Chapter 4

Families of Legendre-Sheffer Polynomials

4.1 Introduction

The Legendre polynomials are being used by mathematicians and engineers for a variety of mathematical and numerical solutions. For example, the Legendre polynomials are applicable in fluid dynamics to study the flow around the outside of a puff of hot gas rising through the air, see for details [105]. The Legendre polynomials are closely related with physical phenomena for which spherical geometry is important. In particular, these polynomials first arose in the problem of expressing the newtonian potential of a conservative force field in an infinite series involving the distance variables of two points and their included central angle.

Further, research on Legendre polynomials [17] discusses and explores the use of Legendre series and best leading coefficients of Legendre polynomials for different applications. Also, Cornille and Martin [20] discussed the applications and an extension of the Szeg-Szász inequality for Legendre polynomials ($\alpha = \beta = 0$) to obtain global bounds on the variation of the phase of an elastic scattering amplitude. In numerical analysis and methods, Legendre polynomials are used to efficiently calculate numerical integrations by Gaussian quadrature method. This method is very effective in approximating integrals with accuracy and in small time.

Many properties of conventional and generalized polynomials have been shown to be derivable in a straightforward way within an operational framework, which is a consequence of the monomiality principle. We recall that the 2-variable Legendre polynomials $S_n(x, y)$ and $\frac{R_n(x, y)}{n!}$ are quasi-monomial under the action of the operators [39, p. 365

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and \( p. 367 \):

\[
\hat{M}_1 = y + 2D_x^{-1}D_y, \quad (4.1.1)
\]

\[
\hat{P}_1 = D_y \quad (4.1.2)
\]

and

\[
\hat{M}_2 = -D_x^{-1} + D_y^{-1}, \quad (4.1.3)
\]

\[
\hat{P}_2 = -D_x xD_x \quad (4.1.4)
\]

respectively, where

\[
D_x^{-n}\{f(x)\} = \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi) d\xi, \quad D_x^{-n}\{1\} = \frac{x^n}{n!}. \quad (4.1.5)
\]

In view of equations (1.4.6), (4.1.1) and (4.1.3), the generating functions for \( S_n(x, y) \) and \( R_n(x, y) \) are given as:

\[
e^{xt}C_0(-xe) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!} \quad (4.1.6)
\]

and

\[
C_0(xt)C_0(-yt) = \sum_{n=0}^{\infty} R_n(x, y) \frac{t^n}{n!}, \quad (4.1.7)
\]

respectively, where \( C_0(x) \) denotes the Tricomi function of order zero. The Tricomi function of order \( n \) is defined by means of generating function (1.2.20). We note that

\[
\exp(-\alpha D_x^{-1}) \{1\} = C_0(\alpha x). \quad (4.1.8)
\]

Further, the series definitions of \( S_n(x, y) \) and \( R_n(x, y) \) are given as [39, p. 366, 367]:

\[
S_n(x, y) = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^k y^{n-2k}}{(n-2k)!(k!)^2} \quad (4.1.9)
\]

and

\[
R_n(x, y) = n! \sum_{k=0}^{n} \frac{(-1)^{n-k} x^{n-k} y^k}{(n-k)!(k!)^2} \quad (4.1.10)
\]

respectively. Consequently from equations (4.1.9) and (4.1.10), it follows that

\[
S_n(x, 0) = \begin{cases} 
  n! \frac{x^{(n/2)}}{[(n/2)!]^2}, & \text{if } n \text{ is even,} \\
  0, & \text{if } n \text{ is odd } 
\end{cases} \quad (4.1.11)
\]

\[S_n(0, y) = y^n\]
and

\[ R_n(x, 0) = (-1)^n x^n ; \quad R_n(0, y) = y^n, \]

or, equivalently

\[ \frac{R_n(x, 0)}{n!} = (-D_x^{-1})^n \{1\} ; \quad \frac{R_n(0, y)}{n!} = (D_y^{-1})^n \{1\}. \quad (4.1.12) \]

Now, since we have

\[ D_x D_x S_n(x, y) = D_y^2 S_n(x, y) \quad (4.1.13) \]

and

\[ D_x D_x R_n(x, y) = -D_y y D_y R_n(x, y), \quad (4.1.14) \]

In view of the relation [41, p. 32(8)]

\[ \frac{\partial}{\partial D_x^{-1}} = D_x D_x, \quad (4.1.15) \]

equations (4.1.13) and (4.1.14) can be written as

\[ \frac{\partial}{\partial D_x^{-1}} S_n(x, y) = D_y^2 S_n(x, y) \quad (1.1.16) \]

and

\[ \frac{\partial}{\partial D_x^{-1}} R_n(x, y) = -\frac{\partial}{\partial D_y^{-1}} R_n(x, y) \quad (1.1.17) \]

respectively.

Using equations (4.1.11), (4.1.12), (4.1.16) and (4.1.17), we find the following operational definitions for the polynomials \( S_n(x, y) \) and \( R_n(x, y) \):

\[ S_n(x, y) = \exp(D_x^{-1} D_y^2) \{y^n\} \quad (4.1.18) \]

and

\[ \frac{R_n(x, y)}{n!} = \exp(-D_x^{-1} \frac{\partial}{\partial D_y^{-1}}) \{(D_y^{-1})^n\} = \exp(-D_y^{-1} \frac{\partial}{\partial D_x^{-1}}) \{(-D_x^{-1})^n\}, \quad (4.1.19) \]

respectively.

Also, we note the following relation [39, p. 366]:

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\[ S_n(x, y) = H_n(y, D^{-1}_x), \quad (4.1.20) \]

where \( H_n(x, y) \) denotes the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) [6], defined by the generating function (3.1.3) and by series (3.1.6).

In view of relation (4.1.20), we can write equation (4.1.16) as:

\[ \frac{\partial}{\partial D^{-1}_x} H_n(y, D^{-1}_x) = D_x^2 H_n(y, D^{-1}_x), \quad (4.1.21) \]

which is equivalent to the heat equation

\[ \frac{\partial}{\partial t} f(y, t) = D_y^2 f(y, t). \quad (4.1.22) \]

Moreover, the 2-variable Legendre polynomials \( S_n(x, y) \) and \( R_n(x, y) \) are related with the ordinary Legendre polynomials \( P_n(x) \) defined by equation (1.2.25), as [39, p. 366 and p. 367]:

\[ P_n(x) = S_n \left( \frac{1 - x^2}{4}, x \right) \quad (4.1.23) \]

and

\[ P_n(x) = R_n \left( \frac{1 - x}{2}, \frac{1 + x}{2} \right), \quad (4.1.24) \]

respectively.

In view of generating function (1.2.29) of the Hermite polynomials \( H_n(x) \) and from Table 1.4.2, we note that the Hermite polynomials \( H_n(x) \) are Sheffer for

\[ g(t) = \exp \left( \frac{t^2}{4} \right), \quad f(t) = \frac{t}{2}, \quad (4.1.25) \]

or, equivalently for

\[ A(t) = \exp(-t^2), \quad H(t) = 2t. \quad (4.1.26) \]

Thus, making use of equations (1.4.20a) (or (1.4.20b)) and (1.4.21a) (or (1.4.21b)), we find the multiplicative and derivative operators for \( H_n(x) \) as:

\[ \hat{M} = 2x - D_x \quad (4.1.27) \]

and

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\[ \hat{P} = \frac{1}{2} D_x, \quad (4.1.28) \]

respectively.

Further, in view of the generating function (1.2.40) of the associated Laguerre polynomials \( L_n^{(\alpha)}(x) \) and from Table 1.4.2(II), we note that the associated Laguerre polynomials \( L_n^{(\alpha)}(x) \) are Sheffer for

\[ g(t) = (1 - t)^{-\alpha - 1}, \quad f(t) = \frac{t}{t - 1}, \quad (4.1.29) \]

or, equivalently for

\[ A(t) = \frac{1}{(1 - t)^{\alpha + 1}}, \quad H(t) = \frac{-t}{1 - t}. \quad (4.1.30) \]

The multiplicative and derivative operators for \( L_n^{(\alpha)}(x) \) are given as:

\[ \hat{M} = -xD_x^2 + (2x - \alpha - 1)D_x - x + \alpha + 1, \quad (4.1.31) \]

\[ \hat{P} = \frac{D_x}{D_x - 1}, \quad (4.1.32) \]

respectively.

For \( \alpha = 0 \), equations (4.1.29)-(4.1.32) reduce to the corresponding expressions for the Laguerre polynomials \( L_n(x) \).

Recently, it has been shown that the Sheffer polynomials and the monomiality principle, along with the underlying operational formalism provide a powerful tool for the investigation of the properties of a wide class of polynomials, see for example [37, 106]. In this chapter, families of Legendre-Sheffer polynomials are introduced and some special properties of these families are derived by using operational methods. In Section 4.2, two different forms of the Legendre polynomials are taken as base to introduce families of Legendre-Sheffer polynomials. Operational rules providing correspondence between Sheffer and Legendre-Sheffer families are also established. In Section 4.3, Legendre-Laguerre and Legendre-Hermite polynomials are considered and their properties are obtained. In Section 4.4, the operational correspondence between Sheffer and Legendre-Sheffer families is applied to derive the results for some members belonging to the Legendre-Sheffer families. In Section 4.5, some concluding remarks are given.
4.2 Legendre-Sheffer Polynomials

We denote the Legendre-Sheffer polynomials by $ss_n(x, y)$ and $RS_n(x, y)$. Thus, in order to introduce the Legendre-Sheffer polynomials corresponding to the multiplicative operators (4.1.1) and (4.1.3) of $S_n(x, y)$ and $\frac{R_n(x,y)}{n!}$ respectively, we consider the generating functions

\[ A(t) \exp(\tilde{M}_1 H(t)) = \sum_{n=0}^{\infty} ss_n(x, y) \frac{t^n}{n!} \]  
\[ A(t) \exp(\tilde{M}_2 H(t)) = \sum_{n=0}^{\infty} RS_n(x, y) \frac{t^n}{n!} \]

so that, we have

\[ A(t) \exp\left((y + 2D^{-1}_x D_y) H(t)\right) = \sum_{n=0}^{\infty} ss_n(x, y) \frac{t^n}{n!} \]  
\[ A(t) \exp\left((-D^{-1}_x + D^{-1}_y) H(t)\right) = \sum_{n=0}^{\infty} RS_n(x, y) \frac{t^n}{n!} \]

respectively.

Now, decoupling the exponential operators in the l.h.s. of equations (4.2.3) and (4.2.4), by using identity (2.2.13), we find

\[ A(t) \exp(y H(t)) \exp\left((H(t))^2 D^{-1}_x \right) = \sum_{n=0}^{\infty} ss_n(x, y) \frac{t^n}{n!} \]  
\[ A(t) \exp\left(-H(t) D^{-1}_x \right) \exp\left(H(t) D^{-1}_y \right) = \sum_{n=0}^{\infty} RS_n(x, y) \frac{t^n}{n!} \]

respectively. Finally, using relation (4.1.8) in equations (4.2.5) and (4.2.6), we find the following generating functions for Legendre-Sheffer polynomials $ss_n(x, y)$ and $RS_n(x, y)$:

\[ A(t) \exp(y H(t)) C_0(-x (H(t))^2) = \sum_{n=0}^{\infty} ss_n(x, y) \frac{t^n}{n!} \]  
\[ A(t) C_0(x H(t)) C_0(-y H(t)) = \sum_{n=0}^{\infty} RS_n(x, y) \frac{t^n}{n!} \]
respectively. Also, in view of equations (4.1.1), (4.1.3), (4.1.14), (4.2.1) and (4.2.2), it follows that the Legendre-Sheffer polynomials $ss_n(x, y)$ and $rs_n(x, y)$ can also be obtained as:

$$ss_n(x, y) = s_n(y + 2D_x^{-1}D_y)$$  \hspace{1cm} (4.2.9)

and

$$\frac{rs_n(x, y)}{n!} = s_n(-D_x^{-1} + D_y^{-1})$$  \hspace{1cm} (4.2.10)

From equations (1.4.20b), (1.4.21b), (4.2.9) and (4.2.10), the multiplicative and derivative operators for the Legendre-Sheffer polynomials $ss_n(x, y)$ and $rs_n(x, y)$ can be obtained as:

$$\hat{M}_i = M_i H'(H^{-1}(D_{\hat{M}_i})) + \frac{A'(H^{-1}(D_{\hat{M}_i}))}{A(H^{-1}(D_{\hat{M}_i}))}; \quad i := R, S$$  \hspace{1cm} (4.2.11)

and

$$\hat{P}_i = H^{-1}(D_{\hat{M}_i}); \quad i := R, S,$$  \hspace{1cm} (4.2.12)

respectively. Differentiating equations (4.2.7) and (4.2.8) partially with respect to $x$ and $y$, we find that $ss_n(x, y)$ and $rs_n(x, y)$ are solutions of equations:

$$D_x^2 ss_n(x, y) = D_x xD_x ss_n(x, y)$$  \hspace{1cm} (4.2.13)

and

$$D_y yD_y rs_n(x, y) = -D_x xD_x rs_n(x, y)$$  \hspace{1cm} (4.2.14)

respectively. Also, in view of relation (4.1.15), equations (4.2.13) and (4.2.14) can be expressed as:

$$D_x^2 ss_n(x, y) = \frac{\partial}{\partial D_x^{-1}} ss_n(x, y)$$  \hspace{1cm} (4.2.15)

and

$$\frac{\partial}{\partial D_y^{-1}} rs_n(x, y) = -\frac{\partial}{\partial D_x^{-1}} rs_n(x, y)$$  \hspace{1cm} (4.2.16)

respectively. Now, from equations (4.1.8), (4.2.7), (4.2.8) and (1.4.14), it follows that

$$ss_n(0, y) = s_n(y)$$  \hspace{1cm} (4.2.17)

and

$$\frac{rs_n(x, 0)}{n!} = s_n(-D_x^{-1}); \quad \frac{rs_n(0, y)}{n!} = s_n(D_y^{-1})$$  \hspace{1cm} (4.2.18)
Finally, solving equations (4.2.15) and (4.2.16) with conditions (4.2.17) and (4.2.18), we get the following operational definitions for $s_{n}(x, y)$ and $r_{n}(x, y)$:

$$s_{n}(x, y) = \exp \left( D^{-1}_{x} D^{2}_{y} \right) \{ s_{n}(y) \}$$  \hspace{1cm} (4.2.19)

and

$$\frac{r_{n}(x, y)}{n !} = \exp \left( -D^{-1}_{x} \frac{\partial}{\partial D^{-1}_{y}} \right) \{ s_{n}(D_{y}^{-1}) \},$$  \hspace{1cm} (4.2.20a)

or, equivalently

$$\frac{r_{n}(x, y)}{n !} = \exp \left( -D^{-1}_{y} \frac{\partial}{\partial D^{-1}_{x}} \right) \{ s_{n}(-D_{x}^{-1}) \}$$  \hspace{1cm} (4.2.20b)

respectively. The operational rules (4.2.19), (4.2.20a) and (4.2.20b) provide a correspondence between the Sheffer and Legendre-Sheffer families.

A simple computation shows that the operational rules (4.2.19), (4.2.20a) and (4.2.20b) can be written in the following generalized forms:

$$s_{n}(m^{2} x, m(y + z)) = \exp \left( D^{-1}_{x} \frac{\partial^{2}}{\partial y^{2}} \right) \{ s_{n}(m(y + z)) \}$$  \hspace{1cm} (4.2.21)

and

$$\frac{r_{n}(m x, m y)}{n !} = \exp \left( -D^{-1}_{y} \frac{\partial}{\partial D^{-1}_{y}} \right) \{ s_{n}(m D_{y}^{-1}) \},$$  \hspace{1cm} (4.2.22a)

or, equivalently

$$\frac{r_{n}(m x, m y)}{n !} = \exp \left( -D^{-1}_{y} \frac{\partial}{\partial D^{-1}_{x}} \right) \{ s_{n}(-m D_{x}^{-1}) \}.$$  \hspace{1cm} (4.2.22b)

For $z = 0$, equation (4.2.21) becomes

$$s_{n}(m^{2} x, m y) = \exp \left( D^{-1}_{x} \frac{\partial^{2}}{\partial y^{2}} \right) \{ s_{n}(m y) \}.$$  \hspace{1cm} (4.2.23)

By using relations (1.4.19a), (1.4.19b) between $A(t)$, $H(t)$ and $f(t)$, $g(t)$, we can write the equivalent forms of the results derived above. Also, since for $H(t) = f^{-1}(t) = t$, the Sheffer polynomials $s_{n}(x)$ reduce to the Appell polynomials. Thus, by taking $H(t) = t$ in the results of the Legendre-Sheffer polynomials, we can obtain the corresponding results for the Legendre-Appell polynomials.

In the next section, we consider the Legendre-Laguerre and Legendre-Hermite poly-
nomials as two important members of the Legendre-Sheffer polynomials family.

4.3 Legendre-Laguerre and Legendre-Hermite Polynomials

The Laguerre and Hermite polynomials are two important members of the Sheffer family. We take these two polynomials and generate the Legendre-Laguerre and Legendre-Hermite polynomials by using the formalism developed in previous sections. First, we establish the generating functions for the Legendre-Laguerre and Legendre-Hermite polynomials.

Taking $A(t) = \frac{1}{(1-t)^{a+1}}$ and $H(t) = \frac{-t}{(1-t)^{a+1}}$ (Table 4.2(II)) of the associated Laguerre polynomials $L_n^{(a)}(x)$ in the l.h.s. of equations (4.2.7) and (4.2.8) and denoting the resultant Legendre-Laguerre polynomials in the r.h.s. by $sL_n^{(a)}(x,y)$ and $rL_n^{(a)}(x,y)$, respectively, we get the generating functions for the Legendre-Laguerre polynomials $sL_n^{(a)}(x,y)$ and $rL_n^{(a)}(x,y)$ as:

$$\frac{1}{(1-t)^{a+1}} \exp \left( \frac{-yt}{1-t} \right) C_0 \left( \frac{-xt^2}{(1-t)^2} \right) = \sum_{n=0}^{\infty} sL_n^{(a)}(x,y) \frac{t^n}{n!} \quad (4.3.1)$$

and

$$\frac{1}{(1-t)^{a+1}} C_0 \left( \frac{-xt}{1-t} \right) C_0 \left( \frac{yt}{1-t} \right) = \sum_{n=0}^{\infty} rL_n^{(a)}(x,y) \frac{t^n}{n!} \quad (4.3.2)$$

respectively. Similarly, taking $A(t) = \exp(-t^2)$, $H(t) = 2t$ (Table 4.2(II)) of the Hermite polynomials $H_n(x)$ in the l.h.s. of equations (4.2.7) and (4.2.8) and denoting the resultant Legendre-Hermite polynomials in the r.h.s. by $sH_n(x,y)$ and $rH_n(x,y)$, respectively, we get the generating functions for the Legendre-Hermite polynomials $sH_n(x,y)$ and $rH_n(x,y)$ as:

$$\exp(-t^2) \exp(2yt) C_0(-4xt^2) = \sum_{n=0}^{\infty} sH_n(x,y) \frac{t^n}{n!} \quad (4.3.3)$$

and

$$\exp(-t^2) C_0(2xt) C_0(-2yt) = \sum_{n=0}^{\infty} rH_n(x,y) \frac{t^n}{n!} \quad (4.3.4)$$

respectively.

Now, we proceed to derive the results for the Legendre-Sheffer polynomials from the ones given for Sheffer polynomials. First, we establish the series definitions for the
Legendre-Sheffer polynomials. We operate \( \exp \left( D_x^{-1} D_y^2 \right) \) on both sides of series definition (1.2.41) of the associated Laguerre polynomials \( L_n^{(\alpha)}(y) \), so that we have

\[
\exp \left( D_x^{-1} D_y^2 \right) \{L_n^{(\alpha)}(y)\} = \sum_{k=0}^{n} \frac{(-1)^k(1 + \alpha)_n}{(n-k)!k!(1 + \alpha)_k} \exp \left( D_x^{-1} D_y^2 \right) \{y^k\},
\]

which on using equation (4.1.18) in the r.h.s. and equation (4.2.19) (for \( s_n(y) = L_n^{(\alpha)}(y) \)) in the l.h.s., yields the following series definition for \( s L_n^{(\alpha)}(x, y) \) in terms of \( S_n(x, y) \):

\[
s L_n^{(\alpha)}(x, y) = \sum_{k=0}^{n} \frac{(-1)^k(1 + \alpha)_n}{(n-k)!k!(1 + \alpha)_k} S_k(x, y).
\]

Again, replacing \( y \) by \( -D_x^{-1} \) in equation (1.2.41) and then operating \( \exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) \) on both sides of the resultant equation, we have

\[
\exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) \{L_n^{(\alpha)}(-D_x^{-1})\} = \sum_{k=0}^{n} \frac{(-1)^k(1 + \alpha)_n}{(n-k)!k!(1 + \alpha)_k} \exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) \{(-D_x^{-1})^k\},
\]

which on using the equation (4.1.19) in the r.h.s. and equation (4.2.20b) (for \( s_n(-D_x^{-1}) = L_n^{(\alpha)}(-D_x^{-1}) \)) in the l.h.s., yields the following series definition for \( R L_n^{(\alpha)}(x, y) \) in terms of \( R_n(x, y) \):

\[
R L_n^{(\alpha)}(x, y) = \sum_{k=0}^{n} \frac{(-1)^k(1 + \alpha)_n}{(n-k)!k!(1 + \alpha)_k} R_k(x, y).
\]

For \( \alpha = 0 \), equations (4.3.6) and (4.3.8) give the series definitions for the Legendre-Laguerre polynomials \( s L_n(x, y) \) and \( R L_n(x, y) \) respectively.

Similarly, considering the series definition (1.2.30) of the Hermite polynomials \( H_n(x) \), we find the series definitions of the Legendre-Hermite polynomials \( s H_n(x, y) \) and \( R H_n(x, y) \) as:

\[
s H_n(x, y) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2)^{n-2k}}{(n-2k)!k!} S_{n-2k}(x, y)
\]

and

\[
R H_n(x, y) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2)^{n-2k}}{(n-2k)!k!} R_{n-2k}(x, y)
\]

respectively.

Next, we derive the expressions for the multiplicative and derivative operators for
the Legendre-Hermite polynomials $sH_n(x, y)$ and $rH_n(x, y)$. In view of operational rules (4.2.19) and (4.2.20b), we have

$$sH_n(x, y) = \exp \left( D_x^{-1} D_y^2 \right) H_n(y)$$  \hspace{1cm} (4.3.11)

and

$$\frac{rH_n(x, y)}{n!} = \exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) H_n(-D_x^{-1}).$$  \hspace{1cm} (4.3.12)

Using the operational definition for Hermite polynomials $H_n(x)$ \cite[609(34)]{27},

$$H_n(x) = \exp \left( -\frac{1}{4} D_x^2 \right) \{(2x)^n\},$$  \hspace{1cm} (4.3.13)

in equation (4.3.11), we have

$$sH_n(x, y) = \exp \left( D_x^{-1} D_y^2 \right) \exp \left( -\frac{1}{4} D_y^2 \right) \{(2y)^n\},$$

which on using identity (2.2.13) (with $k = 0$) in the r.h.s., becomes

$$sH_n(x, y) = \exp \left( D_x^{-1} - \frac{1}{4} D_y^2 \right) \{(2y)^n\}.$$  \hspace{1cm} (4.1.11)

Again, using the operational identity \cite[12]{38}:

$$\exp \left( \lambda D_x^2 \right) \{f(x)\} = f(x + 2\lambda D_x) \exp \left( \lambda D_x^2 \right),$$  \hspace{1cm} (4.3.15)

in the r.h.s. of equation (4.3.14), we find

$$sH_n(x, y) = (2y + (4D_x^{-1} - 1)D_y)^n \{1\}.$$

In view of the above equation and the fact that $sH_0(x, y) = 1$, we find

$$(2y + (4D_x^{-1} - 1)D_y) sH_n(x, y) = sH_{n+1}(x, y).$$  \hspace{1cm} (4.3.16)

Also, from equation (4.3.3), it follows that

$$\frac{1}{2} D_y sH_n(x, y) = n \ sH_{n-1}(x, y).$$  \hspace{1cm} (4.3.17)

Thus, from equations (4.3.16), (4.3.17) and in view of monomiality principal equa-
tions (1.4.1), (1.4.2), we conclude that the Legendre-Hermite polynomials $sH_n(x,y)$ are quasi-monomial under the action of following multiplicative and derivative operators:

$$\hat{M}_{SH} = 2y + (4D_x^{-1} - 1)D_y$$

(4.3.18)

and

$$\hat{P}_{SH} = \frac{1}{2}D_y,$$

(4.3.19)

respectively.

Again, to derive the expressions for the multiplicative and derivative operators for $\frac{R^n_0(x,y)}{n!}$, we use operational definition (4.3.13) (with $x \to -D_x^{-1}$) in equation (4.3.12), so that we have

$$\frac{R^n_0(x,y)}{n!} = \exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) \left\{ 1 \right\}.$$

(4.3.20)

Now, applying the operational identity (4.3.15) to the second exponential in the r.h.s. of equation (4.3.20), we find

$$\frac{R^n_0(x,y)}{n!} = \exp \left( -D_y^{-1} \frac{\partial}{\partial D_x^{-1}} \right) \left( -2D_x^{-1} + \frac{\partial}{\partial D_x^{-1}} \right)^n,$$

(4.3.21)

which on using the shift identity [38, p. 4]

$$\exp (\lambda D_x f(x)) = f(x + \lambda) \exp (\lambda D_x),$$

(4.3.22)

in the r.h.s. becomes

$$\frac{R^n_0(x,y)}{n!} = \left( 2(-D_x^{-1} + D_y^{-1}) + \frac{\partial}{\partial D_x^{-1}} \right)^n.$$

In view of the above equation and the fact that $R_0(x,y) = 1$, we find

$$\left( 2(-D_x^{-1} + D_y^{-1}) + \frac{\partial}{\partial D_x^{-1}} \right) \frac{R^n_0(x,y)}{n!} = \frac{R^{n+1}_0(x,y)}{(n + 1)!}.$$

(4.3.23)

Also, from equation (4.3.4), it follows that

$$-\frac{1}{2} \frac{\partial}{\partial D_x^{-1}} \frac{R^n_0(x,y)}{n!} = \frac{R^{n-1}_0(x,y)}{(n - 1)!}.$$

(4.3.24)
Therefore, from equations (4.3.23), (4.3.24) and in view of monomiality principal equations (1.4.1), (1.4.2), we note that Legendre-Hermite polynomials $H_n^{(x,y)}$ are quasi-monomial under the action of following multiplicative and derivative operators:

$$\hat{M}_{RH} = 2(-D_x^{-1} + D_y^{-1}) + \frac{\partial}{\partial D_x^{-1}}$$

(4.3.25)

and

$$\hat{P}_{RH} = -\frac{1}{2} \frac{\partial}{\partial D_x^{-1}}$$

(4.3.26)

In the next section, we consider the applications of the operational correspondence between Sheffer and Legendre-Sheffer families to derive the results for Legendre-Sheffer polynomial families.

### 4.4 Applications

In order to derive the results for the members of Legendre-Sheffer families from the results given for the special polynomials belonging to Sheffer family, we use the following operations:

$(O_1)$: Operating $\exp(D_x^{-1}D_y^2)$ on both sides of a given result,

$(O_2)$: Replacing $x$ by $D_x^{-1}$ in a given result and then operating $\exp(-D_x^{-1}D_y^{-1})$ on both sides of the resultant equation.

First, we recall the following Sheffer identity for the associated Laguerre polynomials $L_n^{(\alpha)}(x)$ [112, p. 110]:

$$L_n^{(\alpha+\beta+1)}(y + w) = \sum_{k=0}^{n} \binom{n}{k} L_k^{(\alpha)}(y) L_{n-k}^{(\beta)}(w).$$

(4.1.1)

Performing operation $(O_1)$ on the above identity and using the appropriate operational definitions in the resultant equation, we get the following identity for Legendre-associated Laguerre polynomials $sL_n^{(\alpha+\beta+1)}(x, y)$:

$$sL_n^{(\alpha+\beta+1)}(x, y + w) = \sum_{k=0}^{n} \binom{n}{k} sL_k^{(\alpha)}(x, y) L_{n-k}^{(\beta)}(w).$$

(4.4.2)

Taking $\alpha = \beta = 0$ in equation (4.4.2), we get
\( sL_n^{(1)}(x, y + w) = \sum_{k=0}^{n} \binom{n}{k} sL_k(x, y) L_{n-k}(w) \) \tag{4.4.3}

and taking \( \alpha + \beta = -1 \) in equation (4.4.2), we get

\( sL_n(x, y + w) = \sum_{k=0}^{n} \binom{n}{k} sL_k^{(\alpha)}(x, y) L_k^{(-1-\alpha)}(w). \) \tag{4.4.4}

Again, performing operation (\( O_1 \)) on the following functional equation involving associated Laguerre polynomials \( L_n^{(\alpha)}(x) \) [109, p. 209(5)]:

\[
L_n^{(\alpha)}(wy) = \sum_{k=0}^{n} \frac{(1 + \alpha)n(1 - w)^{n-k}w^k}{(n-k)!(1 + \alpha)_k} L_k^{(\alpha)}(y)
\] \tag{4.4.5}

and using the appropriate operational definitions in the resultant equation, we get

\[
sL_n^{(\alpha)}(w^2x, wy) = \sum_{k=0}^{n} \frac{(1 + \alpha)n(1 - w)^{n-k}w^k}{(n-k)!(1 + \alpha)_k} sL_k^{(\alpha)}(x, y),
\] \tag{4.4.6}

which for \( \alpha = 0 \), becomes

\[
sL_n(w^2x, wy) = \sum_{k=0}^{n} \binom{n}{k}(1 - w)^{n-k}w^k sL_k(x, y).
\] \tag{4.4.7}

Next, performing operation (\( O_1 \)) on the following relation for the Shively's pseudo Laguerre polynomials \( R_n(a, y) \) [109, p. 298(1, 2)]:

\[
R_n(a, y) = \frac{1}{(a - 1)_n} \sum_{k=0}^{n} \frac{(a - 1)_{n+k} L_{n-k}(y)}{k!}
\] \tag{4.4.8}

and using the appropriate operational definitions in the resultant equation, we get the following result:

\[
sR_n(a, x, y) = \frac{1}{(a - 1)_n} \sum_{k=0}^{n} \frac{(a - 1)_{n+k} sL_{n-k}(x, y)}{k!},
\] \tag{4.4.9}

where \( sR_n(a, x, y) \) denotes the Legendre-Shively's pseudo Laguerre polynomials.

Also, performing operation (\( O_1 \)) on the following summation formulae [1, p. 169, p. 176], [109, p. 207(3)]:
\[ y^n = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{n!}{2^n k! (n-2k)!} H_{n-2k}(y) \]  \hspace{1cm} (4.4.10)

\[ H_n(y + w) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(\sqrt{2y}) H_k(\sqrt{2w}), \]  \hspace{1cm} (4.4.11)

\[ H_n(y) = 2^n (1 + \alpha) n \sum_{k=0}^{n} \binom{n}{k} 2F_2 \left[ \begin{array}{c} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ -\frac{1}{2}(\alpha+n), -\frac{1}{2}(\alpha+n-1); \end{array} \right] \frac{(-n)_k L_k^{(\alpha)}(y)}{(1+\alpha)_k} \]  \hspace{1cm} (4.4.12)

and making use of appropriate operational definitions in the resultant equations, we get the following results:

\[ S_n(x, y) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{n!}{2^n k! (n-2k)!} S_{n-2k}(x, y) \]  \hspace{1cm} (4.4.13)

\[ sH_n(x, y + w) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} sH_{n-k}(2x, \sqrt{2y}) H_k(\sqrt{2w}), \]  \hspace{1cm} (4.4.14)

\[ sH_n(x, y) = 2^n (1 + \alpha) n \sum_{k=0}^{n} 2F_2 \left[ \begin{array}{c} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ -\frac{1}{2}(\alpha+n), -\frac{1}{2}(\alpha+n-1); \end{array} \right] \times \frac{(-n)_k sL_k^{(\alpha)}(x, y)}{(1+\alpha)_k} \]  \hspace{1cm} (4.4.15)

Further, we make use of operation \((O_2)\) to derive the results. Performing operation \((O_2)\) on equation (4.4.5) and using equations (4.2.22a) and (4.2.20a) (for \( s_{\gamma}(D_y^{-1}) = L_n^{(\alpha)}(D_y^{-1}) \)) in the l.h.s. and r.h.s. of the resultant equation, we get the following identity involving \( R L_n^{(\alpha)}(x, y) \):

\[ \frac{R L_n^{(\alpha)}(wx, wy)}{n!} = \sum_{k=0}^{n} \frac{(1 + \alpha)_n (1 - w)^{n-k} w^k}{(n-k)! (1+\alpha)_k} \frac{R L_k^{(\alpha)}(x, y)}{k!}, \]  \hspace{1cm} (4.4.16)

which for \( \alpha = 0 \), gives

\[ \frac{R L_n(wx, wy)}{n!} = \sum_{k=0}^{n} \binom{n}{k} (1 - w)^{n-k} w^k \frac{R L_k(wx, wy)}{k!}. \]  \hspace{1cm} (4.4.17)

Next, performing operation \((O_2)\) on relation (4.4.8) and using equation (4.2.20a)
(for \( s(D^{-1}) = R_n(a, D^{-1}) \)) in the l.h.s. and again equation (4.2.20a) (for \( s(D^{-1}) = L_n(D^{-1}) \)) in r.h.s., we get the following identity involving \( R_n(a, x, y) \) and \( R_L(x, y) \):

\[
\frac{R_n(a, x, y)}{n!} = \frac{1}{(a-1)^n} \sum_{k=0}^{n} \frac{(a-1)_{n+k}}{k!} \frac{R_L_{n-k}(x, y)}{(n-k)!},
\]

(4.4.18)

where \( R_n(a, x, y) \) denotes the Legendre-Shively’s pseudo Laguerre polynomials.

Finally, performing operation \( \mathcal{O}_2 \) on equations (4.4.10) and (4.4.12) and using appropriate operational definitions in the resultant equations, we get the following summation formulae:

\[
\frac{R_n(x, y)}{n!} = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{2^n k! (n-2k)!} R_H_{n-2k}(x, y) \]

(4.4.19)

and

\[
\frac{R_H_{n}(x, y)}{n!} = 2^n (1 + \alpha)_n \sum_{k=0}^{n} \binom{n}{k} 2F_2 \left[ \begin{array}{c} -\frac{1}{2}(n-k), \quad -\frac{1}{2}(n-k-1); \\
-\frac{1}{2}(\alpha+n), \quad -\frac{1}{2}(\alpha+n-1); \\
\end{array} \right] \frac{R_L_k(x, y)}{(1+\alpha)_k} \]

(4.4.20)

respectively.

The above examples show that the operational formalism developed in the previous section provide a mechanism to derive the results for the Legendre-Sheffer polynomials from the ones given for Sheffer polynomials. Also, by using the correspondence between the Legendre-Appell and Appell polynomials, we can derive the results for the Legendre-Appell polynomials from results of the Appell polynomials.

### 4.5 Concluding Remarks

In the previous sections, we have seen that the monomiality principle provides a useful tool to introduce new families of special polynomials and to establish their properties. In order to further stress the importance of monomiality principle, Dattoli and Khan [29] have combined the monomiality principle with Lie algebraic techniques [103]. This combination referred as Lie-Mon method has been used to deal with Laguerre, Hermite and Legendre polynomials and has been extended to their multi-variable and multi-index counterparts, see for example [26,29,40]. Here, we derive certain results for
the Legendre-Hermite polynomials by making use of Lie-mon method.

We recall that a realization of the Lie algebra $G(0, 1)$ generating the ordinary monomials is given by the following operators [103]:

\[ J^+ = xt, \quad J^- = \frac{1}{t} D_x, \quad J^3 = x D_x, \quad \hat{E} = \hat{1}, \]

which satisfy the commutation relations:

\[ [J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = -\hat{E}. \]

We note that commutation relations (4.5.2) are identical with commutation relations (1.5.14) satisfied by the basis elements of the Lie algebra $G(0, 1)$.

In Section 4.3, it has been shown that the Legendre-Hermite polynomials $H_n(x, y)$ and $H_n(x, y)$ are quasi-monomial under the action of operators given in equations (1.3.18), (4.3.19) and (4.3.25), (4.3.26) respectively. Moreover, since these operators satisfy the commutation relation (1.4.3), therefore we can embed them to form the following $G(0, 1)$ algebra:

\[ J^+ = \hat{M} t, \quad J^- = \frac{1}{t} \hat{P}, \quad J^3 = \hat{M} \hat{P}, \quad \hat{E} = \hat{1}, \]

with the relevant commutation relations as given in equation (4.5.2).

In order to illustrate how a known result for Hermite polynomials $H_n(c)$ can be used to derive a new result involving the Legendre-Hermite polynomials $H_n(x, y)$ or $H_n(x, y)$, we consider the following generating relation for $H_n(x)$ [103, p. 106(1.76)], (for $t = 1$):

\[ \exp(2bx - b^2) H_k \left( x - b + \frac{c}{2} \right) = \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) H_l(x), \]

obtained by using the Lie algebraic methods.

Replacing $x$ by the multiplicative operator $\hat{M}_1$ of $S_n(x, y)$ given in equation (4.1.1) in the above generating relation, we have

\[ \exp \left( 2b \left( y + 2D_x^{-1} D_y \right) - b^2 \right) H_k \left( y + 2D_x^{-1} D_y - b + \frac{c}{2} \right) \]

\[ = \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) H_l(y + 2D_x^{-1} D_y) \]

which on using equation (4.2.9) (for $s_n(x) = H_n(x)$) in the r.h.s., becomes
\[
\exp \left(2b(y + 2D^{-1}D_y) - b^2\right) H_k \left(y + 2D^{-1}D_y - b + \frac{\varepsilon}{2}\right)
= \sum_{l=0}^{\infty} c^{k-l} L_i^{(k-l)}(-b)c H_l(x, y).
\] (4.5.5)

Multiplying both sides of equation (4.5.5) by \(\frac{t^k}{k!}\) and then taking the summation over \(k\), we have
\[
\exp \left(2b(y + 2D^{-1}D_y) - b^2\right) \sum_{k=0}^{\infty} H_k \left(y + 2D^{-1}D_y - b + \frac{\varepsilon}{2}\right) \frac{t^k}{k!}
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c^{k-l} L_i^{(k-l)}(-b)c H_l(x, y) \frac{t^k}{k!},
\]
which on making use of the generating function (1.2.29) of \(H_n(x)\) in the l.h.s. takes the form
\[
\exp \left(2b(y + 2D^{-1}D_y) - b^2\right) \exp \left(2 \left(y + 2D^{-1}D_y - b + \frac{\varepsilon}{2}\right) t - t^2\right)
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c^{k-l} L_i^{(k-l)}(-b)c H_l(x, y) \frac{t^k}{k!}.
\] (4.5.6)

Now, decoupling the exponential operators in the l.h.s. of equation (4.5.6) by using identity (2.2.13) and then using shift identity (4.3.22) in the resultant equation, we find
\[
\exp(-b + t^2 + ct) \exp(2(b + t)y) \exp(4(b + t)^2 D_{x}^{-1}) \{1\}
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c^{k-l} L_i^{(k-l)}(-b)c H_l(x, y) \frac{t^k}{k!},
\] (4.5.7)
which on using equation (4.1.8) in the l.h.s. yields the following generating relation involving Legendre-Hermite polynomials \(sH_n(x, y)\):
\[
\exp(-b + t^2 + ct) \exp(2(b + t)y) C_0(-4(b + t)^2x)
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c^{k-l} L_i^{(k-l)}(-b)c H_l(x, y) \frac{t^k}{k!}.
\] (4.5.8)

Similarly, replacing \(x\) by the multiplicative operator \(\hat{M}_2\) of \(\frac{R_n(x, y)}{n!}\) given in equation (4.1.3) in generating relation (4.5.4) and following the same procedure, we get the following generating relation involving Legendre-Hermite polynomials \(R H_n(x, y)\):
\[
\exp(- (b + t)^2 + ct) C_0(2(b + t)x) C_0(-2(b + t)y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) \frac{R H_l(x, y) t^k}{l!} \frac{1}{k!}
\]  
(4.5.9)

Taking \( b \to 0 \) in generating relations (4.5.8) and (4.5.9) and making use of the limit [103, p. 88(4.29)]

\[
\left. c^n L_l^n(bc) \right|_{b=0} = \begin{cases} 
\binom{n+l}{n} c^n, & \text{if } n \geq 0, \\
0, & \text{if } n < 0,
\end{cases}
\]
(4.5.10)

we get

\[
\exp(ct - t^2) \exp(2ty) C_0(-4t^2 x) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} c^{k-l} \binom{k}{l} s H_l(x, y) \frac{t^k}{k!}
\]
(4.5.11)

and

\[
\exp(ct - t^2) C_0(2tx) C_0(-2ty) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} c^{k-l} \binom{k}{l} R H_l(x, y) \frac{t^k}{k!}
\]
(4.5.12)

respectively.

Next, taking \( c \to 0 \) in generating relations (4.5.8) and (4.5.9) and making use of the limit [103, p. 88(4.29)]

\[
\left. c^n L_l^n(bc) \right|_{c=0} = \begin{cases} 
0, & \text{if } n > 0, \\
(-b)^n, & \text{if } n \leq 0,
\end{cases}
\]
(4.5.13)

we get

\[
\exp(-(b + t)^2) \exp(2(b + t)y) C_0(-4(b + t)^2 x) = \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} b^{l-k} \frac{1}{(l-k)!} s H_l(x, y) \frac{t^k}{k!}
\]
(4.5.14)

and

\[
\exp(-(b + t)^2) C_0(2(b + t)x) C_0(-2(b + t)y) = \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} b^{l-k} \frac{1}{(l-k)!} R H_l(x, y) \frac{t^k}{k!}
\]
(4.5.15)

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Here, we have obtained generating relations for the Legendre-Hermite polynomials $sH_n(x, y)$ and $rH_n(x, y)$ from a known result for the Hermite polynomials $H_n(x)$. This approach opens new possibilities to deal with other members belonging to the families of Legendre-Sheffer and Legendre-Appell polynomials.