Research publications
COMMUTATIVITY OF RINGS INVOLVING ADDITIVE MAPPINGS

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ABSTRACT. Let \( R \) be an associative ring. In the present paper, we investigate commutativity of a ring admitting an additive mapping \( F \) satisfying any one of the following properties: (i) \( F([x,y]) = F([x^2,y^2]) \), (ii) \( F((x \circ y)^3) = F(x^3 \circ y^3) \), (iii) \( F((xy)^n) = F(x^n y^n) \), (iv) \( F(x^n y^n) = F(y^n x^n) \), (v) \( (F(x)F(y))^n = F(y)^n F(x)^n \) for all \( x, y \in R \), where \( m \) and \( n \) are positive integers greater than 1. Moreover, some related results are also discussed. Finally, some examples are given to demonstrate that the restrictions imposed on the hypotheses of the various results are not superfluous.

Mathematics Subject Classification (2010): 16W25, 16N60, 16U80.
Key words: Prime ring, semiprime ring, ideal, additive mapping, commuting mapping, derivation, generalized derivation.

1. Introduction. This research has been motivated by the work of Herstein [25]. Let \( R \) be an associative ring with center \( Z(R) \). For \( x, y \in R \), the symbol \([x,y]\) will denote the commutator \( xy - yx \) and the symbol \( x \circ y \) will denote the anti-commutator \( xy + yx \). A ring \( R \) is called \( n \)-torsion free, if \( nx = 0, x \in R \), implies \( x = 0 \). The least positive integer \( n \) such that \( nx = 0 \) for all \( x \in R \) is called the...
A note on commutativity of rings with additive mappings

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Abstract. We investigate commutativity of the ring \( R \) involving some additive mapping with necessary torsion restrictions on commutators. We give counter examples which show that the hypotheses of our theorems are not superfluous.

Mathematical subject classification: 47B47; 16U80

Keywords: Additive mapping; Commutativity; Commutator

1. INTRODUCTION

This research is inspired by the work of Ashraf and Quadri [1,2]. Throughout this paper \( R \) will denote an associative ring with the identity 1. A ring \( R \) is said to be \( n \)-torsion free if \( nx = 0 \) implies \( x = 0 \) for all \( x \in R \). For any \( x,y \in R \), the symbol \([x,y]\) will denote the commutator \( xy - yx \). An additive mapping \( d:R \to R \) is said to be a derivation of \( R \) if \( d(xy) = d(x)y + xd(y) \) holds for all \( x,y \in R \). We say that a map \( f:R \to R \) preserves commutativity if \([f(x),f(y)] = 0 \) whenever \([x,y] = 0 \) for \( x,y \in R \). In [3], Bell and Daif investigated a certain kind of commutativity preserving maps as follows: Let \( S \) be a subset of \( R \). A map \( f:S \to R \) is called strong commutativity preserving (SCP) on \( S \) if \([f(x),f(y)] = [x,y] \) for all \( x,y \in S \). Precisely, they proved that if a semiprime ring \( R \) admits a derivation which is SCP on a right ideal \( \rho \), then \( \rho \subseteq Z(R) \). In particular, \( R \) is

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ON ORTHOGONAL \((\sigma, \tau)\)-DERIVATIONS IN SEMIPRIME \(\Gamma\)-RINGS

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ABSTRACT. Let \(M\) be a \(\Gamma\)-ring and \(\sigma, \tau\) be endomorphisms of \(M\). An additive mapping \(d : M \rightarrow M\) is called a \((\sigma, \tau)\)-derivation if \(d(xay) = d(x)a\sigma(y) + \tau(x)a\tau(y)\) holds for all \(x, y \in M\) and \(a \in \Gamma\). An additive mapping \(F : M \rightarrow M\) is called a generalized \((\sigma, \tau)\)-derivation if there exists a \((\sigma, \tau)\)-derivation \(d : M \rightarrow M\) such that \(F(xay) = F(x)a\sigma(y) + \tau(x)a\tau(y)\) holds for all \(x, y \in M\) and \(a \in \Gamma\). In this paper, some known results on orthogonal derivations and orthogonal generalized derivations of semiprime \(\Gamma\)-rings are extended to orthogonal \((\sigma, \tau)\)-derivations and orthogonal generalized \((\sigma, \tau)\)-derivations. Moreover, we present some examples which demonstrate that the restrictions imposed on the hypotheses of some of our results are not superfluous.

Mathematics Subject Classification (2010): 16W25, 16N60

Keywords: semiprime \(\Gamma\)-ring, derivation, orthogonal derivation, orthogonal \((\sigma, \tau)\)-derivation, orthogonal generalized derivation, orthogonal generalized \((\sigma, \tau)\)-derivation

1. Introduction

The study of \(\Gamma\)-ring goes back to Nobusawa [10] and further generalized by Barnes [6]. Following [6], a \(\Gamma\)-ring is a pair \((M, \Gamma)\), where \(M\) and \(\Gamma\) are additive abelian groups for which there exists a map from \(M \times \Gamma \times M \rightarrow M\) (the image of \((a, \gamma, b)\) will be denoted by \(a\gamma b\) for all \(a, b \in M\) and \(\gamma \in \Gamma\)) satisfying (i) \((a + b)\gamma c = a\gamma c + b\gamma c\), (ii) \(a(\alpha + \beta)b = a\alpha b + a\beta b\), (iii) \(a(a + b) = aab + aac\) and (iv) \((ab)\beta c = a(a(b\beta c))\) for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\). A \(\Gamma\)-ring \(M\) is said to be prime if \(x\Gamma\Gamma y = \{0\}\) implies \(x = 0\) or \(y = 0\) and \(M\) is said to be semiprime if \(x\Gamma\Gamma x = \{0\}\) implies \(x = 0\). \(M\) is said to be 2-torsionfree if \(2x = 0\) implies \(x = 0\) for all \(x \in M\). For any \(x, y \in M\) and \(\alpha \in \Gamma\), the symbol \([x, y]_\alpha\) stands for the commutator \(xay - yax\). If \(x\alpha yz = x\beta yaz\) holds for all \(x, y, z \in M\) and \(\alpha, \beta \in \Gamma\),

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On *-bimultipliers, Generalized *-biderivations and Related Mappings

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ABSTRACT. In this paper we define the notions of left *-bimultiplier, *-bimultiplier and generalized *-biderivation, and to prove that if a semiprime *-ring admits a left *-bimultiplier M, then M maps $R \times R$ into $Z(R)$. In Section 3, we discuss the applications of theory of *-bimultipliers. Further, it was shown that if a semiprime *-ring $R$ admits a symmetric generalized *-biderivation $G : R \times R \rightarrow R$ with an associated nonzero symmetric *-biderivation $B : R \times R \rightarrow R$, then $G$ maps $R \times R$ into $Z(R)$. As an application, we establish corresponding results in the setting of $C^*$-algebra.

1. Introduction

Throughout the discussion, unless otherwise mentioned, $R$ will denote an associative ring with center $Z(R)$, and $A$ will represent a $C^*$-algebra. However, $A$ may not have unity with center $Z(A)$. For any $x, y \in A$, the symbol $[x, y]$ (resp. $x \circ y$) will denote the commutator $xy - yx$ (resp. the anti-commutator $xy + yx$). Recall that an algebra $A$ is prime if $xAy = \{0\}$ implies $x = 0$ or $y = 0$, and $A$ is semiprime if $xAx = \{0\}$ implies $x = 0$. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with norm $\| \cdot \|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all $x$ and $y$ in $A$. An additive mapping $x \mapsto x^*$ of $A$ into itself is called an involution if the following conditions are satisfied: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(\lambda x)^* = \bar{\lambda}x^*$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$, where $\bar{\lambda}$ is the conjugate of $\lambda$. An algebra (ring) equipped with an involution is called a *-algebra (ring) or algebra with involution (ring with involution). A $C^*$-algebra $A$ is a Banach *-algebra with the additional norm condition $\|x^*x\| = \|x\|^2$ for all $x \in A$.

Let $S$ be a nonempty subset of $R$. A function $f : R \rightarrow R$ is said to be centralizing

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On Generalized Jordan Triple
\((\alpha, \beta)^\ast\)-Derivations and Related Mappings

Shakir Ali, Ajda Fošner, Maja Fošner* and
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Abstract. Let \( R \) be a 2-torsion free semiprime \(*\)-ring and let \( \alpha, \beta \) be surjective endomorphisms of \( R \). The aim of the paper is to show that every generalized Jordan triple \((\alpha, \beta)^\ast\)-derivation on \( R \) is a generalized Jordan \((\alpha, \beta)^\ast\)-derivation. This result makes it possible to prove that every generalized Jordan triple \((\alpha, \beta)^\ast\)-derivation on a semisimple \(H^\ast\)-algebra is a generalized Jordan \((\alpha, \beta)^\ast\)-derivation. Finally, we prove that every Jordan triple left \(\alpha^\ast\)-centralizer on a 2-torsion free semiprime ring is a Jordan left \(\alpha^\ast\)-centralizer.

Mathematics Subject Classification (2010). 16N60, 16W10, 16W25.

Keywords. Semiprime \(*\)-ring, \(H^\ast\)-algebra, Jordan triple \((\alpha, \beta)^\ast\)-derivation, generalized Jordan triple \((\alpha, \beta)^\ast\)-derivation, Jordan triple left \(\alpha^\ast\)-centralizer.

1. Introduction

Throughout the paper, \( R \) will represent an associative ring. Let \( n \geq 2 \) be an integer. A ring \( R \) is said to be \( n \)-torsion free if for \( x \in R \), \( nx = 0 \) implies \( x = 0 \). Recall that \( R \) is prime if \( aRb = \{0\} \) implies \( a = 0 \) or \( b = 0 \). A ring \( R \) is called semiprime if \( aRa = \{0\} \) implies \( a = 0 \). An additive mapping \( x \mapsto x^\ast \) satisfying \((xy)^\ast = y^\ast x^\ast \) and \((x^\ast)^\ast = x \) for all \( x, y \in R \) is called an involution. A ring equipped with an involution \(*\) is called a \(*\)-ring or ring with involution. If \( R \) is an algebra we assume additionally that \((\lambda x)^\ast = \bar{\lambda} x^\ast \) for all \( x \in R \) and \( \lambda \in F \), where \( \bar{\lambda} \) denotes the complex conjugate of \( \lambda \). An algebra equipped with an involution is called a \(*\)-algebra or algebra with involution. The radical of \( A \), denoted by \( \text{rad}(A) \), is the intersection of all maximal left (or right) ideals of \( A \). An algebra \( A \) is called semisimple if

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Dear Professor Shakir Ali,

It is my pleasure to inform you that we have just received referee’s positive opinion on your manuscript, jointly written with D. Filippis and M.S. Khan, entitled "Pair of derivations on semiprime rings with application to Banach algebras" submitted for publication in our Journal of Algebra and Computational Applications (JACA).

Sincerely yours

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PAIR OF DERIVATIONS ON SEMIPRIME RINGS WITH APPLICATIONS TO BANACH ALGEBRAS

SHAKIR ALI, VINCENZO DB PILIPPIS AND MOHAMMAD SALAHUDDIN KHAN

ABSTRACT. Let $R$ be an associative ring. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The objective of the present paper is to characterize a semiprime ring $R$ which admits pair of derivations $d$ and $g$ such that $[d(x'''), g(y'')'] = \pm x'''y''$ for all $x, y \in R$ or $d(x'') o g(y'') = \pm x'''y''$ for all $x, y \in R$ or $[d(x''), d(y'')'] = \pm g(x'''y'')$ for all $x, y \in R$, where $m$ and $n$ are positive integers. With this, several results can be either deduced or generalized. Finally, we apply these purely algebraic results to obtain some range inclusion results of continuous derivations on Banach algebras.

1. INTRODUCTION

This research has been motivated by the work of Wei and Xiao [52]. Throughout this paper $R$ will denote an associative ring with center $Z(R)$ and $A$ will represent an associative algebra over a complex field $C$. Recall that a ring $R$ is said to be prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and $R$ is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. A ring $R$ is said to be $n$-torsion free, where $n > 1$ is an integer, in case $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x o y$ will denote the anti-commutator $xy + yx$. We have extensive use of the basic commutator identities: $[xy, z] = z[y, x] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$.

An additive mapping $d : R \rightarrow R$ is said to be a derivation on $R$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. If $R$ is an algebra we assume additionally that $d$ is linear i.e., $d(ax) = ad(x)$ for all $x \in R$ and $a$ is in some field $F$. Let $S$ be a non empty subset of $R$. A mapping $f : R \rightarrow R$ is called centralizing on $S$ if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on $S$ if $[f(x), x] = 0$ for all $x \in S$. The study of such mappings were initiated by Posner. In [44, Lemma 3], Posner proved that if a prime ring $R$ has a nonzero commuting derivation on $R$, then $R$ is commutative. This result was subsequently refined and extended by a number of algebraists; we refer the reader to [7], [9] and [12] for a state-of-art account and a comprehensive bibliography.

We say that a map $f : R \rightarrow R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for $x, y \in R$. Starting with the paper by Watkins [51], the study of describing maps that preserve commutativity becomes an active research area in matrix theory, operator theory and ring theory (see for instance [1], [3].

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Key words and phrases. Semiprime ring, Banach algebra, derivation and strong commutativity preserving.
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