Chapter 2

On commutativity of rings involving additive mappings

2.1 Introduction

A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. Pursuit of this line of inquiry was inspired by celebrated Jacobson's theorem that any ring in which every element $x$ satisfies an equation of the form $x^{n(x)} = x$, where $n(x) \in \mathbb{N} \setminus \{1\}$, must be commutative [113], a result which generalized the theorem that every finite division ring is commutative as well as the theorem that every Boolean ring is commutative. There are now more than hundred papers in which conditions are given that determine commutativity for a ring or a special type of ring. Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work [98–100]. A significant contributor has been Bell (see [43–48]) who individually, or with co-authors has written more than two dozen articles. Other strong contributors have been Ashraf and Yaqub with a variety of co-authors (viz.; [3], [24–26], [32], [117], [150–152], [162] and [174], where further references can be found).

Another technique for investigating commutativity of rings (algebras) is the use of additive mappings like derivations and automorphisms of the ring $R$. To indicate how strongly related a derivation is to commutativity, we say a derivation (or other function) $d : R \rightarrow R$ is commuting if $d(x)x = xd(x)$ for all $x \in R$, and centralizing if $xd(x) - d(x)x \in Z(R)$ for all $x \in R$. The study of such mappings was initiated by Posner (Posner second theorem). In [148, Theorem 2], Posner proved that if a prime ring $R$ ad-

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mits a nonzero derivation \( d \) such that \( [d(x), x] \in Z(R) \) for all \( x \in R \), then \( R \) is commutative. The analogous result for centralizing automorphisms on prime rings was obtained by Mayne [140]. A number of authors have extended these theorems of Posner and Mayne, and have shown that derivations, automorphisms, and some related mappings cannot be centralizing on certain subsets of noncommutative prime (and some other) rings. For these kind of results we refer the reader to [28], [30], [50], [51], [58], [62], [87] and [124], where further references can be found. There has been a great deal of work recently concerning the relationship between the commutativity of a ring \( R \) and the existence of certain specified additive mappings like derivations and automorphisms of \( R \). Chung, Herstein, Ikeda, Koç, Luh, Martindale, Procesei, Putcha, Richoux, Schacher, Wilson and Yaqub (viz.; [82–84], [102], [106], [109], [111], [135] and [149]) have studied conditions on commutators which imply the commutativity of rings.

In [103], Herstein proved that a ring \( R \) satisfying the identity \((xy)^n = x^n y^n\) for all \( x, y \in R \), where \( n \) is a fixed positive integer greater than 1, must have nil commutator ideal. Further, Bell [47] showed that if \( R \) is an \( n \)-torsion free ring with identity 1 satisfying the identities \((xy)^n = x^n y^n\) and \((xy)^{n+1} = x^{n+1} y^{n+1}\) for all \( x, y \in R \), then \( R \) is commutative. In the year 1980, Hazar Abu-Khuzam [1], proved that if \( R \) is an \( n(n-1) \)-torsion free ring with identity satisfying the identity \((xy)^n = x^n y^n\) for all \( x, y \in R \), then \( R \) is commutative. Further, Ashraf and Quadri [24] obtained commutativity of rings with identity in which commutators are \( n(n+1) \)-torsion free and \( R \) satisfies the identity \((xy)^n = y^n x^n\) for all \( x, y \in R \setminus J(R) \) or for all \( x, y \in R \setminus N(R) \), where \( J(R) \) is the Jacobson radical of \( R \) and \( N(R) \) is the set of nilpotent elements of \( R \). Very recently, Andima and Pajoohesh [13] established the commutativity of \( R \) involving derivations satisfying the above mentioned identities in the setting of prime rings. Similar related results can be found in [2], [11], [19], [19], [23] and [152].

In Section 2.2, we study the commutativity of semiprime ring which admits the additive mappings \( F \) and \( d \) satisfying certain identities viz.; (i) \( F(xy) = F(yx) \), (ii) \( F((xy)^2) = F(x^2 y^2) \), (iii) \( F((xy)^2) = F(y^2 x^2) \), (iv) \( F((xy)^2) = F(xy^2 x) \), (v) \( F((xy)^2) = F(x y^2 x) \), (vi) \( F((xy)^2) = F(x y^2 y) \), (vii) \( F((xy)^2) = F(x^2 y^2) \), (viii) \( F((xy)^2) = F(x^2 y^2) \) for all \( x, y \) in some appropriate subsets of \( R \). In fact, our results extend and unify some known theorems for derivations or generalized derivations to additive mappings in rings viz.; [50, Theorem 3], [87, Theorem 3.3] and [4, Theorem 1] etc.

In the year 1990, Ashraf and Quadri [24] established that a ring \( R \) is commutative if it satisfies \((xy)^n = y^n x^n\) for all \( x, y \) in some appropriate subsets of \( R \) under certain
mild conditions on commutators in \( R \). We continue the similar study involving additive mappings in Section 2.3 and investigate commutativity of rings satisfying any one of the properties: (i) \( F((xy)^n) = F(x^n y^n) \), (ii) \( F(x^m y^n) = F(y^n x^m) \), (iii) \( F([x^n, y]) = F([x, y^n]) \), (iv) \( (F(x)F(y))^n = (F(y)F(x))^n \) for all \( x, y \in R \).

In Section 2.4, we shall continue the similar study and extend results proved in [27], [30] and [31] in the setting of semiprime rings. In fact, it is shown that in a semiprime ring which admits the additive mappings \( F \) and \( d \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \) satisfying the relation \( F(x)F(y) \pm xy \in Z(R) \) for all \( x, y \in I \), the nonzero ideal of \( R \) implies \([d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation such that \( d(I) \neq \{0\} \), then \( R \) contains a nonzero central ideal.

Section 2.5 is devoted to the study of commutativity of the ring \( R \) involving arbitrary mappings with necessary torsion restrictions on commutators.

Finally, in Section 2.6 suitable examples are also provided at places to demonstrate that restrictions imposed on the hypotheses of the various results are not superfluous.

### 2.2 The condition \( F(xy) = F(yx) \)

It was shown in [50] that if \( R \) is a prime ring admitting a nonzero derivation \( d \) such that \( d(xy) = d(yx) \) for all \( x, y \in R \), then \( R \) is commutative. In [87], Daif proved the following result: Let \( R \) be a semiprime ring and \( I \) be a nonzero ideal of \( R \). If \( R \) admits a derivation \( d \) which is nonzero on \( I \) and satisfies \( d(xy) = d(yx) \) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal. Further, Albaş and Argaç [4] established same result for generalized derivations in the setting of prime rings. The following theorem is a natural extension of the above mentioned results:

**Theorem 2.2.1.** Let \( R \) be a semiprime ring and \( I \) be a nonzero ideal of \( R \). Next, let \( F, d : R \to R \) be additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). If \( F(xy) = F(yx) \) for all \( x, y \in I \), then \([d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation such that \( d(I) \neq \{0\} \), then \( R \) contains a nonzero central ideal.

**Proof.** By the assumption, we have

\[
F([x, y]) = 0 \text{ for all } x, y \in I. \tag{2.2.1}
\]
Replacing \( y \) by \( yz \) in (2.2.3) and using it, we get
\[
[x, y]zd(x) = 0 \text{ for all } x, y, z \in I. \quad (2.2.4)
\]
This implies that
\[
[x, d(x)]zd(x) = 0 \text{ for all } x, z \in I. \quad (2.2.5)
\]
Right multiplication by \( x \) to (2.2.5) yields that
\[
[x, d(x)]zd(x)x = 0 \text{ for all } x, z \in I. \quad (2.2.6)
\]
Replacing \( z \) by \( zx \) in (2.2.5), we get
\[
[x, d(x)]zxd(x) = 0 \text{ for all } x, z \in I. \quad (2.2.7)
\]
Subtracting (2.2.6) from (2.2.7), we obtain
\[
[x, d(x)]z[x, d(x)] = 0 \text{ for all } x, z \in I.
\]
This implies that
\[
I[x, d(x)]RI[x, d(x)] = (0) \text{ for all } x \in I.
\]
The semiprimeness of \( R \) forces that \( I[x, d(x)] = (0) \) for all \( x \in I \) and hence \([x, d(x)] \in \text{ann}_R(I)\) for all \( x \in I \). Since \( I \) is an ideal of \( R \), it is obvious that \([x, d(x)] \in I \) for all \( x \in I \). Hence, by Lemma 1.3.19, \([x, d(x)] \in \text{ann}_R(I) \cap I = (0) \) for all \( x \in I \). Further, if \( d \) is a derivation such that \( d(I) \neq (0) \), then in view of Lemma 1.3.3, \( R \) contains a nonzero central ideal. This completes the proof. \( \Box \)

This is well-known that a group \( G \) must be commutative if it satisfies the condition \((xy)^2 = x^2y^2\) for all \( x, y \in G \). In [117], Johnsen et al. established a ring-theoretic analogue of the above mentioned result. Here, we study similar condition involving
additive mappings and derivations. Indeed, we prove the following result:

**Theorem 2.2.2.** Let $R$ be a semiprime ring with identity 1 and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F((xy)^2) = F(x^2y^2)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.

**Proof.** By the assumption, we have $F((xy)^2) = F(x^2y^2)$ for all $x, y \in R$. Replacing $x$ by $x + 1$, we get $F(((x + 1)y)^2) = F((x + 1)^2y^2)$ for all $x, y \in R$. This implies that

$$F((xy)^2 + y^2 + (xy)y + y(xy)) = F(x^2y^2 + y^2 + 2xy^2)$$

for all $x, y \in R$.

Our hypothesis yields that $F(yxy) = F(xy^2)$ for all $x, y \in R$. Repeating this argument for $y + 1$ in place of $y$, the above expression gives that $F((y + 1)x(y + 1)) = F(x(y + 1)^2)$ for all $x, y \in R$. Expanding and simplifying the expression yields that $F(xy) = F(yx)$ for all $x \in R$. By Theorem 2.2.1, we conclude that $[d(x), x] = 0$ for all $x \in R$. Further, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative in view of Lemma 1.3.29. □

Using similar approach with necessary variation we can establish the following:

**Theorem 2.2.3.** Let $R$ be a 3-torsion free semiprime ring with identity and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F((xy)^2) = F(y^2x^2)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.

**Theorem 2.2.4.** Let $R$ be a semiprime ring with identity 1 and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F((xy)^2) = F(xy^2x)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.

**Proof.** By hypothesis, we have $F((xy)^2) = F(xy^2x)$ for all $x, y \in R$. Putting $x = x + 1$, we get

$$F(y^2 + (xy)^2 + yxy + xy^2) = F(y^2 + y^2x + xy^2 + xy^2x)$$

for all $x, y \in R$.

Using our hypothesis, we obtain $F(yxy) = F(y^2x)$ for all $x, y \in R$. Replacing $y$ by $y + 1$ in the last expression, we get $F(((y + 1)x(y + 1)) = F((y + 1)^2x)$ for all $x, y \in R$. Expanding and simplifying, we obtain $F(xy) = F(yx)$ for all $x, y \in R$. Therefore, by
Theorem 2.2.1, we conclude that \([d(x), x] = 0\) for all \(x \in R\). Moreover, if \(R\) is prime and \(d\) is a nonzero derivation of \(R\), then \(R\) is commutative in view of Lemma 1.3.29. 

Similarly, we can prove the following:

**Theorem 2.2.5.** Let \(R\) be a semiprime ring with identity and \(F, d : R \to R\) be additive mappings such that \(F(xy) = F(x)y + xd(y)\) for all \(x, y \in R\). If \(F((xy)^2) = F(xy^2y)\) for all \(x, y \in R\), then \([d(x), x] = 0\) for all \(x \in R\). Moreover, if \(R\) is prime and \(d\) is a nonzero derivation of \(R\), then \(R\) is commutative.

Combining above theorems and Lemma 1.3.10, we prove the following:

**Corollary 2.2.1.** Let \(R\) be a prime ring with identity. If \(R\) admits a generalized derivation \(F\) with an associated nonzero derivation \(d\) such that \(F((xy)^2) = F(xy^2x)\) for all \(x, y \in R\) or \(F((xy)^2) = F(yx^2y)\) for all \(x, y \in R\), then \(R\) is commutative.

It is natural to enquire that what happens if the condition \(F((xy)^2) = F(x^2y^2)\) in Theorem 2.2.2 is replaced by \(F((x \circ y)^2) = F(x^2 \circ y^2)\) or \(F([x, y]^2) = F([x^2, y^2])\). The following theorems gives an answer:

**Theorem 2.2.6.** Let \(R\) be a 2-torsion free semiprime ring with identity 1 and \(F, d : R \to R\) be additive mappings such that \(F(xy) = F(x)y + xd(y)\) for all \(x, y \in R\). If \(F((x \circ y)^2) = F(x^2 \circ y^2)\) for all \(x, y \in R\), then \([d(x), x] = 0\) for all \(x \in R\). Moreover, if \(R\) is prime and \(d\) is a nonzero derivation of \(R\), then \(R\) is commutative.

**Proof.** By hypothesis, we have

\[
F((x \circ y)^2) = F(x^2 \circ y^2) \quad \text{for all } x, y \in R. 
\] (2.2.8)

Replacing \(x\) by \(x + 1\) in (2.2.8), we get

\[
F((x \circ y)^2) + 2F(y^2) + 4F(yxy) = F(x^2 \circ y^2) \quad \text{for all } x, y \in R.
\]

Application of (2.2.8) yields that

\[
2F(y^2 + 2yxy) = 0 \quad \text{for all } x, y \in R.
\]

Since \(R\) is 2-torsion free, the above relation forces that

\[
F(y^2 + 2yxy) = 0 \quad \text{for all } x, y \in R. 
\] (2.2.9)
Now, replace \( y \) by \( x + y \) in (2.2.9) to get
\[
F((x + y)^2 + 2(x + y)x(x + y)) = 0
\]
for all \( x, y \in R \). In view of relation (2.2.9), we obtain
\[
2F(x^2y + yx^2) = -F(xy + yx) \quad \text{for all } x, y \in R.
\]  
(2.2.10)

Substituting \( y + 1 \) for \( y \) in (2.2.10) and using it, we find that
\[
2F(2x^2 + x) = 0 \quad \text{for all } x \in R.
\]  
(2.2.11)

Using the fact that \( R \) is 2-torsion free, we obtain
\[
F(2x^2 + x) = 0 \quad \text{for all } x \in R.
\]  
(2.2.12)

Linearization of (2.2.12) yields that
\[
F(2x^2 + x) + F(2y^2 + y) + 2F(xy + yx) = 0 \quad \text{for all } x, y \in R.
\]  
(2.2.13)

This implies that
\[
F(x \circ y) = 0 \quad \text{for all } x, y \in R.
\]  
(2.2.14)

Replacing \( y \) by \( yz \) in (2.2.14), we get
\[
0 = F(x \circ (yz)) = F((x \circ y)z - y[x, z]) = F(x \circ y)z + (x \circ y)d(z) - F(y)[x, z] - yd([x, z]).
\]  
(2.2.15)

Using (2.2.14) in (2.2.15), we obtain
\[
(x \circ y)d(z) - F(y)[x, z] - yd([x, z]) = 0 \quad \text{for all } x, y, z \in R.
\]  
(2.2.16)

In particular for \( x = z \) in (2.2.16), we get
\[
(x \circ y)d(x) = 0 \quad \text{for all } x, y \in R.
\]  
(2.2.17)

Replacing \( y \) by \( yz \) in (2.2.17) and using it, we obtain
\[
[x, y]zd(x) = 0 \quad \text{for all } x, y, z \in R.
\]

In particular, we have
\[
[x, d(x)]zd(x) = 0 \quad \text{for all } x, z \in R.
\]  
(2.2.18)
Right multiplication by $x$ to (2.2.18) yields that

$$[x, d(x)]zd(x)x = 0 \text{ for all } x, z \in R. \quad (2.2.19)$$

Replace $z$ by $zx$ in (2.2.18) to get

$$[x, d(x)]zxd(x) = 0 \text{ for all } x, z \in R. \quad (2.2.20)$$

Subtracting (2.2.19) from (2.2.20), we get

$$[x, d(x)]z[x, d(x)] = 0 \text{ for all } x, z \in R.$$

This implies that $[x, d(x)]R[x, d(x)] = (0)$ for all $x \in R$. Semiprimeness of $R$ forces that $[x, d(x)] = 0$ for all $x \in R$. Further, if $R$ is prime and $d$ is a nonzero derivation of $R$, then by Lemma 1.3.29, $R$ is commutative. This completes the proof of the theorem. \(\Box\)

**Theorem 2.2.7.** Let $R$ be a 2-torsion free semiprime ring with identity 1 and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F([x, y]^2) = F([x^2, y^2])$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.

**Proof.** By the assumption, we have

$$F([x, y]^2) = F([x^2, y^2]) \text{ for all } x, y \in R. \quad (2.2.21)$$

Replacing $x$ by $x + 1$ in (2.2.21), we get

$$F([x, y] + 1)^2 = F([x^2, y^2]) + 2F([x, y]^2) \text{ for all } x, y \in R.$$  

Using (2.2.21) and the fact that $R$ is 2-torsion free, we obtain

$$F([x, y]^2] = 0 \text{ for all } x, y \in R. \quad (2.2.22)$$

Substituting $y + 1$ for $y$ in (2.2.22) and using it, we get

$$2F([x, y] + 1] = 0 \text{ for all } x, y \in R. \quad (2.2.23)$$

Since $R$ is 2-torsion free, the last expression implies that $F(xy) = F(yx)$ for all $x, y \in R$. Therefore by Theorem 2.2.1, we conclude that $[d(x), x] = 0$ for all $x \in R$. Further, if $R$ is
prime and \(d\) is a nonzero derivation of \(R\), then by Lemma 1.3.29, \(R\) is commutative.

Combining Theorems 2.2.6 & 2.2.7 and Lemma 1.3.10, we obtain the following:

**Corollary 2.2.2.** Let \(R\) be a 2-torsion free prime ring with identity. If \(R\) admits a generalized derivation \(F\) with an associated nonzero derivation \(d\) such that \(F((xy)^2) = F(x^2 \circ y^2)\) for all \(x, y \in R\) or \(F([x, y]^2) = F([x^2, y^2])\) for all \(x, y \in R\), then \(R\) is commutative.

### 2.3 The condition \((xy)^n = x^ny^n\)

In [103], Herstein proved that a ring \(R\) must have nil commutator ideal if \((xy)^n = x^ny^n\) holds for all \(x, y \in R\), where \(n\) is a fixed positive integer greater than 1. Further, Abu-Khuzam [1] proved that if \(R\) is \(n(n-1)\)-torsion free ring with 1 satisfying the identity \((xy)^n = x^ny^n\), then \(R\) is commutative. Furthermore, Ashraf and Quadri [24] showed that \(R\) is commutative if and only if it satisfies \((xy)^n = y^nx^n\) for all \(x, y \in R \setminus J(R)\), where \(J(R)\) is Jacobson radical of \(R\) or for all \(x, y \in R \setminus N(R)\), the set of nilpotent elements of \(R\) and commutators in \(R\) are \(n(n+1)\)-torsion free. The objective of this section is to study similar types of conditions involving additive mappings in the setting of prime and semiprime rings. Moreover, our approach is entirely different from those employed by the above mentioned authors.

We begin our discussion with the following theorem:

**Theorem 2.3.1.** Let \(R\) be an \(n!\)-torsion free semiprime ring with identity 1, where \(n \geq 2\) is a fixed integer and let \(F, d : R \rightarrow R\) be additive mappings such that \(F(xy) = F(x)y + xd(y)\) for all \(x, y \in R\). If \(F((xy)^n) = F(x^ny^n)\) holds for all \(x, y \in R\), then \([d(x), x] = 0\) for all \(x \in R\). Moreover, if \(R\) is prime and \(d\) is a nonzero derivation of \(R\), then \(R\) is commutative.

**Proof.** Suppose \(n = 2\), then result follows by Theorem 2.2.1. Henceforth, we assume that \(n > 2\), and \(F((xy)^n) = F(x^ny^n)\) for all \(x, y \in R\). Replacing \(x\) by \(x + 1\) and \(y\) by \(y + 1\) in above expression, we obtain

\[
F(((x + 1)(y + 1))^n) = F((x + 1)^n(y + 1)^n)
\]

for all \(x, y \in R\).

This can be written as

\[
F((xy + x + y + 1)^n) = F\left(\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right) \left(\sum_{i=0}^{n} \binom{n}{i} y^i\right)\right) \quad \text{for all } x, y \in R.
\]  

(2.3.1)
Using the additivity of $F$ and expanding left hand side of (2.3.1), we write left hand side of (2.3.1) as a sum of the additive mapping $F$ of product of $x$ and $y$. Let $A_{ij}(x,y)$ be as the sum of all terms in which $x$ appears exactly $i$ times and $y$ appears exactly $j$ times. In a similar manner, we define $B_{ij}(x,y)$ for the right hand side of (2.3.1). Next, let us define $R_{ij}(x,y) = A_{ij}(x,y) - B_{ij}(x,y)$ for all $x,y \in R$. Since $R_{ij}(x,y) = 0$ when either $i = 0$ or $j = 0$, and therefore (2.3.1) can be written as

$$\sum_{i=1}^{n} R_{i1}(x,y) + \sum_{i=1}^{n} R_{i2}(x,y) + \ldots + \sum_{i=1}^{n} R_{in}(x,y) = 0 \text{ for all } x,y \in R. \quad (2.3.2)$$

Substituting $qy$ for $y$ in (2.3.2), where $q = 1, 2, \ldots, n$, we get

$$q \sum_{i=1}^{n} R_{i1}(x,y) + q^2 \sum_{i=1}^{n} R_{i2}(x,y) + \ldots + q^n \sum_{i=1}^{n} R_{in}(x,y) = 0 \text{ for all } x,y \in R, \quad (2.3.3)$$

which is a homogeneous system of linear equations with $n$ variables $\sum_{i=1}^{n} R_{i1}(x,y)$, $\sum_{i=1}^{n} R_{i2}(x,y), \ldots, \sum_{i=1}^{n} R_{in}(x,y)$. Thus in view of Lemma 1.3.12, we obtain $\sum_{i=1}^{n} R_{ij}(x,y) = 0$ for $j = 1, 2, \ldots, n$. For fixed $j$, we can write

$$R_{1j}(x,y) + R_{2j}(x,y) + \ldots + R_{nj}(x,y) = 0 \text{ for all } x,y \in R. \quad (2.3.4)$$

Replacing $x$ by $px$ in (2.3.4), where $p = 1, 2, \ldots, n$, we obtain

$$pR_{1j}(x,y) + p^2R_{2j}(x,y) + \ldots + p^nR_{nj}(x,y) = 0 \text{ for all } x,y \in R. \quad (2.3.5)$$

Application of Lemma 1.3.12 yields that $R_{ij}(x,y) = 0$ for all $i = 1, 2, \ldots, n$. In particular, we have $R_{11}(x,y) = 0$ and hence $A_{11}(x,y) = B_{11}(x,y)$ for all $x,y \in R$. Thus, we have $A_{11}(x,y) = F(lxy) + F(myx)$ for some positive integers $l$ and $m$. Here we want to find the coefficient of $xy$ and $yx$. It can be observed that the term $xy$ appears in $(xy + x + y + 1)^n$ in two ways. First $xy$ can be found by multiplying $xy$ from one factor and $1$ from the others. Since each factor contains $xy$, so we have to choose $1$ factor from $n$ factors and this can be done in $\binom{n}{1}$ ways. Next, we can find $xy$ by multiply $x$ from one factor with $y$ from another factor to the right of the factor containing $x$ and choose $1$ from rest of the factors. Since each factor contains $x$ and $y$, therefore we have to choose two factors from $n$ factors to get $xy$. The number of ways doing this is $\binom{n}{2}$. Similarly, the number
of ways to get \( yx \) is \( \binom{n}{2} \). Therefore, we have

\[
\ell = \binom{n}{1} + \binom{n}{2} = \binom{n+1}{2} = \frac{(n+1)n}{2} \quad \text{and} \quad m = \binom{n}{2} = \frac{n(n-1)}{2}.
\]

Since \( A_{11}(x,y) = B_{11}(x,y) \), so we have

\[
n^2F(xy) = \frac{(n+1)n}{2}F(xy) + \frac{(n-1)n}{2}F(yx) \quad \text{for all} \quad x,y \in R,
\]

which simplifies to \( (n-1)nF(xy) = (n-1)nF(yx) \) for all \( x,y \in R \). Since \( R \) is \( (n-1)n \)-torsion free, it follows that \( F(xy - yx) = 0 \) for all \( x,y \in R \) that is, \( F(xy) = F(yx) \) for all \( x,y \in R \). In view of Theorem 2.2.1, we conclude that \( [d(x), x] = 0 \) for all \( x \in R \). Moreover, if \( R \) is prime and \( d \) is a nonzero derivation, then by Lemma 1.3.29, \( R \) is commutative. This proves the theorem.

In view of Lemma 1.3.10 and Theorem 2.3.1, we can derive the following:

**Corollary 2.3.1.** Let \( R \) be an \( n! \)-torsion free prime ring with identity, where \( n \geq 2 \) is a fixed integer. If \( R \) admits a generalized derivation \( F \) with an associated nonzero derivation \( d \) such that \( F((xy)^n) = F(x^ny^n) \) for all \( x,y \in R \), then \( R \) is commutative.

**Theorem 2.3.2.** Let \( R \) be a \( (m \lor n)! \)-torsion free semiprime ring with identity 1, where \( m \) and \( n \) are positive integers and \( F,d : R \to R \) be additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x,y \in R \). If \( F(x^ny^n) = F(y^nx^m) \) for all \( x,y \in R \), then \( [d(x), x] = 0 \) for all \( x \in R \). Moreover, if \( R \) is prime and \( d \) is a nonzero derivation of \( R \), then \( R \) is commutative.

**Proof.** By the hypothesis, we have

\[
F([x^m, y^n]) = 0 \quad \text{for all} \quad x,y \in R. \quad (2.3.6)
\]

Replacing \( x \) by \( 1 + x \) in (2.3.6) and using it, we get

\[
\binom{m}{1} F([x, y^n]) + \binom{m}{2} F([x^2, y^n]) + \ldots + \binom{m}{m-1} F([x^{m-1}, y^n]) = 0 \quad \text{for all} \quad x,y \in R. \quad (2.3.7)
\]
Putting $x = px$ in (2.3.7), where $p = 1, 2, \ldots, n$, we obtain
\[
p \left( \begin{array}{c} m \\ 1 \end{array} \right) F([x, y^n]) + p^2 \left( \begin{array}{c} m \\ 2 \end{array} \right) F([x^2, y^n]) + \ldots + p^{m-1} \left( \begin{array}{c} m \\ m-1 \end{array} \right) F([x^{m-1}, y^n]) = 0
\]
for all $x, y \in R$. Application of Lemma 1.3.12 forces that \( \left( \begin{array}{c} m \\ r \end{array} \right) F([x^r, y^n]) = 0 \) for all $r = 1, 2, \ldots, m - 1$. In particular, for $r = 1$, we have $mF([x, y^n]) = 0$ for all $x, y \in R$. Since $R$ is $m$-torsion free, the last expression yields that
\[
F([x, y^n]) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.3.8}
\]
Replacing $y$ by $1 + y$ in (2.3.8) and using it, we find that
\[
\left( \begin{array}{c} n \\ 1 \end{array} \right) F([x, y]) + \left( \begin{array}{c} n \\ 2 \end{array} \right) F([x, y^2]) + \ldots + \left( \begin{array}{c} n \\ n-1 \end{array} \right) F([x, y^{n-1}]) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.3.9}
\]
Using the same argument as above, we find that \( \left( \begin{array}{c} n \\ s \end{array} \right) F([x, y^s]) = 0 \) for all $s = 1, 2, \ldots, n - 1$. For $s = 1$, we obtain $nF([x, y]) = 0$ for all $x, y \in R$. Since $R$ is $n$-torsion free, the last relation gives that $F([x, y]) = 0$ for all $x, y \in R$. Thus by Theorem 2.2.1, we conclude that $[d(x), x] = 0$ for all $x \in R$. Further, if $R$ is prime and $d$ is a nonzero derivation, then by Lemma 1.3.29, $R$ is commutative.

Following are immediate consequences of Theorem 2.3.2:

**Corollary 2.3.2.** Let $R$ be an $n!$-torsion free semiprime ring with identity, where $n$ is a positive integer and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F(x^n y^n) = F(y^n x^n)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.

**Corollary 2.3.3.** Let $R$ be a $(m \land n)!$-torsion free prime ring with identity, where $m$ and $n$ are positive integers. If $R$ admits a generalized derivation $F$ with an associated nonzero derivation $d$ such that $F(x^n y^n) = F(y^n x^n)$ for all $x, y \in R$, then $R$ is commutative.

**Theorem 2.3.3.** Let $R$ be an $n!$-torsion free semiprime ring with identity $I$, where $n > 1$ is a fixed integer and let $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F([x^n, y^n]) = F([x, y^n])$ holds for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$. Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.
Proof. By assumption, we have

\[ F([x^n, y]) = F([x, y^n]) \quad \text{for all } x, y \in R. \tag{2.3.10} \]

Replacing \( x \) by \( x + 1 \) in (2.3.10) and using it, we get

\[
\binom{n}{1} F([x, y]) + \binom{n}{2} F([x^2, y]) + \cdots + \binom{n}{n-1} F([x^{n-1}, y]) = 0 \quad \text{for all } x, y \in R.
\]

Now, using the parallel arguments as we have used in the proof of Theorem 2.3.2, we find that \( \binom{n}{r} F([x^r, y]) = 0 \) for all \( r = 1, 2, \ldots, n-1 \). In particular, for \( r = 1 \), we obtain \( nF([x, y]) = 0 \) for all \( x, y \in R \). Since \( R \) is \( n \)-torsion free, the last relation implies that \( F([x, y]) = 0 \) for all \( x, y \in R \). Thus by Theorem 2.2.1, we find that \( [d(x), x] = 0 \) for all \( x \in R \). Further, if \( R \) is prime and \( d \) is a nonzero derivation, then by Lemma 1.3.29, \( R \) is commutative.

In view of above theorem and Lemma 1.3.10, we have the following:

**Corollary 2.3.4.** Let \( R \) be an \( n \)-torsion free prime ring with identity, where \( n > 1 \) is a fixed integer. If \( R \) admits a generalized derivation \( F \) with an associated nonzero derivation \( d \) such that \( F([x^n, y]) = F([x, y^n]) \) for all \( x, y \in R \), then \( R \) is commutative.

The next theorem is motivated by Lemma 1.3.4.

**Theorem 2.3.4.** Let \( R \) be an \( n \)-torsion free ring with identity 1, where \( n > 1 \) is a fixed integer. Further, let \( F, d : R \to R \) be additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). If the identity is in the image of \( F \) and \( (F(x)F(y))^n = (F(y)F(x))^n \) for all \( x, y \in R \), then \( [F(x), F(y)] = 0 \) for all \( x, y \in R \). Moreover, if \( R \) is prime and \( d \) is a nonzero derivation of \( R \), then one of the following hold:

(i) \( R \) is commutative;

(ii) \( R \) is a noncommutative subring of a division ring \( D \), and there exists \( a \in D \) such that \( F(x) = ax + xa \) for all \( x \in R \);

(iii) \( R \) is a noncommutative subring of a \( 2 \times 2 \) total matrix ring \( M \) over a field, and there exists \( m \in M \) such that \( F(x) = mx + xm \) for all \( x \in R \).

**Proof.** Since identity is in the image of \( F \), there is an element \( x_0 \in R \) such that \( F(x_0) = 1 \). We assume that \( (F(x)F(y))^n = (F(y)F(x))^n \) for all \( x, y \in R \). Replacing \( x \) by \( x + x_0 \),
and \( y \) by \( y + x_0 \) in the last relation, we get

\[
(F(x + x_0)F(y + x_0))^n = (F(y + x_0)F(x + x_0))^n \text{ for all } x, y \in R.
\]

This implies that

\[
((F(x) + 1)(F(y) + 1))^n = ((F(y) + 1)(F(x) + 1))^n \text{ for all } x, y \in R.
\]

That is, \((F(x)F(y) + F(x) + F(y) + 1)^n = (F(y)F(x) + F(y) + F(x) + 1)^n\) for all \( x, y \in R \). Expand both sides of the above expression, define \( A_{ij}(x, y) \), \( B_{ij}(x, y) \) and \( R_{ij}(x, y) \) in a way similar to Theorem 2.3.1. By the same argument as in Theorem 2.3.1, we conclude that \( R_{ij}(x, y) = 0 \) for all \( 1 \leq i, j \leq n \) and in particular, \( R_{11}(x, y) = 0 \), so that \( A_{11}(x, y) = B_{11}(x, y) \) for all \( x, y \in R \). Thus, we have

\[
A_{11}(x, y) = lF(x)F(y) + mF(y)F(x) \quad (2.3.11)
\]

where \( l \) and \( m \) are positive integers. Using the same arguments as we have used in the proof of Theorem 2.3.1, we obtain \( l = m + n \) and since the right hand side of \((F(x)F(y) + F(x) + F(y) + 1)^n = (F(y)F(x) + F(y) + F(x) + 1)^n\) can be formed from the left by reversing the roles of \( F(x) \) and \( F(y) \), it follows that

\[
B_{11}(x, y) = A_{11}(x, y) = mF(x)F(y) + lF(y)F(x) \text{ for all } x, y \in R. \quad (2.3.12)
\]

From (2.3.11) and (2.3.12), we conclude that \((l - m)(F(x)F(y) - F(y)F(x)) = 0 \) for all \( x, y \in R \). Since \( l - m = n \), and \( R \) is \( n \)-torsion free, it follows that \( F(x)F(y) - F(y)F(x) = 0 \) that is, \([F(x), F(y)] = 0 \) for all \( x, y \in R \). Further, if \( R \) is prime ring and \( d \) is a nonzero derivation of \( R \), then in view of Lemma 1.3.4, we get the required result. \( \square \)

As an immediate corollary we have the prime case of Theorem 2.3.4.

**Corollary 2.3.5.** Let \( R \) be an \( n \! \text{-} \text{torsion free} \) prime ring with identity, where \( n > 1 \) is a fixed integer, and let \( F \) be a generalized derivation with an associated nonzero derivation \( d \). If the identity is in the image of \( F \) and \((F(x)F(y))^n = (F(y)F(x))^n \) holds for all \( x, y \in R \), then one of the following hold:

(i) \( R \) is commutative;

(ii) \( R \) is a noncommutative subring of a division ring \( D \), and there exists \( a \in D \) such that \( F(x) = ax + xa \) for all \( x \in R \);
(iii) \( R \) is a noncommutative subring of a \( 2 \times 2 \) total matrix ring \( M \) over a field and there exists \( m \in M \) such that \( F(x) = mx + xm \) for all \( x \in R \).

### 2.4 The condition \( F(x)F(y) \pm xy \in Z(R) \)

Over the last some decades, several authors have explored various identities involving automorphisms, derivations and multipliers on an appropriate subset of a prime and semiprime ring (viz.: [19], [28], [50], [96], [137], [140], [141] and [176], where further references can be found). In [27], Ashraf and Rehman proved that a prime ring \( R \) must be commutative if it admits a derivation \( d : R \to R \) such that \( d(x)d(y) \pm xy \in Z(R) \) for all \( x, y \in I \), the nonzero ideal of \( R \). This result was further explored by many authors in various directions (see for instance [30], [137]).

In this section, we continue our study in the similar direction and obtain rather a more general result in the setting of semiprime rings. In fact, the result which we want to refer states as follows:

**Theorem 2.4.1.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \), and let \( F, d : R \to R \) be two additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). If \( F(x)F(y) \pm xy \in Z(R) \) for all \( x, y \in I \), then \([d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation such that \( d(I) \neq (0) \), then \( R \) contains a nonzero central ideal.

**Proof.** We begin with the situation

\[
F(x)F(y) - xy \in Z(R) \quad \text{for all } x, y \in I. \tag{2.4.1}
\]

Replacing \( y \) with \( yz \), \( z \in I \), we have

\[
F(x)F(yz) - x(yz) \in Z(R) \tag{2.4.2}
\]

which gives

\[
F(x)(F(y)z + yd(z)) - xyz \in Z(R) \quad \text{for all } x, y, z \in I. \tag{2.4.3}
\]

Commuting both sides of (2.4.3) with \( z \) and then using (2.4.1), we get

\[
[F(x)yd(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{2.4.4}
\]
Putting \( y = zy \) in above relation we obtain
\[
[F(x)zyd(z), z] = 0 \text{ for all } x, y, z \in I. \tag{2.4.5}
\]
Now putting \( x = xz \) in (2.4.4), we get
\[
[F(x)zyd(z), z] + [xd(z)yd(z), z] = 0 \text{ for all } x, y, z \in I. \tag{2.4.6}
\]
In view of (2.4.5), above relation reduces to
\[
[xd(z)yd(z), z] = 0 \text{ for all } x, y, z \in I. \tag{2.4.7}
\]
Substituting \( d(z)x \) for \( x \) in (2.4.7) and using it, we obtain
\[
[d(z), z]xd(z)yd(z) = 0 \text{ for all } x, y, z \in I. \tag{2.4.8}
\]
This implies \( [d(z), z]x[d(z), z]y[d(z), z] = 0 \) that is, \( (I[d(z), z])^3 = (0) \) for all \( z \in I \). Since \( R \) is semiprime, it contains no nonzero nilpotent left ideals, implying \( I[d(z), z] = (0) \) for all \( z \in I \). Thus, \( [d(z), z] \in \text{ann}_R(I) \) for all \( z \in I \). On the other hand, since \( I \) is an ideal of \( R \), \( [d(z), z] \in I \) for all \( z \in I \). Thus \( [d(z), z] \in I \cap \text{ann}_R(I) \) for all \( z \in I \). In view of Lemma 1.3.19, \( [d(z), z] = 0 \) for all \( z \in I \). Moreover, if \( d \) is a derivation such that \( d(I) \neq (0) \), then by Lemma 1.3.3, \( R \) contains a nonzero central ideal.

By the same argument, we obtain the same conclusion in case \( F(x)F(y) + xy \in Z(R) \) for all \( x, y \in I \). This completes the proof of theorem. \( \square \)

Following corollaries are the immediate consequences of the above theorem:

**Corollary 2.4.1** ([30, Theorem 2.5]). Let \( R \) be a prime ring and \( I \) a nonzero ideal of \( R \). If \( R \) admits a generalized derivation \( F \) associated with a nonzero derivation \( d \) such that \( F(x)F(y) \pm xy \in Z(R) \) for all \( x, y \in I \), then \( R \) is commutative.

**Corollary 2.4.2.** Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \), and let \( F, d : R \to R \) be two additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). If \( F(x)F(y) \pm xy \in Z(R) \) for all \( x, y \in I \), then \( [d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation, then one of the following holds:

(i) \( d = 0 \) and \( F(x) = \lambda x + \zeta(x) \) for all \( x \in I \), where \( \lambda \in C \) and \( \zeta : I \to C \) is a left multiplier mapping.

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(ii) $R$ is commutative.

Proof. By Theorem 2.4.1, we obtain $[d(x), x] = 0$ for all $x \in I$. If $d$ is a derivation, then by Lemma 1.3.3, either $d = 0$ or $R$ is commutative. If $R$ is commutative, we obtain our conclusion (ii). So we assume that $R$ is noncommutative. Then $d = 0$. In this case, for all $r, s \in R$, we get $F(rs) = F(r)s + rd(s) = F(r)s$. In other words, $F$ is a left multiplier on $R$. We have

$$F(x)F(y) \pm xy \in Z(R) \text{ for all } x, y \in I.$$  \hfill (2.4.9)

Replacing $y$ with $yz$, and using the fact that $F$ is left multiplier on $R$, we find that

$$(F(x)F(y) \pm xy)z \in Z(R) \text{ for all } x, y, z \in I.$$  \hfill (2.4.10)

If for some $x, y \in I$, $0 \neq F(x)F(y) \pm xy \in Z(R)$, then by Lemma 1.3.28, we get $I \subseteq Z(R)$. Hence, $R$ is commutative by Lemma 1.3.13(b), a contradiction. Thus we are forced to conclude that

$$F(x)F(y) \pm xy = 0 \text{ for all } x, y \in I.$$  \hfill (2.4.11)

Replacing $x$ by $xy$ in (2.4.11), we get

$$F(x)yF(y) \pm xy^2 = 0 \text{ for all } x, y \in I.$$  \hfill (2.4.12)

Right multiplying (2.4.11) by $y$ and then subtracting from (2.4.12), we have $F(x)[F(y), y]$ = 0 for all $x, y \in I$. Putting $xz$ for $x$ in the last relation, we obtain $F(x)z[F(y), y] = 0$ for all $x, y, z \in I$. This implies $[F(x), x]z[F(x), x] = 0$ for all $x, z \in I$ that is, $[F(x), x]I[F(x), x] = 0$ for all $x \in I$. Since $R$ is prime, it follows that $[F(x), x] = 0$ for all $x \in I$. Then by Lemma 1.3.24, $F(x) = \lambda x + \zeta(x)$ for all $x \in I$, where $\lambda \in C$ and $\zeta : I \to C$. Since $F$ is additive, $\zeta$ is also additive mapping. Moreover, since $F$ is left multiplier, so the mapping $\zeta(x) = F(x) - \lambda x$ is also a left multiplier on $I$. This implies that $F(x) = \lambda x + \zeta(x)$ for all $x \in I$. Hence, we obtain our conclusion (i). \hfill $\square$

Theorem 2.4.2. Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$, and let $F, d : R \to R$ be two additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$. Moreover, if $d$ is a derivation such that $d(I) \neq (0)$, then $R$ contains a nonzero central ideal.
Proof. First we consider the case

\[ F(x)F(y) - yx \in Z(R) \text{ for all } x, y \in I. \]  

(2.4.13)

Replacing \( y \) by \( yz \), we get

\[ F(x)(F(y)z + yd(z)) - yzx \in Z(R) \text{ for all } x, y, z \in I. \]  

(2.4.14)

This implies that

\[ (F(x)F(y) - yx)z + y[x, z] + F(x)yd(z) \in Z(R) \text{ for all } x, y, z \in I. \]  

(2.4.15)

Application of (2.4.13) yields that

\[ [y[x, z], z] + [F(x)yd(z), z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.16)

Substituting \( xz \) for \( x \) in (2.4.16), we obtain

\[ [y[x, z], z]z + [(F(x)z + xd(z))yd(z), z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.17)

Putting \( y = zy \) in (2.4.16), we get

\[ z[y[x, z], z] + [F(x)zyd(z), z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.18)

Subtracting (2.4.18) from (2.4.17), we have

\[ [[y[x, z], z], z] + [xd(z)yd(z), z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.19)

Replacing \( x \) by \( xz \) in (2.4.19), we obtain

\[ [[y[x, z], z], z]z + [xz(z)yd(z), z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.20)

Right multiplying (2.4.19) by \( z \) and then subtracting from (2.4.20), we get

\[ [x[d(z)yd(z), z], z] = 0 \text{ for all } x, y, z \in I. \]  

(2.4.21)
Now, we substitute $d(z)yd(z)x$ for $x$ in (2.4.21) and get

\[
0 = [d(z)yd(z)x[d(z)yd(z), z], z] = d(z)yd(z)[x[d(z)yd(z), z], z] + [d(z)yd(z), z]x[d(z)yd(z), z].
\]  

(2.4.22)

By using (2.4.21), it reduces to

\[
[d(z)yd(z), z]x[d(z)yd(z), z] = 0 \text{ for all } x, y, z \in I.
\]  

(2.4.23)

Since $I$ is an ideal, it follows that $x[d(z)yd(z), z]Rx[d(z)yd(z), z] = (0)$ and hence

\[
x[d(z)yd(z), z] = 0 \text{ for all } x, y, z \in I
\]  

(2.4.24)

that is,

\[
x\{d(z)yd(z)z - zd(z)yd(z)\} = 0 \text{ for all } x, y, z \in I.
\]  

(2.4.25)

Now we put $y = yd(z)u$ and then obtain

\[
x\{d(z)yd(z)ud(z)z - zd(z)yd(z)ud(z)\} = 0 \text{ for all } x, y, z, u \in I.
\]  

(2.4.26)

By (2.4.25), this can be written as

\[
x\{d(z)yzd(z)ud(z) - d(z)yd(z)zd(z)\} = 0 \text{ for all } x, y, z, u \in I
\]  

(2.4.27)

that is,

\[
xd(z)y[d(z), z]ud(z) = 0 \text{ for all } x, y, z, u \in I.
\]  

(2.4.28)

This implies $x[d(z), z]y[d(z), z]u[d(z), z] = 0$ for all $x, y, z, u \in I$ and so $(I[d(z), z])^3 = (0)$ for all $z \in I$. Since semiprime ring contains no nonzero nilpotent left ideals, it follows that $I[d(z), z] = (0)$ for all $z \in I$. Therefore, for all $x \in I$ we have $[d(x), x] \in I \cap ann_R(I)$. Since $R$ is semiprime, by Lemma 1.3.19, we conclude that $[d(x), x] = 0$ for all $x \in I$, as desired. Moreover, if $d$ is a derivation such that $d(I) \neq (0)$, then by Lemma 1.3.3, $R$ contains a nonzero central ideal.

In the same manner the conclusion can be obtained in case $F(x)F(y) + yx \in Z(R)$ for all $x, y \in I$. Hence, the theorem is now proved. □

**Corollary 2.4.3** ([30, Theorem 2.6]). Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ such that $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$, then $R$ is commutative.
Corollary 2.4.4. Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and let \( F, d : R \to R \) be two additive mappings such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). If \( F(x)F(y) \pm yx \in Z(R) \) for all \( x, y \in I \), then \([d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation, then \( R \) is commutative.

Proof. In view of Theorem 2.4.2, we have \([d(x), x] = 0 \) for all \( x \in I \). Moreover, if \( d \) is a derivation, then by Lemma 1.3.3, either \( d = 0 \) or \( R \) is commutative. If \( R \) is commutative, then we have done. So, let \( R \) be noncommutative. Then \( d = 0 \). In this case, for any \( r, s \in R \), we have \( F(rs) = F(r)s + rd(s) = F(r)s \). In other words, \( F \) is a left multiplier on \( R \). Replacing \( y \) with \( yx \), we get by our hypothesis that \( (F(x)F(y)x \pm yx)x \in Z(R) \) for all \( x, y \in I \). This implies by Lemma 1.3.28 that for each \( x \in I \), either \( F(x)F(y) \pm yx = 0 \) for all \( y \in I \) or \( x \in Z(R) \). The sets \( x \in I \) for which these two conditions hold are additive subgroups of \( I \) whose union is \( I \), therefore \( F(x)F(y) \pm yx = 0 \) for all \( x, y \in I \) or \( I \subseteq Z(R) \). The last condition implies by Lemma 1.3.13(6) that \( R \) is commutative, a contradiction. Hence we conclude that \( F(x)F(y) \pm yx = 0 \) for all \( x, y \in I \). Now replacing \( y \) with \( yz \), we get \( F(x)[F(y)x, z] = 0 \) for all \( x, y, z \in I \). Substituting \( yF(z) \) for \( y \) in the last relation, we get \( F(x)[F(y)F(z), x] = 0 \) for all \( x, y, z \in I \). Since \( R \) is prime, for each \( x \in I \), either \( F(x) = 0 \) or \( [R, x] = 0 \). Since both of these two conditions form two additive subgroups of \( I \) whose union is \( I \), we conclude that either \( F(I) = 0 \) or \( I \subseteq Z(R) \). The last condition implies that \( R \) is commutative, a contradiction. Hence \( F(I) = 0 \). This case gives \( 0 = F(RI) = F(R)I \), implying \( F(R) = 0 \). In this case, again our hypothesis yields that \( I^2 \subseteq Z(R) \). Since \( I^2 \) is a nonzero central ideal of \( R \), so \( R \) must be commutative by Lemma 1.3.13(6). This completes the proof. \( \square \)

Corollary 2.4.5. Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \), and let \( F : R \to R \) be a generalized derivation of \( R \) induced by a nonzero derivation \( d \). If \( F(x^2) = ax^2 \) for all \( x \in I \), where \( a \in \{0, 1, -1\} \), then \([I, I]d(I) = 0 \). Moreover, if \( R \) is prime, then \( R \) is commutative.

Proof. Linearizing \( F(x^2) = ax^2 \), we get \( F(x^2 + y^2 + xy + yx) = a(x^2 + y^2 + xy + yx) \) for all \( x, y \in I \). By the hypothesis, we obtain \( F(x \circ y) = a(x \circ y) \) for all \( x, y \in I \). Thus, by Lemma 1.3.16, we conclude that \([I, I]d(I) = 0 \). Further, if \( R \) is prime, from the last
relation, we have \([I, I]Id(R) = (0)\). This implies that \([I, I]RId(R) = (0)\). The primeness of \(R\) forces that either \([I, I] = (0)\) or \(Id(R) = (0)\). If \([I, I] = (0)\), then \([I, R]I = (0)\).

Since \(R\) is prime and \(I\) a nonzero ideal of \(R\), the last expression yields that \([I, R] = (0)\) that is, \(I\) is a central ideal of \(R\). Thus, by Lemma 1.3.13(b), \(R\) is commutative. On the other hand, if \(Id(R) = (0)\), then \(IRd(R) = (0)\). Since \(R\) is prime, we conclude that \(d(R) = (0)\), which is a contradiction. This completes the proof. \(\square\)

**Corollary 2.4.6.** Let \(R\) be a 2-torsion free ring with identity 1, and let \(F : R \rightarrow R\) be a generalized derivation induced by \(d\). If \(F(x)F(x) = \pm x^2\) for all \(x \in R\), then there exists \(c \in Z(R)\) such that \(c^2 = \pm 1\) and \(F(x) = cx\) for all \(x \in R\).

**Proof.** Linearization of \(F(x)F(x) = \pm x^2\) gives \(F(x)F(x) + F(y)F(y) + F(x)F(y) + F(y)F(x) = \pm(x^2+y^2+xy+yx)\) for all \(x, y \in R\). Our hypothesis yields that \(F(x)F(y) = \pm(xy)\) for all \(x, y \in R\). Hence, by Lemmas 1.3.5 and 1.3.6, we get the required result \(\square\)

### 2.5 The condition \([x, y]^n = \pm[x^n, y^n]\)

In [43], Bell presented a simple alternate proof of a long standing result due to Herstein [103], which states that a ring \(R\) satisfying \((x + y)^n = x^n + y^n\), where \(n > 1\) is a fixed integer, must have nil commutator ideal and the set of nilpotent elements of \(R\) form an ideal. The proof due to Bell depends on the observations that the rings satisfying \((x + y)^n = x^n + y^n\) also satisfy the identity \([x^n, y] = [x, y^n]\) and (at least in the absence of zero divisors) the identity \([x, y]^n = [x^n, y^n]\). Later in [47, Theorem 5] it was established that if \(R\) is an \(n\)-torsion free ring with identity 1 satisfying \([x^n, y] = [x, y^n]\), then \(R\) is commutative. In the year 1991, Ashraf and Quadri [25] prove that a ring \(R\) with identity must be commutative if there are positive integers \(k, m, n\) with \(m + n > 2\) such that \([x, y]^k = [x^m, y^n]\) for all \(x, y \in R\) and commutators in \(R\) are \(m!n!\)-torsion free. In particular, they proved that a ring \(R\) with identity must be commutative if there is an integer \(n > 1\) such that \([x, y]^n = [x^n, y^n]\) for all \(x, y \in R\) and commutators in \(R\) are \(nl\)-torsion free. In the present section, we investigate the commutativity of ring satisfying similar identities involving arbitrary mappings. The following lemma is essential for developing the proof of the main results in the present section:

**Lemma 2.5.1.** Let \(R\) be a ring with identity 1 in which commutators are \(nl\)-torsion free. If there is a positive integer \(n\) such that \([x, y^n] = 0\) for all \(x, y \in R\), then \(R\) is commutative.
Proof. By the hypothesis, we have

$$[x, y^n] = 0 \text{ for all } x, y \in R.$$  \hspace{1cm} (2.5.1)

Replacing $y$ by $1 + y$ in (2.5.1), we obtain

$$\binom{n}{1} [x, y] + \binom{n}{2} [x, y^2] + \ldots + \binom{n}{n-1} [x, y^{n-1}] = 0 \text{ for all } x, y \in R. \hspace{1cm} (2.5.2)$$

Substituting $qy$ for $y$ in (2.5.2), where $q = 1, 2, \ldots, (n-1)$, we get

$$q \binom{n}{1} [x, y] + q^2 \binom{n}{2} [x, y^2] + \ldots + q^{n-1} \binom{n}{n-1} [x, y^{n-1}] = 0 \text{ for all } x, y \in R. \hspace{1cm} (2.5.3)$$

The above equation produces the system of $(n-1)$ homogeneous equations, the coefficient matrix of this system is Vandermonde matrix

$$\begin{pmatrix}
1 & 1 & \ldots & 1 \\
2 & 2^2 & \ldots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & (n-1)^2 & \ldots & (n-1)^{n-1}
\end{pmatrix}$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n$, it follows that $\binom{n}{r} [x, y^r] = 0$ for all $x, y \in R$ and $r = 1, 2, \ldots, (n-1)$. In particular, for $r = 1$, we have $n[x, y] = 0$ for all $x, y \in R$. Since commutators in $R$ are $n$-torsion free, the last expression implies that $[x, y] = 0$ for all $x, y \in R$. Hence, $R$ is commutative. This proves the lemma. $\Box$

Now, we prove the following theorem:

**Theorem 2.5.1.** Let $R$ be a ring with identity $1$ and $f : R \to R$ be any mapping. Then the following are equivalent:

(i) There is an integer $n > 1$ such that $(f[x, y])^n = \pm [x^n, y^n]$ for all $x, y \in R$ and commutators in $R$ are $n!$-torsion free.

(ii) There is an integer $n > 1$ such that $f([x, y]^n) = \pm [x^n, y^n]$ for all $x, y \in R$ and commutators in $R$ are $n!$-torsion free.
(iii) There are positive integers \( m \) and \( n \) with \( m + n > 2 \) such that \( (f^m[x,y])^n = \pm [x^m, y^n] \) for all \( x, y \in R \) or \( (f^m[x,y])^n = \pm [x^n, y^m] \) for all \( x, y \in R \) and commutators in \( R \) are \((m \lor n)!\)-torsion free.

(iv) There are positive integers \( m \) and \( n \) with \( m + n > 2 \) such that \( f^m([x,y]^n) = \pm [x^m, y^n] \) for all \( x, y \in R \) or \( f^m([x,y]^n) = \pm [x^n, y^m] \) for all \( x, y \in R \) and commutators in \( R \) are \((m \lor n)!\)-torsion free.

(v) \( R \) is commutative.

Proof. It is immediate that commutativity of \( R \) implies each of the conditions (i) through (iv). Now, we show that each of the conditions implies commutativity of \( R \).

(i) \( \Rightarrow \) (v) We assume that

\[
(f[x,y])^n = \pm [x^n, y^n] \text{ for all } x, y \in R. \tag{2.5.4}
\]

Substituting \( x \) by \( 1 + x \) in (2.5.4), we get

\[
(f[x,y])^n = \pm \binom{n}{1} [x, y^n] \pm \binom{n}{2} [x^2, y^n] \pm \cdots \pm \binom{n}{n-1} [x^{n-1}, y^n] \pm [x^n, y^n]
\]

for all \( x, y \in R \). Application of (2.5.4) yields that

\[
\binom{n}{1} [x, y^n] + \binom{n}{2} [x^2, y^n] + \cdots + \binom{n}{n-1} [x^{n-1}, y^n] = 0 \tag{2.5.5}
\]

for all \( x, y \in R \). Replacing \( x \) by \( px \) in (2.5.5), where \( p = 1, 2, \ldots, (n - 1) \), we obtain

\[
p \binom{n}{1} [x, y^n] + p^2 \binom{n}{2} [x^2, y^n] + \cdots + p^{n-1} \binom{n}{n-1} [x^{n-1}, y^n] = 0 \text{ for all } x, y \in R.
\]

This represents a system of homogeneous linear equations with \( n - 1 \) variables \( \binom{n}{1} [x, y^n] \), \( \binom{n}{2} [x^2, y^n] \), \( \cdots \), \( \binom{n}{n-1} [x^{n-1}, y^n] \), the coefficient matrix of this system is Vandermonde.
Since the determinant of the matrix is equal to a product of positive integers, each of which is less than \( n \), it follows that
\[
\left( \begin{array}{cccc}
1 & 1 & \ldots & 1 \\
2 & 2^2 & \ldots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & (n-1)^2 & \ldots & (n-1)^{n-1}
\end{array} \right).
\]

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than \( n \), it follows that \( \det \left( \begin{array}{c} x^r \end{array} \right) = 0 \) for all \( x, y \in R \) and \( r = 1, 2, \ldots, (n - 1) \). In particular, we have \( n[x^n, y^n] = 0 \) for all \( x, y \in R \). Since commutators in \( R \) are \( n \)-torsion free, the last relation yields that \( [x, y^n] = 0 \) for all \( x, y \in R \). Thus, in view of Lemma 2.5.1, we conclude that \( R \) is commutative.

\( (ii) \Rightarrow (v) \) is similar to \( (i) \Rightarrow (v) \).

\( (iii) \Rightarrow (v) \) First we consider the case
\[
([x^n, y^n])^n = \pm [x^n, y^n] \quad \text{for all } x, y \in R. \tag{2.5.6}
\]

Replacing \( x \) by \( 1 + x \) in (2.5.6), we get
\[
\left( \begin{array}{c}
m \\
1
\end{array} \right) [x^n, y^n] + \left( \begin{array}{c}
m \\
2
\end{array} \right) [x^{2^n}, y^n] + \cdots + \left( \begin{array}{c}
m \\
 m - 1
\end{array} \right) [x^{m-1}, y^n] = 0 \quad \text{for all } x, y \in R. \tag{2.5.7}
\]

Using the same arguments as we have used to prove \( (i) \Rightarrow (v) \), we conclude that \( \left( \begin{array}{c} m \\
 r \end{array} \right) [x^r, y^n] = 0 \) for all \( x, y \in R \) and \( r = 1, 2, \ldots, (m - 1) \). In particular, for \( r = 1 \) we have \( m[x, y^n] = 0 \) for all \( x, y \in R \). The last expression implies that \( [x, y^n] = 0 \) for all \( x, y \in R \), since commutators in \( R \) are \( m \)-torsion free. Thus, by Lemma 2.5.1, \( R \) is commutative.

Similar conclusion holds for the case \( (f^m[x, y])^n = \pm [x^n, y^m] \) for all \( x, y \in R \).

\( (iv) \Rightarrow (v) \) Using the parallel arguments as we have used in the proof of \( (iii) \Rightarrow (v) \).

This completes the proof of the theorem. \( \square \)

**Theorem 2.5.2.** Let \( R \) be a ring with identity 1 and \( f, g : R \to R \) be two mappings such that \( f \) is additive and \( f(1) = 1 \). Then the following are equivalent:

\( (i) \) There is an integer \( n > 1 \) such that \( [f(x), g(y)]^n = \pm [x^n, y^n] \) for all \( x, y \in R \), and commutators in \( R \) are \( n! \)-torsion free.

\( (ii) \) There are positive integers \( m \) and \( n \) with \( m + n > 2 \) such that \( [f^m(x), g^n(y)] = \)
\[ \pm [x^n, y^n] \text{ for all } x, y \in R \text{ or } [f^m(x), g^n(y)] = \pm [x^n, y^m] \text{ for all } x, y \in R, \text{ and commutators in } R \text{ are } (m \lor n)\text{-torsion free.} \]

(iii) There are positive integers \( m \) and \( n \) with \( m + n > 2 \) such that \([f^m(x), g^n(y)] = \pm [x^n, y^n] \text{ for all } x, y \in R \text{ or } [f^m(x), g^n(y)] = \pm [x^n, y^n] \text{ for all } x, y \in R, \text{ and commutators in } R \text{ are } (m \lor n)\text{-torsion free.} \]

(iv) \( R \) is commutative.

**Proof.** Clearly, (iv) \( \Rightarrow \) (i), (iv) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (iii). Now, we will prove that (i) \( \Rightarrow \) (iv) By the assumption, we have

\[ [f(x), g(y)]^n = \pm [x^n, y^n] \text{ for all } x, y \in R. \]  

(2.5.8)

Replacing \( x \) by \( 1 + x \) in (2.5.8), we get

\[ [f(1 + x), g(y)]^n = \pm \binom{n}{1} [x, y^n] \pm \binom{n}{2} [x^2, y^n] \pm \cdots \pm \binom{n}{n-1} [x^{n-1}, y^n] \pm [x^n, y^n] \]

for all \( x, y \in R \). Using (2.5.8) and the fact that image of identity is identity under \( f \), we conclude that

\[ \binom{n}{1} [x, y^n] + \binom{n}{2} [x^2, y^n] + \cdots + \binom{n}{n-1} [x^{n-1}, y^n] = 0 \text{ for all } x, y \in R \]

Using parallel arguments as we have used to prove (i) \( \Rightarrow \) (v) in Theorem 2.5.1, we find that \( \binom{n}{r} [x^r, y^n] = 0 \text{ for all } x, y \in R \text{ and } r = 1, 2, \ldots, (n - 1) \). In particular, we have

\[ n[x, y^n] = 0 \text{ for all } x, y \in R. \]

Since commutators in \( R \) are \( n \)-torsion free, last expression yields that \([x, y^n] = 0 \text{ for all } x, y \in R\). Hence, \( R \) is commutative by Lemma 2.5.1.

(ii) \( \Rightarrow \) (iv) First we assume that

\[ [f^m(x), g^n(y)] = \pm [x^m, y^n] \text{ for all } x, y \in R. \]  

(2.5.9)

Replacing \( x \) by \( 1 + x \) in (2.5.9), we obtain

\[ [f^m(1 + x), g^n(y)] = \pm \binom{m}{1} [x, y^n] \pm \binom{m}{2} [x^2, y^n] \pm \cdots \pm \binom{m}{m-1} [x^{m-1}, y^n] \pm [x^m, y^n]. \]

for all \( x, y \in R \). Using (2.5.9) and the fact that image of identity is identity under \( f \), we
get

\[
\binom{m}{1} [x, y^n] + \binom{m}{2} [x^2, y^n] + \ldots + \binom{m}{m-1} [x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.
\]

The above expression is similar to the relation (2.5.7) and henceforth using the same approach as we have used to obtain commutativity of \( R \) from expression (2.5.7) in the proof of Theorem 2.5.1, we get the required result.

Similarly, we can prove the result for the case \([f^m(x), g^n(y)] = \pm [x^n, y^m] \) for all \( x, y \in R \).

\((iii) \Rightarrow (iv)\) It can be proved by using the same techniques with necessary variations. Thereby, the proof is completed.

The next theorem is motivated by [92, Theorem 1].

**Theorem 2.5.3.** Let \( R \) be a ring with identity 1, \( d : R \to R \) be a derivation of \( R \) and \( g \) be any mapping of \( R \). If there are positive integers \( m \) and \( n \) with \( m + n > 2 \) such that \([d(x^m), g(y^n)] = [x^m, y^n] \) for all \( x, y \in R \) and commutators in \( R \) are \((m \lor n)!\)-torsion free, then \( R \) is commutative.

**Proof.** By the assumption, we have

\([d(x^m), g(y^n)] = [x^m, y^n] \) for all \( x, y \in R \). \hspace{1cm} (2.5.10)

Replacing \( x \) by \( 1 + x \) in (2.5.10) and using the fact that \( d(1) = 0 \), we get

\[
\binom{m}{1} [x, y^n] + \binom{m}{2} [x^2, y^n] + \ldots + \binom{m}{m-1} [x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.
\]

The above expression is same as the relation in (2.5.7) and henceforth using the same approach as we have used to obtain commutativity of \( R \) from expression (2.5.7) in the proof of Theorem 2.5.1, we get the required result. This proves the theorem.

\[\square\]

### 2.6 Some examples

We begin this section by the following example which shows that semiprimeness is an essential condition in the hypotheses of Theorem 2.2.1.
Example 2.6.1. Let $S$ be any commutative ring. Next, let

$$R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in S \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix} \mid b, c \in S \right\}.$$

Clearly, $R$ is not semiprime as $R \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = (0)$ for $b \neq 0$. Define mappings $F, d : R \to R$ such that

$$F \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad \text{and} \quad d \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \quad \text{for all} \ a, b, c \in S.$$

Then $R$ is a ring with identity under the natural operations and $I$ is an ideal of $R$. It is straightforward to check that $F$ and $d$ satisfy the requirements of Theorem 2.2.1, but $[d(x), x] \neq 0$ for all $x \in I$. Hence, in Theorem 2.2.1, the hypothesis of semiprimeness is crucial.

Example 2.6.2. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Clearly, $R$ is a ring with identity which is not semiprime. Define $F, d : R \to R$ such that

$$F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \quad \text{for all} \ a, b, c \in \mathbb{Z}.$$

Then it is easy to see that the mappings $F$ and $d$ are additive such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Also $F(xy) = F(x)F(y) = F(y)F(x)$ for all $x, y \in R$. Further, for any $x, y \in R$ the following conditions: $F([x, y]^2) = F([x^2, y^2])$, $F((xy)^n) = F(x^n y^n)$ and $F(x^n y^n) = F(y^n x^n)$ are satisfied, where $m$ and $n$ are positive integers. However, $[d(x), x] \neq 0$ for all $x \in R$. Hence, in Theorems 2.2.7, 2.3.1 and 2.3.2, the hypothesis of semiprimeness can not be omitted.

Our next example shows that the identity element is necessary in Lemma 2.5.1 and Theorem 2.5.1.

Example 2.6.3. Let $S$ be any noncommutative ring and

$$R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}.$$
Obviously, $R$ is a ring without identity. Also, it can be easily seen that for any integer $n > 1$, the identity $[x, y^n] = 0$ holds for all $x, y \in R$, but $R$ is not commutative. Hence, in Lemma 2.5.1 identity element is necessary. Further, define a mapping $f : R \to R$ such that

$$f \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$$

for all $a, b \in S$.

It is easy to see that $f$ satisfies all the requirements of Theorem 2.5.1. However, $R$ is not commutative.

**Example 2.6.4.** Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}.$$

Clearly, $R$ is a ring without identity. Consider the mappings $f, g : R \to R$ such that

$$f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

for all $a, b, c \in \mathbb{Z}$.

It is straightforward to check that $f$ and $g$ satisfy all the requirements of Theorem 2.5.2, but $R$ is not commutative.