Preface

Special functions are well known for their equivocating property and great applicability in interdisciplinary areas of sciences. Special functions consist with a great historical background with enormous literature. There was plenty of erudition poured into special functions in the last few decades of the nineteenth century. They could have gone the way of invariant theory, or syzygies, or those other somehow characteristically Victorian mathematical pursuits. The special functions of mathematical physics, such as the classical orthogonal polynomials (the Laguerre, Hermite, Jacobi polynomials), the spherical, cylindrical and hypergeometric functions appear as the solutions of many theoretical and applied problems of physics, engineering, statistics, biology, economics and other most diverse areas of natural, life and social sciences. These functions also emerge in such areas of application as heat conduction, communication systems, electro-optics, nonlinear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory and electric circuit theory, among others. For example, in 1920s the special functions like Laguerre polynomials and Hermite polynomials appeared in even the most basic problems of quantum mechanics. Then there was scattering theory, which seemed to use a wide class of special functions.

The theory of special functions with its countless beautiful formulae is very well suited to an algorithmic approach to mathematics. In fact, the special functions started appearing after the arrival of calculus in the late seventeenth century. Of course the inclination for a broad theory including as many as possible of the known special functions has its intellectual appeal, but it is worth to note other motivations. For a long time, the applications of special functions to the physical sciences and engineering estimated the relative significance of these functions as a part of applied mathematics. In the days before the electronic computer, the ultimate admiring comment to a special function was the computation, by hand, of comprehensive tables of its values. This was a capital-intensive process, intended to make the function accessible by look-up, as for the well-known logarithm tables. There might be the following two aspects of the hypothesis, which were then mattered:
(i) for numerical analysis, discovery of infinite series or other analytical expression allowing rapid calculation,

(ii) reduction of as many functions as possible to the given function.

In contrast, there are approaches typical of the interests of pure mathematics: asymptotic analysis, analytic continuation and monodromy in the complex plane and the discovery of symmetry principles and other structure behind the facade of endless formulae in rows. Actually, these approaches are consistent.

Various generalizations of the special functions of mathematical physics have witnessed a significant evolution during the recent years. This further advancement in the theory of special functions serves as an analytic foundation for the majority of problems in mathematical physics that have been solved exactly and find broad practical applications. For some physical problems the use of new classes of special functions provided solutions hardly achievable with conventional analytical and numerical means. For example, the use of generalized Bessel functions is now a well established tool to treat synchrotron radiation [66], mechanics of solids and heat transfer [89]. Further the importance of generalized Hermite polynomials has been recognized [25,33,40,67,91] and has been exploited to deal with quantum mechanical and optical beam transport problems. The usefulness of the generalized Laguerre polynomials to treat radiation physics problems such as wave propagation and quantum beam life time in storage rings [131] is a well established fact.

An important development in the theory of generalized special functions is the introduction of multi-index and multi-variable special functions. The importance of these functions has been recognized both in purely mathematical and applied frameworks. To solve the problems arising in many branches of mathematics, going from the theory of partial differential equations to the abstract group theory, requirement of multi-index and multi-variable special functions are realized. The theory of multi-index and multi-variable Hermite polynomials was initially developed by Hermite himself [64]. Some research contributions are given in order to develop the theory of multi-index and multi-variable special functions and polynomials, see for example [9,30,40].

Another important generalization of special functions is special matrix functions and polynomials. These functions appear in the literature related to statistics [69], scattering theory [59] and more recently in connection with matrix analogues of Hermite, Laguerre and Legendre differential equations and the corresponding polynomial families [70–72].
In view of the fact that there is a close link between the classical orthogonal polynomials and orthogonal matrix polynomials, the former have been extended to the later.

The study of special matrix functions is important due to their applications in certain areas of statistics, physics and engineering. It has been established that there is a close link between scalar polynomials satisfying higher order recurrence relations and orthogonal matrix polynomials. Keeping in view this fact, the matrix-valued counterparts of special functions such as special functions of matrix arguments and special functions with matrix parameters, have gained increasing interest. Constantine and Muirhead [19] studied the hypergeometric functions of two argument matrices. But it is difficult to develop a theory of special matrix functions for general matrices. Due to certain properties of matrices, the development of the theory of special matrix functions are confined to a certain class of matrices. For the special functions of many matrices, only very little has been achieved even for the case of real symmetric positive definite matrices. However, the theory of special functions with matrix parameter is developed and their properties are also studied by some authors, see for example Defez et al. [47, 48], Jódar et al. [70–77], Sayyed et al. [116, 117].

Operational techniques are important because they are closer to implementations and language definitions than more abstract mathematical techniques. It is well known from the literature that operational techniques include integral, differential and exponential operators and provide a systematic and analytic approach to study special functions, see for example [110, 120]. The operational techniques are based upon single, double and multiple integral transforms and upon certain operators involving derivatives. Methods connected with the use of integral transforms have been successfully applied to the solution of differential and integral equations, the study of generalized special functions and the evaluation of integrals. Further, it should be noted that the technique of inverse operator, applied for derivatives of various orders, combined with integral transforms, provides easy and straightforward solutions of various types of differential equations. Dattoli and his co-workers have shown that by using operational techniques, many properties of ordinary and multi-variable special functions are simply derived and framed in a more general context, see for example [22, 23, 26–30, 34–38, 40–44]. Operational techniques provide a general framework to derive generating relations and summation formulae involving multi-variable special functions.

It has been shown by Saad and Hall [114] that many integrals containing products of confluent hypergeometric functions follow directly from one single integral that has a
very simple formula in terms of Appell’s double series $F_2$. They studied some techniques for computing such series and also presented the applications requiring matrix elements of singular potentials and the perturbed Kratzer potential.

The operational methods combined with the monomiality principle [22,121] open new possibilities to deal with the theoretical foundations of special functions and also to introduce new families of special functions. The concepts and formalism associated with the monomiality treatment can be exploited in different ways. They can be used to simplify the derivation of the properties of ordinary or generalized special polynomials and to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations of evolution type.

Sequences of polynomials play a fundamental role in applied mathematics. One of the important classes of polynomial sequences is the class of Sheffer sequences. The Appell polynomial sequences, which is a subclass of Sheffer polynomial sequences, is also important. The Appell and Sheffer polynomial sequences appear in different applications in pure and applied mathematics. These polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. The typical examples of Appell polynomial sequences are the Bernoulli and Euler polynomials, while the Sheffer class contains important sequences such as the Hermite and Laguerre polynomials. In the past few decades, there has been a renewed interest in Appell and Sheffer polynomials. Di Bucchianico [50] summarized and documented more than five hundred old and new findings related to the study of Appell and Sheffer polynomial sequences. One aspect of such study is to find differential equation and recursive formulae for Sheffer polynomial sequences. For instance, He and Ricci [62] developed the differential equation and recursive formula for Appell polynomials.

Recently, Subuhi Khan and her co-authors presented a systematic study of certain new classes of mixed special polynomials related to the Appell and Sheffer polynomial sequences, see for example [83,84,88]. These mixed polynomials are important due to the fact that they possess important properties such as differential equations, generating functions, series definitions, integral representations etc. Also, these polynomials can be framed within the context of monomiality principle. The problems arising in different areas of science and engineering are usually expressed in terms of differential equations, which in most of the cases have special functions as their solutions. Saad et al. [115]
studied the criterion for polynomial solutions to a class of linear differential equations of second order. The differential equations satisfied by the mixed special polynomials may be used to express the problems arising in new and emerging areas of sciences. Due to this fact, these new mixed special polynomials are important from the viewpoint of applications.

The theory of Lie groups is one of the classical well established chapters of mathematics. This theory was developed by the Norwegian mathematician, Sophus Lie [95] in the late nineteenth century in connection with his work on systems of differential equations. During the past one hundred years, the concepts and methods of the theory of Lie groups entered into many areas of mathematics and theoretical physics and become inseparable from them. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid-state physics, cosmology and control theory.

The theory of group representations and its relation to special functions give a powerful tool to the development of mathematical physics. This is a rapidly growing field, through which one can bring to bear many powerful methods of modern mathematics. One of the numerous consequences of the relationship of the theory of group representations and special functions is to investigate properties of special functions including generating relations. For example, one may refer to the works of Miller [103], Wawrzynczyk [127], Weisner [128-130], Vilenkin [126] and Srivastava and Manocha [120]. Recently, the contributions showing Lie theoretical representations of Laguerre 2D polynomials [79], Hermite 2D polynomials [86] and 2-variable Laguerre matrix polynomials [85] are given.

In this thesis, certain mixed type and matrix special functions are studied. Operational methods are used to introduce certain new families of mixed special polynomials associated with Appell and Sheffer polynomial sequences. These families are also framed within the context of monomiality principle. Further, the mixed special matrix polynomials are introduced related to the Hermite and Laguerre matrix polynomials and their properties are established.

The objectives of the work presented in this thesis are:

1. To introduce the following mixed type and matrix special functions:
2-Iterated Appell polynomials
(ii) Families of mixed Appell polynomials
(iii) Families of Legendre-Sheffer polynomials
(iv) Family of Laguerre-Sheffer polynomials
(v) 2-Variable generalized Hermite matrix polynomials
(vi) Families of mixed Hermite matrix polynomials
(vii) Hermite-Laguerre matrix polynomials.

2. To establish the properties of mixed type and matrix special functions mentioned in 1(i)-(vii).

3. To derive certain results for the mixed type and matrix special functions mentioned in 1(i)-(vii).

The present thesis is a part of the research work carried out by the author during the last three and half years concerning certain mixed type and matrix special functions. The thesis comprises of eight chapters and each chapter is subdivided into five sections.

In Chapter 1, the necessary background material of special functions, special matrix polynomials, operational methods and Lie algebraic techniques is given. The definitions and properties of hypergeometric functions, certain classical and matrix special functions are reviewed. The technicalities associated with the monomiality principle are presented and Appell and Sheffer polynomial sequences are defined. Further, the definitions and the concepts related to Lie algebraic approach to special functions are given. However, the definitions, concepts and results reviewed in this chapter are only those, which are required in carrying out the research work presented in the thesis.

In Chapter 2, the Appell type sets of polynomials and related numbers are introduced by using monomiality principle formalism. The 2-iterated Appell polynomials are introduced and their properties are established. Certain results for the 2-iterated Bernoulli and Euler polynomials are obtained. The Bernoulli and Euler based Appell polynomials are introduced and different sets of polynomials, namely, the Bernoulli-Euler (or Euler-Bernoulli), Bernoulli and Euler based generalized Euler polynomials are obtained as particular cases of these polynomials. Further, the 2-iterated Bernoulli, 2-iterated Euler and Bernoulli-Euler numbers are introduced. The determinantal approach to define Bernoulli-Appell polynomials is also explored.

In Chapter 3, the monomiality principle is used to introduce mixed type special polynomials of two variables related to the Appell polynomials sequence. Some results
for the 2-variable general polynomials are derived. The 2-variable mixed Appell polynomials are introduced and framed within the context of monomiality principle. Further, the Gould-Hopper Appell polynomials are considered and results for some members belonging to the Gould-Hopper Appell polynomials family are obtained. Examples of some members belonging to the Gould-Hopper-Appell and Hermite-Appell families are also given and their plots are drawn for suitable values of the parameters and indices.

In Chapter 4, the families of Legendre-Sheffer polynomials are introduced and some special properties of these families are established by using operational methods. Two different forms of the Legendre polynomials are taken as base to introduce families of Legendre-Sheffer polynomials. Operational rules providing correspondence between Sheffer and Legendre-Sheffer families are established. Further, the Legendre-Laguerre and Legendre-Hermite polynomials are considered and their properties are obtained. The operational correspondence between Sheffer and Legendre-Sheffer families is applied to derive the results for some members belonging to the Legendre-Sheffer polynomial families.

In Chapter 5, the Laguerre-Sheffer polynomials family is introduced and framed within the context of monomiality principle. An operational correspondence between the Sheffer and the Laguerre-Sheffer families is established. Further, the multiplicative and derivative operators, recurrence relations and the differential equations for the Laguerre-Sheffer and Laguerre-Appell polynomial families are obtained. Certain relations, identities and expansions for the Laguerre-Sheffer polynomials are also derived.

In Chapter 6, the 2-variable generalized Hermite matrix polynomials are introduced and framed within the context of an irreducible representation of the harmonic oscillator Lie algebra $\mathcal{G}(0, 1)$. Series definition, differential equation and some other important properties of these matrix polynomials are derived. Further, generating relations involving these matrix polynomials are derived by using the representation theory of the Lie algebra $\mathcal{G}(0, 1)$. Certain new and known generating relations involving other forms of the Hermite matrix and scalar polynomials are also obtained as applications.

In Chapter 7, the Hermite-general matrix polynomials are introduced by making use of operational identities for decoupling of the exponential operators. The operational rules are established, which provide connections of these matrix polynomials with the 2-variable general polynomials as well as with the 2-index 2-variable Hermite matrix polynomials. The multiplicative and derivative operators, matrix recurrence relations and the matrix differential equations for these matrix polynomials are derived by using
the concepts associated with the monomiality principle. Further, some members belonging to the Hermite-general matrix polynomials family are considered and certain results for these special polynomials are derived. Summation formulae connecting the Hermite-general matrix polynomials with certain other matrix and scalar special polynomials are also established.

In Chapter 8, the 2-variable Hermite Kampe de Feriet polynomials are used to introduce the mixed type special matrix polynomials. Operational representations, properties and generating relations for these matrix polynomials are derived. The Hermite-Laguerre matrix polynomials and Hermite-modified Laguerre matrix polynomials are introduced by means of the generating functions and series definitions. Special cases of these matrix polynomials are mentioned. Operational representations, recurrence relations and differential equations of these matrix polynomials are established. Further, generating relations involving the Hermite-Laguerre matrix polynomials are derived by using operational methods. Certain results for new and known matrix as well as scalar special polynomials are also obtained as particular cases.

Each chapter (except Chapter 1) of the thesis ends with some concluding remarks. A comprehensive list of references of books, monographs, proceedings and research papers is provided at the end of the thesis.

The published and submitted research papers based on the work of this thesis are as follows:


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6. Some Properties of Hermite-General Matrix Polynomials (Submitted for publication in the *Ukrainian Mathematical Journal*).

7. Hermite-Laguerre Matrix Polynomials and Generating Relations (Submitted for publication in the *Reports on Mathematical Physics*).

The results of this thesis have been presented in the following National and International Conferences:

1. **Workshop on Special Functions and Applications**, organized by Department of Mathematics, College of Science, King Saud University, Riyadh, KSA (20th - 21st February, 2013).

2. **International Conference on Special Functions and their Applications (ICSFA-2012) and Symposium on Works of Ramanujan**, organized by Department of Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Surat, India and Society for Special Functions and their Applications (27th-29th June, 2012).


4. **International Conference on Special Functions and their Applications (ICSFA-2011) and Symposium on Works of Ramanujan**, organized by Department of Mathematics and Statistics, J. N. Vyas University, Jodhpur, India and Society for Special Functions and their Applications in association with JIET Group of Institutions, Jodhpur, India (28th-30th June, 2011).

5. **Conference on Special Functions and their Applications and Symposium on Computational and Biological Mathematics**, organized by School of Mathematics and Allied Sciences, Jiwaji University, Gwalior, India (21st-23rd June, 2010).