Chapter 6

Extreme Value Distributions and Processes

6.1 Introduction

Extreme value theory is a blend of an enormous variety of applications involving natural phenomena such as rainfall, floods, wind gusts, air pollution, and corrosion, and delicate mathematical results on point processes and regularly varying functions. Extreme value theory seems to have originated mainly from the needs of astronomers in utilizing or rejecting outlying observations. The early papers by Fuller (1914) and Griffith (1920) on the subject were highly specialized both in fields of applications and in methods of mathematical analysis.

The importance of the Gumbel distribution in practice is due to its extreme value behaviour. It has been applied either as the parent distribution or as an asymptotic approximation, to describe extreme wind speeds, sea wave heights, floods, rainfall, age at death, minimum temperature, rainfall during droughts, electrical strength of materials, air pollution problems, geological problems, naval engineering etc.

The importance of the Weibull distribution is due to its extreme value behaviour. It has been applied to model fatigue strength of materials, lifetime of vacuum tubes, electrical insulations, ballbearings, human reliability, semiconductor devices, motors, capacitors, photoconductive cells, corrosion resistance, leakage failure of batteries etc. The importance of the Frechet distribution in practice is due to its
extreme value behaviour. It has been applied to data on characteristics of sea waves, wind speeds, etc.

In section 6.2, the new Marshall – Olkin Gumbel maximum (MO-GUMX) distribution is introduced and its distributional properties are investigated. The graphs corresponding to probability density function, distribution function and hazard rate function are given. Some characteristic properties are established. In section 6.3, we construct the first order and $k^{\text{th}}$ order autoregressive model having minification structure. The sample path behaviour is also considered. In section 6.4, we considered the new Marshall – Olkin Gumbel minimum (MO-GUMN) distribution and its properties are studied. In section 6.5, we consider AR (1) model and AR(k) model with MO-GUMN marginal. In section 6.6, new Marshall – Olkin Frechet distribution is considered and its properties are studied. In section 6.7, the first and $k^{\text{th}}$ order autoregressive minification structure with new Marshall – Olkin Frechet maximum (MO-FRMX) marginal is constructed. In section 6.8, we find MO-FRMN distribution and its properties are studied. In section 6.9, we construct the AR (1) and AR (k) model with MO-FRMN marginal distribution.

6.2 Marshall – Olkin generalized Gumbel family

Consider the Gumbel maximum distribution with survival function (see Castillo (1988))

$$\tilde{F}(x) = 1 - \exp (-\exp\left(-\frac{x-\lambda}{\delta}\right)); \ -\infty < x < \infty ; \delta > 0$$

where $\lambda$ and $\delta$ are constants known as the location and scale parameters. Substituting this in (2.2.1) we get a new family of
distributions which we shall refer to as MO-GUMX family, whose
survival function is given by
\[ G(x) = \frac{\alpha(1 - \exp[-\exp(-\frac{(x-\lambda)}{\delta})])}{1 - (1 - \alpha)[1 - \exp[-\exp(-\frac{(x-\lambda)}{\delta})]]}; \quad -\infty < x < \infty, \delta, \alpha > 0. \]

The probability density function is
\[ g(x) = \frac{\alpha \exp(-\frac{(x-\lambda)}{\delta})[\exp[-\exp(-\frac{(x-\lambda)}{\delta})]]}{\delta[1 - (1 - \alpha)[1 - \exp[-\exp(-\frac{(x-\lambda)}{\delta})]]]^2}; \quad -\infty < x < \infty, \delta, \alpha > 0. \]

The hazard rate function is
\[ r(x) = \frac{\exp(-\frac{(x-\lambda)}{\delta})[\exp[-\exp(-\frac{(x-\lambda)}{\delta})]]}{\delta[1 - (1 - \alpha)[1 - \exp[-\exp(-\frac{(x-\lambda)}{\delta})]]][1 - \exp[-\exp(-\frac{(x-\lambda)}{\delta})]]}. \]

Figure 6.1 gives the graph of \( r(x) \) for different values of \( \alpha \) when \( \lambda = 38 \) and \( \delta = 7 \). When \( \alpha < .5 \), the curve at first increases and then decreases. When \( \alpha > .5 \), it is increasing.

Let \( Y = (X - \lambda) / \delta \). Then the moment generating function is
\[ M_Y(t) = \alpha e^{-\lambda \delta t} \Gamma(1 - \frac{t}{\delta}) \sum_{i=0}^{\infty} \left( \frac{\alpha}{\lambda} \right)^i (i + 1)^{\delta t}. \]

**Theorem 6.2.1**

Marshall-Olkin Gumbel Maximum distribution is geometric extreme stable.

Proof is similar to Theorem 3.4.1.

**Theorem 6.2.2**

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function
\(F(x)\) and \(N\) be a geometric random variable with parameter \(p\) and 
\(P(N = n) = p \cdot q^{n-1}; \ n = 1, 2, \ldots, \ 0 < p < 1, \ q = 1-p,\) which is independent of \(\{X_i\}\) for all \(i \geq 1\). Let \(U_N = \min_{i \leq N} X_i\). Then \(\{U_N\}\) is distributed as MO-GUMX if and only if \(\{X_i\}\) is distributed as GUMX.

Proof is similar to theorem 3.4.1

6.3 An AR (1) model with MO-GUMX marginal distribution

Theorem 6.3.1

Consider an AR (1) structure given by (2.3.1), where \(\{\epsilon_n\}\) is a sequence of independent and identically distributed random variables independent of \(\{X_{n-1}, X_{n-2}, \ldots\}\), then \(\{X_n\}\) is a stationary Markovian AR (1) process with MO-GUMX marginals if and only if \(\{\epsilon_n\}\) is distributed as GUMX distribution.

Proof is similar to theorem 2.3.1

Theorem 6.3.2

Consider an autoregressive minification process \(X_n\) of order \(k\) with structure (2.5.1). Then \(\{X_n\}\) has stationary marginal distribution as MO-GUMX if and only if \(\{\epsilon_n\}\) is distributed as GUMX.

The sample path of the process for different values of \(p, \lambda\) and \(\delta\) are given in Figure 6.7.

6.4 Marshall – Olkin Gumbel distribution for minimum.

Consider Gumbel minimum distribution with survival function (See Castillo (1988))
\[ F(x) = \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right]; \quad -\infty < x < \infty; \delta > 0 \]

where \( \lambda \) and \( \delta \) are constants known as the location and scale parameters. Substituting this in (2.2.1), we get a new family of distributions, which we shall refer to as MO-GUMN family, whose survival function is given by

\[ G(x) = \frac{\alpha \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right]}{1 - (1 - \alpha) \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right]; \quad -\infty < x < \infty, \delta, \alpha > 0. \]

The probability density function is

\[ g(x) = \frac{\alpha \exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right]}{\delta (1 - (1 - \alpha) \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right])}; \quad -\infty < x < \infty, \delta, \alpha > 0. \]

The moment generating function is

\[ M,(t) = (\alpha t / \delta) e^{\chi \cdot \delta \eta} \sum_{i=0}^{\infty} \left( \frac{\alpha}{(i+1)^{1/\delta}}. \right) \]

The hazard rate function is given by

\[ r(x) = \frac{\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right)}{\delta (1 - (1 - \alpha) \exp \left[ -\exp \left( \frac{-\left( \frac{\lambda - x}{\delta} \right)}{\delta} \right) \right])}. \]

Figure 6.2 gives the graph of \( r(x) \), which is an increasing curve for all values of \( \alpha \).

**Theorem 6.4.1**

Marshall-Olkin Gumbel Minimum distribution is geometric extreme stable.

Proof is similar to Theorem 3.4.1.
Theorem 6.4.2

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function \( \overline{F}(x) \) and \( N \) be a geometric random variable with parameter \( p \) and \( P(N = n) = p q^{n-1}; n=1,2,\ldots, 0 < p < 1, q = 1-p \), which is independent of \( \{X_i\} \) for all \( i \geq 1 \). Let \( U_N = \min_{i \leq N} X_i \). Then \( \{U_N\} \) is distributed as MO-GUMN if and only if \( \{X_i\} \) is distributed as GUMN

Proof is similar to Theorem 3.4.1

6.5 An AR (1) model with MO-GUMN marginal distribution

Theorem 6.5.1

Consider an AR (1) structure given by (2.3.1), where \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables independent of \( \{X_{n-1}, X_{n-2},\ldots\} \), then \( \{X_n\} \) is stationary Markovian AR(1) process with MO-GUMN marginals if and only if \( \{\varepsilon_n\} \) is distributed as GUMN distribution.

Proof is similar to Theorem 2.3.1

Theorem 6.5.2

Consider an autoregressive minification process \( X_n \) of order \( k \) with structure (2.5.1). Then \( \{X_n\} \) has stationary marginal distribution as MO-GUMN if and only if \( \{\varepsilon_n\} \) is distributed as GUMN.

6.6 Marshall – Olkin Frechet distribution for maximum

Consider Frechet distribution for maximum (FRMX) with survival function (see Castillo (1988))

\[
\overline{F}(x) = 1 - \exp\left[-\left(\frac{\delta}{x - \lambda}\right)^\beta\right]; x \geq \lambda, \delta, \beta > 0,
\]

\[= 1 \quad \text{otherwise}
\]
where \( \lambda, \delta, \beta \) are location and scale and shape parameters.

Substituting this in (2.2.1) we get a new family of distributions which we shall refer to as MO-FRMX family, whose survival function is given by

\[
G(x) = \frac{\alpha[1 - \exp\left(-\frac{\delta}{x - \lambda}\right)]}{1 - (1 - \alpha)[1 - \exp\left(-\frac{\delta}{x - \lambda}\right)]}.
\]

= 1 otherwise.

The probability density function is

\[
g(x) = \frac{\alpha \beta \left(\frac{\delta}{x - \lambda}\right)^{\beta + 1} \exp\left(-\frac{\delta}{x - \lambda}\right)}{\delta[1 - (1 - \alpha)[1 - \exp\left(-\frac{\delta}{x - \lambda}\right)]]^2}.
\]

The hazard rate function is

\[
r(x) = \frac{\beta \left(\frac{\delta}{x - \lambda}\right)^{\beta + 1} \exp\left(-\frac{\delta}{x - \lambda}\right)}{\delta[1 - (1 - \alpha)[1 - \exp\left(-\frac{\delta}{x - \lambda}\right)]]^2}\left[1 - \exp\left(-\frac{\delta}{x - \lambda}\right)\right].
\]

Figure 6.3 gives the graph of \( r(x) \). For all values of \( \alpha \) at first it increases and then decreases.

**Theorem 6.6.1**

Marshall – Olkin Frechet Maximum distribution is geometric extreme stable.

Proof is similar to Theorem 3.4.1.

**Theorem 6.6.2**

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function \( \bar{F}(x) \) and \( N \) be a geometric random variable with parameter \( p \) and \( P(N = n) = p q^{n-1}; n = 1,2,\ldots, 0 < p < 1, q =1-p, \) which is independent
of \( \{X_i\} \) for all \( i \geq 1 \). Let \( U_N = \min_{i \leq N} X_i \). Then \( \{U_N\} \) is distributed as MO-FRMX if and only if \( \{X_i\} \) is distributed as FRMX.

Proof is similar to Theorem 3.4.1

6.7 An AR (1) model with MO-FRMX marginal distribution

Theorem 6.7.1

Consider an AR (1) structure given by (2.3.1), where \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables independent of \( \{X_{n-1}, X_{n-2}, \ldots\} \), then \( \{X_n\} \) is stationary Markovian AR(1) process with MO-FRMX marginals if and only if \( \{\varepsilon_n\} \) is distributed as FRMX distribution.

Proof is similar to theorem 2.3.1

Theorem 6.7.2

Consider an autoregressive minification process \( X_n \) of order \( k \) with structure (2.5.1). Then \( \{X_n\} \) has stationary marginal distribution as MO-FRMX if and only if \( \{\varepsilon_n\} \) is distributed as FRMX.

The sample path of the process for different values of \( p, \lambda \) and \( \delta \) are given in Figure 6.8.

6.8 Marshall–Olkin Frechet distribution for minimum

Consider Frechet distribution for maximum (FRMN) with survival function (see Castillo (1988))

\[
\bar{F}(x) = \exp\left[-\left(\frac{\delta}{x-\lambda}\right)^\beta\right]; x \leq \lambda, \delta, \beta > 0,
\]

\[
= 0 \quad \text{otherwise}
\]

where \( \lambda, \delta, \beta \) are location and scale and shape parameters.
Substituting this in (2.2.1) we get a new family of distribution which we shall refer to as MO-FRMN family, whose survival function is given by

\[ G(x) = \frac{\alpha [1 - \exp\left(-\frac{\delta}{x-\lambda}\right)^\beta]}{1 - (1 - \alpha)\exp[-\left(-\frac{\delta}{x-\lambda}\right)^\beta]} \cdot \]

\[ = 0 \quad \text{otherwise.} \]

The probability density function is

\[ g(x) = \frac{\alpha \beta \left(\frac{\delta}{\lambda-x}\right)^{\beta+1} \exp[-\left(-\frac{\delta}{\lambda-x}\right)^\beta]}{\delta[1 - (1 - \alpha)\exp[-\left(-\frac{\delta}{\lambda-x}\right)^\beta]]^2}. \]

The hazard rate function is

\[ r(x) = \frac{\beta \left(\frac{\delta}{\lambda-x}\right)^{\beta+1}}{\delta[1 - (1 - \alpha)\exp[-\left(-\frac{\delta}{\lambda-x}\right)^\beta]]}. \]

Figure 6.4 gives the graph of \( r(x) \) which is increasing for all values of \( \alpha \).

**Theorem 6.8.1**

Marshall – Olkin Frechet Minimum distribution is geometric extreme stable.

Proof is similar to theorem 3.4.1.

**Theorem 6.8.2**

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function \( \tilde{F}(x) \) and \( N \) be a geometric random variable with parameter \( p \) and \( P(N = n) = p \ q \ n^{-1}; n = 1, 2, \ldots, 0 < p < 1, q = 1-p \), which is independent of \( \{X_i\} \) for...
all $i \geq 1$. Let $U_N = \min_{i \leq N} X_i$. Then $\{U_N\}$ is distributed as MO-FRMN if and only if $\{X_i\}$ is distributed as FRMN.

Proof is similar to theorem 3.4.1

6.9 An AR (1) model with MO-FRMN marginal distribution

Theorem 6.9.1

Consider an AR (1) structure given by (2.3.1), where $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_{n-1}, X_{n-2}, \ldots\}$, then $\{X_n\}$ is stationary Markovian AR(1) process with MO-FRMN marginals if and only if $\{\epsilon_n\}$ is distributed as FRMN distribution.

Proof is similar to theorem 2.3.1

Theorem 6.9.2

Consider an autoregressive minification process $X_n$ of order $k$ with structure (2.5.1). Then $\{X_n\}$ has stationary marginal distribution as MO-FRMN if and only if $\{\epsilon_n\}$ is distributed as FRMN.

6.10 Marshall – Olkin Weibull distribution for maximum

Consider a Weibull maximum (WEMX) distribution with survival function (see Castillo (1988))

$$F(x) = 1 - \{\exp[-(\frac{x - \lambda}{\delta})^\beta]\}; x \leq \lambda,$$

$$= 0 \text{ otherwise}$$

where $\lambda$, $\delta$ and $\beta$ are constants known as the location, scale and shape parameters.

Substituting this in (2.2.1) we get a new family of distributions, which we shall refer to as MO-WEMX family, whose survival function is given by
\[ \hat{G}(x) = \frac{\alpha [1 - \exp\left(-\frac{\lambda - x}{\delta}\right)]}{1 - (1 - \alpha) [1 - \exp\left(-\frac{\lambda - x}{\delta}\right)]}; x \leq \lambda. \]
\[ = 0 \quad \text{otherwise.} \]

The probability density function is
\[ g(x) = \frac{\alpha \beta \left(\frac{\lambda - x}{\delta}\right)^{\beta-1} \exp\left(-\frac{\lambda - x}{\delta}\right)}{\delta [1 - (1 - \alpha) [1 - \exp\left(-\frac{\lambda - x}{\delta}\right)]]^2}; x \leq \lambda. \]
\[ E(X^r) = -\lambda + \frac{\delta r}{\alpha \beta} \Gamma(r / \beta) \sum_{i=0}^{\infty} \frac{(-1)^{i+2}}{(i+1)^{r+2}} \left(\frac{\alpha}{\lambda}\right)^i. \]

The hazard rate function is
\[ r(x) = \frac{\beta \left(\frac{\lambda - x}{\delta}\right) \exp\left(-\frac{\lambda - x}{\delta}\right)}{\delta [1 - (1 - \alpha) [1 - \exp\left(-\frac{\lambda - x}{\delta}\right)]] [1 - \exp\left(-\frac{\lambda - x}{\delta}\right)]}; x \leq \lambda. \]

Figure 6.5 gives the graph of \( r(x) \) which is increasing for all values \( p \).

**Theorem 6.10.1**

Marshall – Olkin Weibull Maximum distribution is geometric extreme stable.

Proof is similar to theorem 3.4.1.

**Theorem 6.10.2**

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables with common survival function \( \bar{F}(x) \) and \( N \) be a geometric random variable with parameter \( p \) and \( P(N = n) = p q^{n-1}; n=1,2,\ldots, 0 < p < 1, q = 1-p \), which is independent of \( \{X_i\} \) for all \( i \geq 1 \). Let \( U_N = \min_{I \leq N} X_i \). Then \( \{U_N\} \) is distributed as MO-WEMX if and only if \( \{X_i\} \) is distributed as WEMX.
Proof is similar to theorem 3.4.1

**Theorem 6.10.3**
Consider an AR (1) structure given by (2.3.1), where \( \{e_n\} \) is a sequence of independent and identically distributed random variables independent of \( \{X_{n-1}, X_{n-2}, \ldots\} \), then \( \{X_n\} \) is stationary Markovian AR(1) process with MO-FRMX marginals if and only if \( \{e_n\} \) is distributed as FRMX distribution.
Proof is similar to theorem 2.3.1

**Theorem 6.10.4**
Consider an autoregressive minification process \( X_n \) of order \( k \) with structure (2.5.1). Then \( \{X_n\} \) has stationary marginal distribution as MO-WEMX if and only if \( \{e_n\} \) is distributed as WEMX.

The sample path of the process for different values of \( p, \lambda \) and \( \delta \) are given in Figure 6.9.

### 6.11 Marshall – Olkin Weibull distribution for minimum

Consider Weibull distribution for minimum (WEMN) with survival function (see Castillo(1988))

\[
\tilde{F}(x) = \exp\left\{ -\left( \frac{x - \lambda}{\delta} \right)^{\beta} \right\}; \quad x \geq \lambda \\
= 1 \quad \text{otherwise}
\]

where \( \lambda, \delta \) and \( \beta \) are constants known as the location, scale and shape parameters. Substituting this in (2.2.1), we get a new family of distributions, which we shall refer to as MO-WEMN family whose survival function is given by
\[
G(x) = \frac{\alpha \exp\left(-\frac{x - \lambda}{\delta}\right)}{1 - (1 - \alpha) \exp\left[-\left(\frac{x - \lambda}{\delta}\right)\right]}; x \geq \lambda.
\]

= 1 otherwise.

The probability density function is

\[
g(x) = \frac{\alpha \beta \left(\frac{x - \lambda}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x - \lambda}{\delta}\right)\right]}{\delta \left[1 - (1 - \alpha) \exp\left[-\left(\frac{x - \lambda}{\delta}\right)\right]\right]^2}; x \leq \lambda.
\]

\[
E(X^r) = \lambda + \frac{\delta r \Gamma(r / \beta)}{\alpha \beta} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{(i+1)^{\beta}}\left(\frac{\alpha}{\alpha - 1}\right).
\]

The hazard rate function is

\[
r(x) = \frac{\beta \left(\frac{x - \lambda}{\delta}\right)^{\beta-1}}{\delta \left[1 - (1 - \alpha) \exp\left[-\left(\frac{x - \lambda}{\delta}\right)\right]\right]}.
\]

Figure 6.6 gives the graph of \(r(x)\). For \(\alpha < .5\), it is at first increases and then decreases and increases. For all other values \(\alpha > .5\) it increases.

**Theorem 6.11.1**

Let \(\{X_i, i \geq 1\}\) be a sequence of independent and identically distributed random variables with common survival function \(\overline{F}(x)\) and \(N\) be a geometric random variable with parameter \(p\) and \(P(N = n) = p q^{n-1}; n = 1, 2, ..., 0 < p < 1, q = 1 - p\), which is independent of \(\{X_i\}\) for all \(i \geq 1\). Let \(U_N = \min_{i \leq N} X_i\). Then \(\{U_N\}\) is distributed as MO-WEMN if and only if \(\{X_i\}\) is distributed as WEMN.

Proof is similar to theorem 3.4.1
Theorem 6.11.2

Consider an AR(1) structure given by (2.3.1), where \( \{e_n\} \) is a sequence of independent and identically distributed random variables independent of \( \{X_{n-1}, X_{n-2}, \ldots\} \), then \( \{X_n\} \) is a stationary Markovian AR(1) process with MO-WEMN marginals if and only if \( \{e_n\} \) is distributed as a WEMN distribution.

Proof is similar to theorem 2.3.1

Theorem 6.11.3

Consider an autoregressive minification process \( X_n \) of order \( k \) with structure (2.5.1). Then \( \{X_n\} \) has stationary marginal distribution as MOWEMN if and only if \( \{e_n\} \) is distributed as a WEMN.

6.12 Important functions of some bivariate exponentials.

Consider the survival function of Mardias distribution (Castillo, 1988) as

\[
\bar{F}(x, y) = \left(e^x + e^y - 1\right)^{-1}.
\]

From (3.6.1) we can see that the new survival function is

\[
\bar{G}(x, y) = \frac{1}{1 + (1/\alpha)(e^x + e^y - 2)},
\]

which is known as Marshall – Olkin bivariate Mardias distribution (MO-BMD).

Consider the survival function of Morgentern’s (Castillo, 1988) family given by

\[
\bar{F}(x, y) = e^{-\alpha x} (1 + \alpha (1 - e^{-x})(1 - e^{-y})).
\]

Then from (3.6.1) the new survival function is

\[
\bar{G}(x, y) = \frac{1}{1 + (1/\alpha)(e^{-x+y}(1 + \alpha (1 - e^{-x})(1 - e^{-y})))^{-1}},
\]
which is refered to as Marshall – Olkin bivariate Morgenstern’s (MO-BMR) distribution.

Consider the survival function of Gumbel type I (Castillo, 1988) distribution

$$\bar{F}(x, y) = e^{-x^{-1/\alpha}}.$$ 

From (3.6.1) we can see that the new survival function is

$$\bar{G}(x, y) = \frac{1}{1 + (1/\alpha)(e^{-x^{-1/\alpha}} - 1)},$$

which is known as MO-BGU Type I distribution.

Consider the survival function of Gumbel's type II (Castillo, 1988).

$$\bar{F}(x, y) = \exp(-(x^m + y^m)^{1/m}).$$

From (3.6.1) the new survival function is

$$\bar{G}(x, y) = \frac{1}{1 + (1/\alpha)(\exp(-(x^m + y^m)^{1/m}) - 1)},$$

which is known as MO-BGU Type II distribution.
Hazard rate function of MOGUMX for various values of $\alpha$
Fig 6.2

Hazard rate function of MOGUMN for various values of $\alpha$. 

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Fig 6.3

Hazard rate function of MOFRMX for various values of \( \alpha \)
Fig 6.4

Hazard rate function of MOFRMN for various values of $\alpha$. 
Fig 6.5

Hazard rate function of MOWEMX for various values of $\alpha$
Fig 6.6

Hazard rate function of MOWEMN for various values of $\alpha$
Fig 6.7

Sample paths of MOGUMAXAR (1) process for different values of $p$, $\lambda$ and $\delta$. 
Sample paths of MOFRMAXAR (1) process for different values of $p, \lambda, \delta$ and $\beta$. 
Fig 6.9
Sample paths of MOWEMAXAR(1) process for different values of p, λ, δ and β