y\) in \(A\). A \(C^*\)-algebra \(A\) is a Banach *-algebra with the additional norm condition \(\|x^*x\| = \|x\|^2\) for all \(x \in A\). Throughout the present section, \(C^*\)-algebras are assumed to be non unital unless indicated otherwise. We start by proving some results concerning \(C^*\)-algebras.

**Theorem 4.4.1.** Let \(A\) be a \(C^*\)-algebra. If \(M : A \times A \to A\) is a bilinear mapping such that \(M(xy, z) = M(x, z)y^*\) for all \(x, y, z \in A\) or \(M(xy, z) = x^*M(y, z)\) for all \(x, y, z \in A\), then \(M\) maps \(A \times A\) into \(Z(A)\).
Proof. We are given that $M : A \times A \to A$ is a bilinear mapping such that $M(xy, z) = M(x, z)y^*$ for all $x, y, z \in A$. Since $A$ is $C^*$-algebra, so that $A$ is semiprime $*$-ring by Remark 1.2.14. In view of Theorem 4.2.1, we are forced to conclude that $M$ maps $A \times A$ into $Z(R)$.

Similarly, we can prove the result for the case $M(xy, z) = x^*M(y, z)$ for all $x, y, z \in A$. Thereby the proof of the theorem is completed.

Theorem 4.4.2. Let $A$ be a $C^*$-algebra. If $A$ admits a symmetric bilinear generalized $*$-biderivation $G : A \times A \to A$ with associated a symmetric bilinear $*$-biderivation $B : A \times A \to A$, then $G$ maps $A \times A$ into $Z(A)$.

Proof. As a consequence of Theorem 4.3.2, and of the fact that every $C^*$-algebra is semiprime $*$-ring (viz.; [15]).

Similarly, we can establish the following:

Theorem 4.4.3. Let $A$ be a $C^*$-algebra. If $A$ admits a symmetric bilinear generalized reverse $*$-biderivation $G : A \times A \to A$ with an associated nonzero symmetric bilinear reverse $*$-biderivation $B : A \times A \to A$, then $G$ maps $A \times A$ into $Z(A)$. 