Chapter 4

Univariate and Bivariate Pareto Processes*

4.1 Introduction

A wide variety of socioeconomic data have distributions which are heavy tailed and reasonably well fitted by Pareto or generalized Pareto distribution. Yeh et al (1988) developed an autoregressive minification process with Pareto type III marginals. Pillai (1991) generalized this by developing a semi-Pareto process. The role of the Pareto law in modeling data on income, stock price fluctuations, insurance risks, business failures etc. is well known. Moreover, it is one of the most popular distributions for modeling heavy and long tailed data.

Non-Gaussian time series models with various stationary marginal distributions have been introduced by various authors. Lawrance and Lewis (1982) consider an autoregressive structure of the form \( X_n = \varepsilon_n + V_n X_{n-1} \) where \( \{V_n\} \) are independent identically distributed (i.i.d.) binary variables with \( P(V_n = 1) = \alpha \) and \( P(V_n = 0) = 1 - \alpha \) and \( \{\varepsilon_n\} \) are assumed to be an i. i. d. sequence independent of

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* The results included in this chapter were partly presented in the form of a paper entitled “Marshall – Olkin Pareto Processes” jointly by Alice Thomas and K.K. Jose in the XXII ISPS conference held at Pune in August 2002.
{X_{n-1}, X_{n-2}, \ldots}. They also develop a $p^{th}$ order model in mixed exponential variables.

In the present chapter, we consider a new family of distributions introduced by Marshall and Olkin (1997) and similar to those introduced by Pillai, Jose and Jayakumar (1995). In section 4.2, we introduce the Marshall – Olkin semi – Pareto distributions and its properties are studied. In section 4.3, AR (1) and AR(k) models with MO-SP marginal are constructed and studied. In section 4.4, Marshall-Olkin Pareto distribution is introduced and its properties are considered. In section 4.5, estimation of parameters are discussed. In section 4.6, we introduce the Marshall – Olkin bivariate semi-Pareto distributions as a generalization of the bivariate Semi – Pareto distribution of Balakrishna and Jayakumar (1996, 1997). In section 4.7, we construct a bivariate Semi – Pareto AR (1) model with Marshall – Olkin stationary distribution. We generalize it to the $k^{th}$ order AR model in section 4.8. The model developed here is analogous to the model introduced by Lawrance and Lewis (1982) where the role of addition is taken by minimum. In section 4.9, some characteristic properties of Pareto family are established. The models can be used for modeling time series data from various contexts like stock prices and other economic contexts.

4.2 Marshall – Olkin semi – Pareto family

A random variable $X$ with positive support is said to follow semi – Pareto family of distributions denoted by $\text{SP} (\beta, \rho)$ if its survival function is of the form

$$F(x) = P(X > x)$$
where $\psi(x)$ satisfies the functional equation (3.2.2).

Substituting (4.2.1) in (2.2.1) we get a new family of distributions which we shall refer to as Marshall-Olkin semi-Pareto (MO-SP) family, whose survival function is given by

$$
\tilde{G}(x; \alpha) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)} ; \quad x \geq 0, \alpha > 0.
$$

The probability density function corresponding to $G$ is given by

$$
g(x; \alpha) = \frac{\alpha \psi(x)}{(\alpha + \psi(x))^2}.
$$

The hazard rate is given by

$$
r(x; \alpha) = \frac{\psi'(x)}{\alpha + \psi(x)}.
$$

If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having the survival function of MO-SP $\tilde{G}(x)$, then $U_n = \min(X_1, X_2, \ldots, X_n)$ has the survival function

$$
\tilde{G}_{U_n}(x) = \left(1 + \frac{1}{\alpha} \psi(x)\right)^{-n}.
$$

**Theorem 4.2.1**

Marshall-Olkin semi - Pareto distribution is geometric extreme stable

**Proof**

Proceeding as in Theorem 3.4.1 we have

$$
\tilde{G}(x) = \frac{p F(x)}{1 - (1 - p) \tilde{F}(x)} ; \quad -\infty < x < \infty.
$$

Suppose
\[ \bar{F}(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}, \]

which is the survival function of MO-SP. Then
\[ \bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}, \]

Hence \( U \) is geometric minimum stable.

Similarly
\[ \bar{H}(x) = \frac{1}{1 + p \psi(x)}. \]

Hence \( V \) is geometric maximum stable.

Hence the family of MO-SP distributions with the survival function of the form
\[ \bar{F}(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}, \]

is geometric -extreme stable.

**Theorem 4.2.2**

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function \( \bar{F}(x) \) and \( N \) be a geometric random variable with parameter \( p \) and \( P(N = n) = p q^{n-1}; n=1,2,\ldots, 0 < p < 1, q = 1-p \), which is independent of \( \{X_i\} \) for all \( i \geq 1 \). Let \( U_n = \min_{i \leq n} X_i \). Then \( \{U_n\} \) is distributed as MO-SP if and only if \( \{X_i\} \) is distributed as semi - Pareto.

**Proof**

Proceeding as in Theorem 2.2.2 we have
\[
\overline{H}(x) = \frac{p F(x)}{1 - (1 - p) F(x)}.
\]

Suppose
\[
\overline{F}(x) = \frac{1}{1 + \psi(x)},
\]
then
\[
\overline{H}(x) = \frac{1}{1 + \frac{1}{p} \psi(x)},
\]
which is the survival function of MO-SP. This proves the sufficiency part of the theorem.
Conversely, suppose
\[
\overline{H}(x) = \frac{1}{1 + \frac{1}{p} \psi(x)}.
\]
Then we get
\[
\overline{F}(x) = \frac{1}{1 + \psi(x)},
\]
which is the survival function of semi-Pareto.

The following theorem establishes the relationship between the semi-Weibull distribution and Marshall-Olkin semi-Pareto distribution.

**Theorem 4.2.3**

If \(X_1, X_2, \ldots, X_n\) are identically and independently distributed as MO-SP then \(Z_n = (n / \alpha)^{1/\beta} U_n; \ \alpha, \ \beta > 0, \ n > 1, \ n > \alpha\) is asymptotically distributed as semi-Weibull where \(U_n = \min(X_1, X_2, \ldots, X_n)\).

**Proof**

If \(X\) is distributed as MO-SP then
This establishes the theorem.

As a corollary we have the following result.

Corollary 4.2.1

If $X_1, X_2, \ldots, X_n$ are identically and independently distributed as MOP then $Z_n = (n / \alpha)^{1/\beta} U_n$; $\alpha, \beta > 0$, $n > 1$, $n > \alpha$ is asymptotically distributed as Weibull where $U_n = \min(X_1, X_2, \ldots, X_n)$.

Now using the definition of domain of attraction we have the following theorem.

Theorem 4.2.5

The Marshall–Olkin Semi- Pareto distributions (MO-SP) are in the domain of attraction for minimum of the semi-Weibull distributions.

Proof

We have

$$
\bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}.
$$

$$
F_{Z_n}(x) = P \left( (n / \alpha)^{1/\beta} U_n > x \right)
= \left( \bar{G} \left( (n / \alpha)^{1/\beta} x \right) \right)^n
= \left( 1 + \frac{1}{n} \psi(x)^x \right)^n
$$

Now $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \psi(x) \right)^n = e^{-\psi(x)}$

This establishes the theorem.

As a corollary we have the following result.

Corollary 4.2.1

If $X_1, X_2, \ldots, X_n$ are identically and independently distributed as MOP then $Z_n = (n / \alpha)^{1/\beta} U_n$; $\alpha, \beta > 0$, $n > 1$, $n > \alpha$ is asymptotically distributed as Weibull where $U_n = \min(X_1, X_2, \ldots, X_n)$.

Now using the definition of domain of attraction we have the following theorem.

Theorem 4.2.5

The Marshall–Olkin Semi- Pareto distributions (MO-SP) are in the domain of attraction for minimum of the semi-Weibull distributions.

Proof

We have

$$
\bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}.
$$

Taking $a_n = 0$ and $b_n = (n/\alpha)^{1/\beta} > 0$, we have
Theref

\[ L_n(a_n + b_n x) = 1 - \left( F \left( (n / \alpha)^{-\beta} x \right) \right)^n \]

\[ = 1 - \left( 1 + \frac{1}{n} \psi(x) \right)^{-n} \]

Therefore

\[ \lim_{n \to \infty} L_n(a_n + b_n x) = 1 - e^{-\psi(x)}. \]

Hence \( L(x) = 1 - e^{-\psi(x)} \) which is the cumulative distribution of the semi-Weibull distributions and this establishes the theorem.

4.3 An AR (1) model with MO-SP marginal distributions

Theorem 4.3.1

Consider an AR (1) structure given by (2.3.1), where \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables independent of \( \{X_{n-1}, X_{n-2}, \ldots\} \), then \( \{X_n\} \) is a stationary Markovian AR(1) process with MO-SP marginals if and only if \( \{\varepsilon_n\} \) is distributed as semi-Pareto distributions.

**Proof**

Proceeding as in the case of theorem 2.3.1 if we take

\[ \overline{F}_\varepsilon(x) = \frac{1}{1 + \psi(x)}, \]

then it easily follows that

\[ \overline{F}_X(x) = \frac{1}{1 + \frac{1}{\alpha} \psi(x)}, \]

which is the survival function of MO-SP.

Conversely, if we take,
it is easy to show that $F_{\varepsilon n}(x)$ is distributed as semi-Pareto and the process is stationary. In order to establish stationarity we proceed as follows.

Assume $X_{n-1} \overset{d}{=} \text{MO-SP}$ and $\varepsilon_n \overset{d}{=} \text{semi-Pareto}.$

Then

$$F_{X_n}(x) = \frac{1}{1 + \frac{x}{\alpha}},$$

This establishes that $\{X_n\}$ is distributed as MO-SP. Even if $X_0$ is arbitrary, it is easy to establish that $\{X_n\}$ is stationary and is asymptotically marginally distributed as MO-SP.

**Theorem 4.3.2**

Consider an autoregressive minification process $X_n$ of order $k$ with structure (2.5.1). Then $\{X_n\}$ has stationary marginal distribution as MO-SP if and only if $\{\varepsilon_n\}$ is distributed as semi-Pareto.

### 4.4 Marshall-Olkin Pareto distribution and its properties

We consider the Pareto (type III) distribution with the survival function

$$F(x; \beta) = \frac{1}{1 + x^\beta} : x \geq 0, \beta > 0.$$ 

Substituting this in (2.2.1), we get a new family of distributions which we shall refer to as Marshall-Olkin generalized Pareto (MO-GP) family, whose survival function is given by
\[ \bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta}; \quad x \geq 0, \beta, \alpha > 0. \]

Table 4.1 gives the cumulative distribution function of MO-GP distribution.

The probability density function corresponding to \( G \) is given by

\[ g(x) = \frac{\beta \alpha x^{\beta-1}}{(\alpha + x^\beta)^2}; \quad x \geq 0, \alpha > 0, \beta > 0. \]

The hazard rate is given by

\[ r(x) = \frac{x^{\beta-1}}{(\alpha + x^\beta)}; \quad x \geq 0, \alpha > 0, \beta > 0. \]

The 1\textsuperscript{st} moment about zero is

\[ E(X) = \left(\frac{\Gamma(1/\beta)}{\beta}\right) (1 + \frac{1}{\beta} x^\beta); \quad 0 < 1 \leq 1 \]

and the variance is

\[ V(X) = \alpha^{2/\beta} \left\{ \Gamma(1 + (2/\beta)) \Gamma(1 - (2/\beta)) - \left( \Gamma(1 + (1/\beta)) \Gamma(1 - (1/\beta)) \right)^2 \right\} \]

if \( \beta > 2 \). \hfill (4.4.3)

If \( \{X_i, i = 1, 2, \ldots, n\} \) are independent and identically distributed random variables having the survival function of MO-GP \( \bar{G}(x) \), then \( \{\min X_i, i = 1, 2, \ldots, n\} \) has the survival function \( \left( 1 + \frac{1}{\alpha} x^\beta \right)^n \).

Figure 4.1 gives the graph of the probability density function of \( g(x) \) for various values of \( \alpha \) and \( \beta \). Figure (4.2) gives the graph of the hazard rate \( r(x) \). This points out the wide applicability of the MO-GP distribution for modeling various types of financial and socio-economic data.
Theorem 4.4.1
Marshall-Olkin generalized Pareto distribution is geometric extreme stable

Proof
Proceeding as in Theorem 3.4.1, we have

$$\bar{G}(x; p) = \frac{p F(x)}{1 - (1 - p) F(x)} \quad (-\infty < x < \infty).$$

Suppose

$$\bar{F}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta}.$$ 

which is the survival function of MO-GP. Then

$$\bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha p} x^\beta}.$$ 

Hence U is geometric minimum stable.

Similarly

$$\bar{H}(x) = \frac{1}{1 + \frac{p}{\alpha} x^\beta}.$$ 

Hence V is geometric maximum stable.

Hence the family of MO-GP distributions with the survival function of the form

$$\bar{F}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta}$$

is geometric-extreme stable.
Theorem 4.4.2

Let \{X_i, i \geq 1\} be a sequence of independent and identically distributed random variables with common survival function \(\overline{F}(x)\) and \(N\) be a geometric random variable with parameter \(p\) and \(P(N = n) = p q^{n-1}; n=1,2,\ldots, 0 < p < 1, q = 1-p\), which is independent of \{X_i\} for all \(i \geq 1\). Let \(U_N = \min_{1 \leq i \leq N} X_i\). Then \{U_N\} is distributed as MOGP if and only if \{X_i\} is distributed as Pareto.

Proof

Proceeding as in theorem 2.2.2 we have

\[
\overline{H}(x) = \frac{p F(x)}{1 - (1 - p) F(x)}.
\]

Suppose

\[
\overline{F}(x) = \frac{1}{1 + x^\beta},
\]

then

\[
\overline{H}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta},
\]

which is the survival function of MO-GP. This proves the sufficiency part of the theorem.

Conversely, suppose

\[
\overline{H}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta}.
\]

Then we get

\[
\overline{F}(x) = \frac{1}{1 + x^\beta},
\]

which is the survival function of Pareto.
Theorem 4.4.3

Consider an AR (1) structure given by (2.3.1), where \{\varepsilon_n\} is a sequence of independent and identically distributed random variables independent of \{X_{n-1}, X_{n-2}, \ldots\}, then \{X_n\} is a stationary Markovian AR(1) process with MO-GP marginals if and only if \{\varepsilon_n\} follows the Pareto distribution.

Proof

Proceeding as in the case of Theorem 2.3.1 if we take

\[ \bar{F}_{\varepsilon}(x) = \frac{1}{1 + \alpha x^\beta} , \]

then it easily follows that

\[ \bar{F}_X(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta} , \]

which is the survival function of MO-GP.

Conversely if we take

\[ \bar{F}_{X_n}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta} , \]

it is easy to show that \(F_{\varepsilon,n}(x)\) is distributed as Pareto and the process is stationary. In order to establish stationarity we proceed as follows.

Assume \(X_{n-1} \overset{d}{=} \text{MO-GP} \text{ and } \varepsilon_n \overset{d}{=} \text{Pareto}. \)

Then

\[ \bar{F}_{X_n}(x) = \frac{1}{1 + \frac{1}{\alpha} x^\beta} . \]

This establishes that \{X_n\} is distributed as MO-GP. Even if \(X_0\) is arbitrary, it is easy to establish that \{X_n\} is stationary and is asymptotically marginally distributed as MO-GP.
Theorem 4.4.4
Consider an autoregressive minification process $X_n$ of order $k$ with structure (2.5.1). Then $\{X_n\}$ has stationary marginal distribution as MO-GP if and only if $\{\epsilon_n\}$ is distributed as Pareto.

4.4.1 Sample path behaviour
The sample path of the process for different values of $p$ and $\lambda$ are given in figure 4.4. The simulated sample path using 100 observations generated from the MO-GPAR (1) process with $p=0.3, 0.5$ and $0.9$ and $\alpha = 1.5$ and $10$ are given respectively in the figure 4.5. The sample path behaviour of the MO-GPAR (1) process seems to be distinctive and is adjustable through the parameters $p$ and $\alpha$. This makes the model very rich. Mainly there seems to be three cases:

(1) runs of increasing values – up runs  
(2) runs of decreasing values – down runs  
(3) both (peaks).

These observations can be verified by referring to the Table 4.2 showing $P(X_n < X_{n-1})$. These probabilities are obtained through a Monte Carlo simulation procedure. Sequences of 100, 300, 500, 700, 900 observations from MO-GPAR (1) process are generated repeatedly for ten times and for each sequence the probability is estimated. A table of such probabilities is provided with the average from ten trials along with an estimate of standard error (see Table 4.2).

4.5. Estimation of parameters
In this section we consider the problem of estimating the parameters of the MO-GP distribution which can be tackled in a way
similar to those of Johnson and Kotz (1970). We assume that $X_1, X_2, \ldots, X_n$ are independent random variables each distributed as MO-GP.

### 4.5.1 Estimators from moments

Provided $\beta > 2$, from (4.4.2) and (4.4.3) we have the coefficient of variation (CV) given by

$$CV = \left( \frac{[\Gamma((2/\beta)+1)\Gamma(1-(2/\beta))-(\Gamma((1/\beta)+1)\Gamma(1-(1/\beta)))^2]}{\Gamma((1/\beta)+1)\Gamma(1-(1/\beta))} \right)^{1/2}. \quad (4.5.1)$$

Now we can form a table for various CV by using (4.5.1) for different $\beta$ values. In order to estimate $\alpha$ and $\beta$, we need to calculate the coefficient of variation of the data ((CV)$_d$) on hand. Having done this, we compare (CV)$_d$ with CV using table. The corresponding $\beta$ is the estimated one ($\beta^*$). The parameter $\alpha$ can then be estimated using the following.

$$\tilde{x} = \alpha^{1/\beta} \frac{\Gamma(1+(1/\beta))\Gamma(1-(1/\beta))}{\Gamma(1+(1/\beta))\Gamma(1-(1/\beta))} ; \beta < 1.$$ 

$$\alpha^* = \left( \frac{\tilde{x}}{\Gamma(1+(1/\beta))\Gamma(1-(1/\beta))} \right)^\beta,$$

where $\tilde{x}$ is the mean of the data.

### 4.5.2 Maximum likelihood estimators

The likelihood function for a sample $(x_1, x_2, \ldots, x_n)$ from MO-GP distribution is

$$L = \frac{\beta^n \alpha^s}{\prod_{i=1}^n x_i^{\beta+1} \prod_{i=1}^n (1+\alpha x_i^{-\beta})^2}.$$ 

Taking logarithms on both sides, differentiating partially with respect to the parameters $\alpha$ and $\beta$ and setting the results to zero we have
\[
\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{x_i^{-\beta}}{1 + \alpha x_i^{-\beta}} = 0
\]
\[
\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} \sum_{i=1}^{n} \log x_i + 2 \sum_{i=1}^{n} \frac{x_i^{-\beta} \log x_i}{1 + \alpha x_i^{-\beta}} = 0.
\]
Solving we find the relation
\[
\beta^0 = \frac{n \alpha}{(1 - \alpha) \sum \log x_i} \tag{4.5.2}
\]
between the maximum likelihood estimators \( \hat{\alpha}, \hat{\beta} \) of \( \alpha, \beta \) respectively.

To find an estimate for \( \alpha \) first fix the value of \( \beta \). Then by Newton-Raphson's method find the value of \( \alpha \). This gives the estimate of \( \alpha \). Then using (4.5.2) we can find an estimate of \( \beta \).

### 4.5.3 Estimation from Quantiles

Select two numbers \( P_1 \) and \( P_2 \) between 0 and 1 and obtain estimators of the respective quantiles \( \hat{X}_{P_1} \) and \( \hat{X}_{P_2} \). Estimators of \( \beta \) and \( \alpha \) are then obtained by solving the two simultaneous equations
\[
P_j = 1 - \frac{1}{1 + (\hat{X}_{P_1}/\alpha)} \quad : \quad j = 1, 2. \tag{4.5.3}
\]

The estimator of \( \beta \) is
\[
\beta^0 = \log \left( \frac{P_j (1 - P_{1-j})}{P_{j-1} (1 - P_j)} \right) \cdot \log \left( \frac{\hat{X}_{P_{1-j}}}{\hat{X}_{P_j}} \right).
\]

The corresponding estimator of \( \alpha \) can be obtained from (4.5.3).
4.6 Marshall – Olkin bivariate semi - Pareto distributions

A random vector \((X, Y)\) is said to have the bivariate semi-Pareto distribution with parameters \(\alpha, \beta, p\) if its survival function is of the form

\[
\bar{F}(x, y) = P(X > x, Y > y) = \frac{1}{1 + \psi(x, y)}
\]

(4.6.1)

where \(\psi(x, y)\) satisfies the functional equation (3.6.3).

From (3.6.1) we can see that the new survival function is

\[
\bar{G}(x, y;\alpha) = \frac{1}{1 + \frac{1}{\alpha} \psi(x, y)} ; \ x, y \geq 0, \ \alpha > 0,
\]

(4.6.2)

which we shall refer to as Marshall-Olkin bivariate semi-Pareto distribution denoted as MO-BSP.

**Theorem 4.6.1**

If \(\{(X_i, Y_i), i \geq 1\}\) be a bivariate sequence of non-negative random vectors identically and independently distributed as MO-BSP, then \(Z_n = ((n/\alpha)^{1/\beta_1} U_n, (n/\alpha)^{1/\beta_2} V_n); \ \beta_1, \ \beta_2 > 0, \ n > 1, \ n > \alpha\) is asymptotically distributed as bivariate semi-Weibull where \(U_n = \min (X_1, X_2, ..., X_n)\) and \(V_n = \min (Y_1, Y_2, ..., Y_n)\).

**Proof**

If \((X, Y)\) is distributed as MO-BSP, then from (4.6.2) we have

\[
\bar{G}(x, y) = \frac{1}{1 + \frac{1}{\alpha} \psi(x, y)}
\]

\[
\bar{F}_{Z_n}(x, y) = P[(n/\alpha)^{1/\beta_1} U_n > x, (n/\alpha)^{1/\beta_2} V_n > y]
\]

\[
= \left(\bar{G}((n/\alpha)^{1/\beta_1} x, (n/\alpha)^{-1/\beta_2} y)\right)^n
\]

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Now \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \psi(x, y) \right)^{-n} = e^{-\psi(x, y)} \).

This establishes the theorem. As a corollary we have the following result.

**Corollary 4.6.1**

If \((X_i, Y_i), i \geq 1\) be a sequence of bivariate non-negative random vectors identically and independently distributed as MO-BP, then \( Z_n = ((n/\alpha)^{1/\beta_1} U_n, (n/\alpha)^{1/\beta_2} V_n); \beta_1, \beta_2 > 0, n > 1, n > \alpha \) is asymptotically distributed as bivariate Weibull where

\[
U_n = \min(X_1, X_2, \ldots, X_n) \quad \text{and} \quad V_n = \min(Y_1, Y_2, \ldots, Y_n).
\]

As in Castillo (1988) we can extend the concept of domain of attraction to the multivariate setup.

**Theorem 4.6.2**

The Marshall-Olkin bivariate semi-Pareto distributions (MO-BSP) are in the domain of attraction for minimum of the bivariate semi-Weibull distributions.

**Proof**

We have

\[
\bar{F}(x, y) = \frac{1}{1 + \frac{1}{\alpha} \psi(x, y)}.
\]

Taking \( a_n = (0, 0)' \) and \( b_n = ((n/\alpha)^{1/\beta_1} > 0, (n/\alpha)^{-1/\beta_2} > 0) \), we have

\[
L_n(a_n + b_n, X) = 1 - \left( G((n/\alpha)^{-1/\beta_1} x, (n/\alpha)^{-1/\beta_2} y) \right)^n
= 1 - \left( 1 + \frac{1}{n} \psi(x, y) \right)^{-n}
\]

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Therefore
\[ \lim_{n \to \infty} L_n(a_n + b_n x) = 1 - e^{-\psi(x, y)}. \]

Hence \( L(x) = 1 - e^{-\psi(x, y)} \), which is the cumulative distribution of the bivariate semi-Weibull distribution and this establishes the theorem.

Now we consider a bivariate AR (1) model \{\((X_n, Y_n)\)\} having stationary marginal distribution as the MO-BSP and innovations \{\((U_n, V_n)\)\} distributed as bivariate semi-Pareto.

### 4.7 Marshall – Olkin bivariate semi-Pareto AR (1) model

**Theorem 4.7.1**

Consider a bivariate autoregressive minification process \{\((X_n, Y_n)\)\} having a structure (3.7.1). Then \{\((X_n, Y_n)\)\} has stationary marginal distribution as MO-BSP if and only if \{\((U_n, V_n)\)\} is jointly distributed as bivariate semi-Pareto distribution.

**Proof**

From (3.7.1)
\[ \tilde{F}_{X_n,Y_n}(x,y) = P(X_n > x, Y_n > y) \]
\[ = p G_{U_n,V_n}(x,y) + (1 - p) \tilde{F}_{X_{n-1},Y_{n-1}}(x,y) \tilde{G}_{U_n,V_n}(x,y) \]  \( (4.7.1) \)

Under stationarity
\[ \tilde{F}_{X,Y}(x,y) = p \tilde{G}_{U,V}(x,y) / [1 - (1-p) \tilde{G}_{U,V}(x,y)] \]

If we take
\[ \tilde{G}_{U,V}(x,y) = 1 / \{1 + \psi(x,y)\} \]
\[ \tilde{F}_{X,Y}(x,y) = p / \{p + \psi(x,y)\} \]

which is the survival function of MO-BSP.

Conversely if we take
\[ F_{X,Y}(x, y) = p / \{ p + \psi(x, y) \} \]

it is easy to show that \( G_{U,V}(x, y) \) is distributed as bivariate semi- Pareto distribution and the process is stationary. In order to establish stationarity we proceed as follows.

Assume \( \{(X_n, Y_n)\} \overset{d}{=} \text{MO-BSP} \) and \( \{(U_n, V_n)\} \overset{d}{=} \text{BSP} \).

Then from (4.7.1)

\[ F_{X,Y}(x, y) = p / \{ p + \psi(x, y) \} \]

This establishes that \( \{(X_n, Y_n)\} \) is distributed as \( \text{MO-BSP} \). Even if \( (X_0, Y_0) \) is arbitrary, it is easy to establish that \( \{(X_n, Y_n)\} \) is stationary and is asymptotically marginally distributed as \( \text{MO-BSP} \).

### 4.8 Generalization to the k-th order autoregressive model

A \( k^{th} \) order autoregressive model has the structure

\[
\begin{align*}
X_n & = \begin{cases} 
U_n & \text{w.p.} \ p_0 \\
\min(X_{n-1}, U_n) & \text{w.p.} \ p_1 \\
\min(X_{n-2}, U_n) & \text{w.p.} \ p_2 \\
\vdots & \\
\min(X_{n-k}, U_n) & \text{w.p.} \ p_k 
\end{cases} \\
Y_n & = \begin{cases} 
V_n & \text{w.p.} \ p_0 \\
\min(Y_{n-1}, V_n) & \text{w.p.} \ p_1 \\
\min(Y_{n-2}, V_n) & \text{w.p.} \ p_2 \\
\vdots & \\
\min(Y_{n-k}, V_n) & \text{w.p.} \ p_k 
\end{cases}
\end{align*}
\]

(4.7.1)

where \( 0 < p_i < 1, \sum_{i=1}^{k} p_i = 1 - p_0 \). Then \( \{(X_n, Y_n)\} \) has stationary marginal distribution as \( \text{MO-BSP} \) if and only if \( \{(U_n, V_n)\} \) is jointly distributed as bivariate semi-Pareto distribution.
4.9 Characterizations

In the following section we consider the special case namely bivariate Pareto distribution having the survival function

$$F(x, y) = \frac{1}{1 + x^{\beta_1} + y^{\beta_2}}; \quad x, y \geq 0, \beta_1, \beta_2 > 0.$$  

From (3.6.1) the new survival function is

$$G(x, y) = \frac{1}{1 + \alpha (x^{\beta_1} + y^{\beta_2})}; \quad x, y \geq 0, \beta_1, \beta_2 > 0, 0 < \alpha < 1,$$

which is known as Marshall-Olkin bivariate Pareto(MO-BP) distribution.

The density function is given by

$$g(x, y) = 2\beta_1, \beta_2, x^{\beta_1 - 1} y^{\beta_2 - 1} \frac{1}{\alpha^2} \left(1 + (1/\alpha)(x^{\beta_1} + y^{\beta_2})\right)^{-3}; \quad x, y \geq 0, \beta_1, \beta_2,$$

$$0 < \alpha < 1.$$  

The marginal distributions of $X$ and $Y$ are

$$g(x) = \frac{\beta_1}{\alpha} x^{\beta_1 - 1} \left(1 + (1/\alpha)x^{\beta_1}\right)^{-2}; \quad x \geq 0, \beta_1, 0 < \alpha < 1.$$  

and

$$g(y) = \frac{\beta_2}{\alpha} y^{\beta_2 - 1} \left(1 + (1/\alpha)y^{\beta_2}\right)^{-2}; \quad y \geq 0, \beta_2, 0 < \alpha < 1.$$  

$$E(X_i) = \left(\alpha^2/\beta_i\right)^{2/\beta_i} \left(1 + (1/\alpha)r \beta_i^{-1} \alpha^{-1} \beta_i^{-1}\right)^{1/\beta_i}; \quad r < \beta_i.$$  

$$V(X_i) = \alpha^2/\beta_i \left\{\Gamma(1 + (2/\beta_i))\left(\Gamma(1 - (2/\beta_i)) - (\Gamma(1 + (1/\beta_i))\Gamma(1 - (1/\beta_i)))\right)^2\right\};$$

$$\text{if } (\beta_i > 2).$$

$$E(XY) = 2\alpha^{\beta_1 + \beta_2} \left(\beta_i + 1, 2 - \frac{1}{\beta_i}\right)\beta_i \left(\frac{1}{\beta_i} + 1, 1 - \frac{1}{\beta_i} - \frac{1}{\beta_i}\right).$$

Using these the correlation between $X$ and $Y$ can be obtained.

Figure 4.4 gives the graph of the survival function of MO-BP.
Theorem 4.9.1

Consider a bivariate autoregressive minification process \((X_n, Y_n)\) having structure (3.7.1). Then \(\{(X_n, Y_n)\}\) has stationary marginal distribution as MO-BP distribution if and only if \(\{(U_n, V_n)\}\) is jointly distributed as bivariate Pareto distribution.

Proof

Proceeding as in Theorem 3.7.1 if we take
\[
\tilde{G}_{U,Y}(x, y) = \frac{1}{1 + x^\beta + y^\beta},
\]
we get
\[
\tilde{F}_{X,Y}(x, y) = \frac{1}{1 + \frac{1}{\alpha}(x^\beta + y^\beta)}
\]
which is the survival function of MO-BP.

Conversely if we take
\[
\tilde{F}_{X,Y}(x, y) = \frac{1}{1 + \frac{1}{\alpha}(x^\beta + y^\beta)}
\]
it is easy to show that \(G_{U,Y}(x, y)\) is distributed as bivariate Pareto distribution and the process is stationary.

Theorem 4.9.2

Let \(N\) be a geometric random variable with parameter \(p\) and \(P\{N = n\} = p \cdot q^{n-1}, n = 1, 2, \ldots, 0 < p < 1, q = 1 - p.\) Consider a sequence \(\{(X_i, Y_i), i \geq 1\}\) be independently and identically distributed random vectors with common survival function \(\tilde{F}(x, y).\) \(N\) and \((X_i, Y_i)\) are independent for all \(i \geq 1.\) Let \(U_N = \min_{i \leq N} X_i\) and \(V_N = \min_{i \leq N} Y_i\). Then the random vectors \((U_N, V_N)\) are distributed as MO-BP if and only if \((X_i, Y_i)\) have BP distribution.
Proof

Consider

\[ \bar{S}(x, y) = P\left[U_N > x, V_N > y\right] \]

\[ = \sum_{n=1} \left[ \bar{F}(x, y)\right]^n p q^{n-1} \]

\[ = p \bar{F}(x, y) / [1 - (1-p) \bar{F}(x, y)] \]

Let \( \bar{F}(x, y) = \frac{1}{1 + x^{\beta_1} + y^{\beta_2}} \)

which is the survival function of bivariate Pareto.

Substituting this in the above equation we have

\[ \bar{S}(x, y) = \frac{1}{1 + \frac{1}{\alpha} (x^{\beta_1} + y^{\beta_2})} = \bar{G}(x, y) \]

which is the survival function of MO-BP.

Conversely suppose that

\[ \bar{S}(x, y) = \frac{1}{1 + \frac{1}{\alpha} (x^{\beta_1} + y^{\beta_2})} \]

Then \( p \bar{F}(x, y) / [1 - (1-p) \bar{F}(x, y)] = \frac{1}{1 + \frac{1}{\alpha} (x^{\beta_1} + y^{\beta_2})} \).

On simplifying we get

\[ \bar{F}(x, y) = \frac{1}{1 + x^{\beta_1} + y^{\beta_2}}. \]

Hence the proof is complete.

Theorem 4.9.3

Let \( N \) be a geometric random variable with parameter \( p \) and \( P\{N = n\} = p q^{n-1}, n = 1, 2, \ldots, 0 < p < 1, q = 1-p. \) Consider a sequence \( \{(X_i, Y_i), i \geq 1\} \) be independently and identically distributed random vectors with common survival function \( \bar{F}(x, y). \) \( N \) and \( (X_i, Y_i) \) are
independent for all \(i \geq 1\). Let \(U_N = \min_{i \leq N} X_i\) and \(V_N = \min_{i \leq N} Y_i\). Then the random vectors \((U_N, V_N)\) are distributed as MO-BSP if and only if \((X_i, Y_i)\) have BSP distribution.

**Proof**

Proceeding as in Theorem 4.9.2 we have

\[
\overline{S}(x, y) = p \overline{F}(x, y) / [1 - (1 - p) \overline{F}(x, y)]
\]

Let \(\overline{F}(x, y) = 1 / \{1 + \psi(x, y)\}\),

which is the survival function of bivariate semi-Pareto.

Substituting this in the above equation we have

\[
\overline{S}(x, y) = p / \{p + \psi(x, y)\} = \overline{G}(x, y),
\]

which is the survival function of MOBSP.

Conversely suppose that

\[
\overline{S}(x, y) = p / \{p + \psi(x, y)\}.
\]

Then

\[
p \overline{F}(x, y) / [1 - (1 - p) \overline{F}(x, y)] = p / \{p + \psi(x, y)\}
\]

On simplifying we get

\[
\overline{F}(x, y) = 1 / \{1 + \psi(x, y)\}.
\]

Hence the proof is complete.

Now we shall establish another characterization of the MO-BSP distribution.

Let \(\{N_k, k \geq 1\}\) be a sequence of geometric random variables with parameters \(p_k, 0 \leq p_k < 1\).

Define

\[
\overline{F}_k(x, y) = P[U_{N_{k-1}} > x, V_{N_{k-1}} > y], k = 2, 3, \ldots
\]

\[
= p_{k-1} \overline{F}_{k-1}(x, y) / \{1 - (1 - p_{k-1}) \overline{F}_{k-1}(x, y)\}
\]

(4.9.3)

Here we refer \(\overline{F}_k\) as the survival function of the geometric \((p_{k-1})\)minimum of independent and identically distributed random vectors with \(\overline{F}_{k-1}\) as the common survival function.
Theorem 4.9.4

Let \( \{(X_i, Y_i), \ i \geq 1\} \) be a sequence of independent and identically distributed non-negative random vectors with common survival function \( \bar{G}(x,y) \). Define \( \bar{F}_1 = \bar{G} \) and \( \bar{F}_k \) as the survival function of the geometric \((p_{k-1})\) minimum of independent and identically distributed random vectors with common survival function \( \bar{F}_{k-1}, \ k = 2,3,\ldots \) Then
\[
\bar{F}_k(x, y) = \bar{G}(x, y) \quad (4.9.4)
\]
if and only if \((X_i, Y_i)\) have MO-BSP distributions.

Proof

By definition, the survival function \( \bar{F}_k \) satisfies the equation (4.9.3).

We have
\[
\bar{G}(x, y) = \frac{1}{1 + \frac{1}{p} \gamma(x, y)} = \frac{1}{1 + \phi(x, y)},
\]
where \( \phi(x, y) \) is a monotonically increasing function in both \( x \) and \( y \) \((x \geq 0, y \geq 0)\) and
\[
\lim_{x \to 0} \lim_{y \to 0} \phi(x, y) = 0 \text{ and } \lim_{x \to \infty} \lim_{y \to \infty} \phi(x, y) = \infty.
\]

So we can write
\[
\bar{F}_k(x, y) = \frac{1}{1 + \phi_k(x, y)}; \ k = 1, 2,\ldots
\]

Substituting this in (4.9.3), we get
\[
\phi_k(x, y) = \frac{\phi_{k-1}(x, y)}{p_{k-1}}; \ k = 2, 3, 4\ldots
\]
Recursively using this relation we have
\[ \theta_k(x, y) = \frac{\theta_1(x, y)}{p_1 p_2 \ldots p_{k-1}}, \]
\[ \text{since } \overline{F}_1 = \overline{G} \text{ implies that } \phi_1 = \phi. \]

This implies that
\[ \theta_k(x, y) = \frac{\theta_1(x, y)}{p_1 p_2 \ldots p_{k-1}} \quad (4.9.5) \]
Hence \[ \overline{F}_k(x, y) = \overline{G}(x, y) \]

This proves the sufficiency part.

Conversely assume that equation (4.9.4) is true. By the hypothesis of the theorem equation (4.9.5) follows.

Thus equation (4.9.4) and equation (4.9.5) together lead to the equation
\[ |1 + \frac{1}{p_1 p_2 \ldots p_{k-1}} \phi_1[(x, y)]^{-1} = \overline{G}(x, y) \]
\[ = \frac{1}{1 + \phi(x, y)} \]
This implies that
\[ \phi(x, y) = \frac{\phi_1(x, y)}{p_1 p_2 \ldots p_{k-1}}. \]

Hence the proof is complete.
Table 4.1 c.d.f. table for the MOGP distribution for $\beta = 1$

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p.d.f. of MOGP for various values of $\alpha$ and $\beta = 2$
Fig 4.2

Hazard rate function of MOGP for various values of $\alpha$ and $\beta = 2$
Table 4.2  \( P( X_n < X_{n-1}) \) for the MO-GPAR(1) process with \( \beta = 2 \).

(Standard error in brackets)

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<th>700</th>
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Fig 4.3

Sample paths of MOGPAR (1) Process for different values of \( p \) and \( \beta \).
$\rho = 0.9$, $\alpha = 2$, $\beta = 5$

Fig 4.4

Survival function - MOBP