CHAPTER I

1. BASIC CONCEPTS IN INVENTORY

1.1. INTRODUCTION:

Inventories are an asset to the firm and from a financial standpoint, inventories represent a capital investment and must, therefore compete with other asset forms for the firm’s limited capital funds. The objective of inventory is to maximize the profit of the firm.
However, in many cases inventory is treated as a major cash flow constraint. Thus making it necessary to optimise inventory using analytical and statistical methods in an integrated approach. One of the biggest challenges in optimising inventory is the fact that it is considered as output of many inter organizational process. Most of the organizations particularly in India are tend to attempt to lower inventory using non- analytical approach with lower service levels. Various process changes can be suggested and be modelled to verify their impact on the inventory levels and service levels. It would be apt to consider real world constraints prior to deciding on the appropriate changes. The theory of inventory control took its roots from the scientific management movement during early 1900’s. Since World War II the inventory control discipline developed further, using the tools of Operations Research. Over hundred years, still many are exploring on inventory models, as this subject has become an interesting research area. The fast revolutionary progress of scientific technology, in the recent years with the advancement of computers challenged many researchers to work extensively and to solve a variety of problems relating to production planning and inventory control. Inventory control is viewed as the lifeblood of concern as it plays a vital role in fields like industry, agriculture, armed
services, banks, hospitals and transport undertakings etc. Carrying of inventories is the most needed and common feature of almost all the organizations that supply goods to the customers. Production management and Inventory management are inseparable like "SIAMESE TWINS" and thus no decisive action can be taken in one area independently of the other and ought to be dealt with the help of inventory control.

1.2 CONCEPT OF INVENTORY:

Inventory may be considered as an accumulation of material(s) maintained for use in future. According to Love[47], inventory may be defined as 'a quantity of goods or materials in the control of an enterprise and held for a time, in a relatively idle or unproductive state, awaiting its intended use or sale'. The subject of inventory control is the body of mathematical and quantitative methods known as inventory theory and this is self-explanatory.

Inventory control aims at managing the timing and the quantities of goods to be ordered and stocked so that demand can be met satisfactorily and economically.

The two key problems, which are inevitable and always bother the managers of inventories, are:
(i) The time of replenishment

(ii) The quantity of replenishment.

These two decisions when and how much, determine the size of the inventory. It is very apt to refer the inventory process as reservoir or dam, as the supply process is responsible for replenishing the inventory, whereas the demand process accounts for the depletion of inventory. Here supplies are made according to a known process and stock depletion due to demand process or predetermined release rules. Essentially finished goods like reservoirs, which will permit the continuous flow of the product to customers.

The need for inventory arises as a consequence of the difference between the rates of supply and demand. If the supply rate is higher than the demand rate, the status of inventory is positive whereas in the other case shortages do appear. According to Johnson and Montgomery [40], if the on-hand inventory is denoted by $I(t)$ at time ‘$t$’ then the relation between supply, inventory and demand can be expressed as

$$I(t) = I(0) + \int_0^t [x(u) - D(u)] \, du \quad \ldots(1.2.1)$$

where
\[ I(0) = \text{initial inventory} \]
\[ x(u) = \text{supply at time 'u'} \text{ and} \]
\[ D(u) = \text{demand at time 'u'} \]

Therefore the problem of inventory is to determine a policy which helps answering the two major problems 'when to order' and 'how much to order' and also economical to adopt. An arbitrary policy can never be economical since, the impact of quantity purchased or produced will be on the stock on-hand, which may lead to either over-stocking resulting in a higher cost of investments or under stocking, resulting in a failure to meet the demand. In these circumstances a prudent stockiest will make decision regarding the stock on-hand that will have the associated cost as the least.

1.3 RELEVANT FACTORS IN INVENTORY CONTROL:

Inventory control is an integral part of production planning. The term "Inventory control" is a key term. The underlying meaning is controlling inventories. By this we mean that the firm must decide what levels of inventory can be economically maintained. The reason for this is that there are certain cost advantages and disadvantages associated with every unit of inventory maintained. Management science paved the way to maintain inventories in an efficient manner using the principles of
operations research to arrive at right decisions. Inventories were seem to be an evil to some managers, but Operations Research provided effective solutions to the varied problems of inventory management was dealt by Edward. A. Silver[74] and a review was published. In that review he has covered the objective of inventory management including the relevant related costs. Further suggestions were made for bridging the gap existed between theory and practice of inventory problems in many organizations. Many books and journals over a period of hundred years included potential writings of actual use of Operations Research in inventory management. The following objectives and constraints needed by the decision makers in an inventory management context are vividly discussed. The most of the inventory management problems that are covered by the majority of the operations research literature mainly used the criteria of minimization of costs. Of course significant portion of the published material ignored the system control costs. More over the criteria used didn’t include discounting. However the cost of the capital reflected was usually the cost of carrying system. Inventory management problems often interact with other areas of the operations management domain which include
i) provision of raw materials for production schedule

ii) production of inventories of finished items

iii) determination of inventory levels for spare parts.

The inventory management attempts to answer the key factors

(1). How often should the inventory status be determined i.e. what is the review interval?

(2) When should a replenishment order be placed?

(3) How large should be the replenishment order?

The author has collaborated on a text Peterson & Silver [59] that contains an extensive bibliography. They have attempted to answer the above three questions with the help of inventory modelling. In the same lines C.Wagner [80], Aggarwal[1] Nahamias [53-57] have attempted the above queries.

1.4. OBJECTIVES AND CONSTRAINTS OF INVENTORY SYSTEM:

Before combating the problem of inventory, the analyst has to identify the controllable and uncontrollable variables along with decision variables for which he is pursuing the solutions. Prior to that it is necessary to study all the objectives to be achieved and the threats
affecting the attainment of the objectives. The construction of the mathematical model is the next step to find out the overall effectiveness of the inventory system, given the values of the uncontrollable variables. Many inventory managers pursue a number of objectives. Some of which are

(i) Profit maximization (with or without discounting)
(ii) Maximization of rate of return on stock investment
(iii) Cost minimization (with or without discounting)
(iv) Maximization of chance of survival
(v) Ensuring flexibility of operation
(vi) Determination of a flexible selection

Three of the major objectives in most manufacturing firms based upon earning profits are

1. Maximum customer invoice
2. Minimum inventory increment
3. Efficient (low-cost) plant operation

In the business world, only very few companies have priority to fulfill these objectives, as they are very important for sustained success. Production and inventory control are concerned basically with providing
the information needed for the day-to-day decisions required in plant operation. The chief objective of any organization is the minimization of costs and maximization of profits. Many decision makers do not use the discounting criteria. There are some possible constraints in decision-making such as minimum order size, maximum order quantity, minimum tolerable customer service levels, storage space limitation and maximum budget available in a period etc. Elison [25] viewed that there is often little difference between certain objectives and internally imposed constraints. However, a trade-off between two or more measures the effectiveness that can be arrived at the effect of uncertainty in the model. Considering them as random variables can incorporate variables and in such cases the inventory model is said to be probabilistic or stochastic. By analysing the inventory model we can arrive at optimum values for the decision variables subject to the constraints, if any. Further the various input parameters in the model will be estimated by making use of the past information and it is necessary to carry out sensitivity analysis on the solution in order to understand the changes in the parameters in response to the solution.
1.5. PURPOSE OF INVENTORY POLICIES:

The purpose of inventory is justified only when the value of the benefits out of its functioning matches the cost of inventory. Several reasons for maintaining inventory are enlisted and some of which are

i) To exploit the economies of scale in production or procurement.

ii) To cope up with the fluctuating demand over time

iii) To withstand speculation on prices and

iv) To face uncertainty about supplies or requirements or both.

The best way to exploit the economics of scale is by making bulk purchase or production and thereby keeping inventory. It is observed in many cases the total cost of production will be a concave function of the quantity produced, where in the marginal cost of production is a non-increasing function of the quantity produced. The same is true with purchased materials also when there is price discount for bulk purchases.

Thus maintenance of inventory will be justified in such cases. The consequence of the difference between the rates of supply and demand will lead to the need for inventory. Therefore the purpose of inventory is to determine a policy indicating "when to order" and "how much to order" so that the policy would be economical to adopt.
1.6. INVENTORY SYSTEM COSTS:

While maintaining inventory, several costs arise among which following three kinds of costs are important. Of these costs any two or all three are subject to control (Naddor [50]). These are (i) Holding costs (ii) Shortage costs; and (iii) Setup costs.

i) Holding costs: This is the cost associated with holding one unit of inventory for one unit of time. The time is generally taken as one year but not necessarily. This cost results from the payments made towards inventory maintenance in the form of taxes, insurance, protection against pilferage, deterioration, damage and cost of storage space. Another important factor of the holding cost is the opportunity loss associated with the money tied up in the inventory. Though this is not out of pocket cost, it is equivalent to the amount, which the investment on inventory could have earned if invested elsewhere. The common method of modelling inventory-holding cost is to assume that it is proportional to the average inventory held during the period. If I(t) is the inventory level at time t, then the average inventory during a period (0,T) is given by

\[ \bar{I} = \frac{\int_0^T I(t)dt}{T} \]  

...(1.6.1)
where \( T \) is the planning period and carrying charge will be usually in the range of 15% to 25% of the stock value per year. Hadley & Whitin [32] calls this cost as ensemble average.

ii) setup cost:

This is also called ordering cost or procurement cost and consists of two components of which one is a fixed cost denoted by \( A \) and is independent of the quantity procured. The other varies with the quantity and is denoted by \( f(q) \). The fixed cost is called cost of ordering in case of purchased items and represents the initial cost of selling up machinery before production is taken up in a batch. While modelling this cost we use the notation \( A + f(q) \). In particular \( f(q) \) may be a linear function so that the cost can be written as \( A + pq \), where \( p \) is the unit variable cost of procurement.

iii) Shortage cost:

This is the cost of unsatisfied demand. When the item is temporarily out of stock and a demand occurs, it may be either cancelled by the customer or back ordered. When the customer cancels an order, the profit is a loss of good will to the management. When the customer is willing to wait until the stocks arrive for the inventory, the demand is said to be back
ordered or back logged. The delayed delivery for the customer may be followed by a penalty cost. If the customer is not willing to wait it is necessary to supply at a higher cost. The unit cost due to shortage when the item is not backlogged is denoted by $\Pi$. In this case, the demand is usually lost. In case of back ordering the unit cost of shortage may be proportional to both the quantity short and the duration of shortage. However, the shortage cost is denoted by $[\Pi b + \Pi \bar{B}]$ where $b$ is the quantity of lost sales and $\bar{B}$ is the average back order during the period $[0, T]$ given by

$$\bar{B} = \frac{1}{T} \left[ \int_{0}^{T} B(t) dt \right] \quad \ldots(1.6.2)$$

Where $B(t)$ is the quantity of back orders at time $t$.

iv). Material cost: The cost of material is least affected by the decision regarding inventory maintenance in many cases. However, with quantity discounts or price fluctuations the material cost becomes one of the variable costs of the system. When the price of the item is subject to inflation hike in price may often be expected in future purchases. In such cases a significant saving on the material cost can be realized by
purchasing extra quantity before the price goes up. An inventory ordering model with this criterion is discussed by Buza Cott[11] and Sambandam [68] contributed a related work in this area.

v) Operating costs / System control costs:

Generally, these costs associated with data processing, forecasting, stock reviewing and placing timely orders, checking up the material flow in the system etc. that are essential for implementing any inventory policy. These costs depend on the inventory policy adopted. According to Silver [74] a major portion of literature of inventory models does not seem to have given due consideration to these costs. The estimation of cost parameter for any inventory system is one of the important phases of inventory decision-making. These costs are to be estimated from the management accounts and methods of determination are discussed by Love [47] and Lambert and Londe [46].

1.7 Supply and Demand Behaviour:

The accumulation of stock due to supply process is also termed as replenishment. In case of production the supply will usually be finite at the rate of p units per unit time. This is called finite replenishment pattern. Controlling p can regulate the inventory position. When the quantity order is
supplied entirely in a single replenishment, the process is called instantaneous replenishment pattern. Some times, the supplies require lead-time, which will be uncertain, many times, in practice. In an inventory system, the demand will be an exogenous factor, causing stock depletion. In addition to this, demand is usually uncertain in nature but a reasonable forecast. The input for the inventory is the supply process where as the output is the demand process. The supply could be made either by purchasing the item from external sources or by producing the same. There is always a limitation on the capacity of the supplier or production facility. Usually, there will be a time lag between placement of an order and the actual receipt of the quantity. This is called as ‘lead time’ and is generally denoted by ‘L’, which may be either a known constant or a random variable. When the lead times are lengthy, it becomes necessary to maintain sufficient stock to avoid ‘out - of – stock’ conditions during the lead time. This stock is generally referred as safety stock or buffer stock.

The demand is an exogenous factor in an inventory system, causing stock depletion. The demand is usually uncertain in nature but it can be predicted using the past demands of the product. If the demand is predictable for the product in the inventory, it is said to be deterministic situation. Major
inventory decisions depend on the accuracy of prediction of demand. A number of forecasting methods for prediction of demand are extensively discussed by Brown [8] and Johnson and Montegomery [40].

In some cases we find demand is not known with certainty, and the uncertain demand may follow a particular probability distribution then these models are known as stochastic inventory models. These models assume probability functions with regard to the behaviour of demand times, quantity demanded and lead time. Models with these features were studied both for the single product and multi-product inventories by several researchers.

1.8 Classification Of Inventory Models:

The procedure for modelling the inventory is just to list out all the assumptions and characteristics involved in the situation, and to form the total cost expression. After that the total cost should be optimised in order to obtain the optimum order quantity and re-order point. The mathematical modelling of inventory system provides the answers for the two crucial questions, how much to order? And when to order? .

The inventory models are classified mainly as deterministic and probabilistic models. The deterministic inventory models assume that demand and lead time are known with certainty. It means that number of units and the time at
which these may be demanded and the duration of lead time if any is known in advance. This does not mean that the demand is always a constant, and it can vary from period to period, but still predictable. Hadley and Whitin [32] discussed the different types of deterministic inventory models in detail.

Sometimes demand is not known certainly. Such uncertain demand may follow a particular probability distribution. In such case the modelling should be as the modelling of stochastic inventory model. The stochastic models assume probability functions with regard to the behaviour of demand times, quantity demand and lead time. Models with these features were studied both in single product and multi product inventories by several researchers.

1.9 DETERMINISTIC AND PROBABILISTIC MODELS:

An inventory models where the future demands are known with certainty are said to be deterministic. In such situation the demand is treated as a known constant or as a variable with known variability overtime. Therefore demand can be specified either as a continuous or discrete function of time. Some times even though the demand is not known with certainty it is possible at least to know the probability distribution governing the situation and then the demand process should be modelled as stochastic process. If the
probability of demand during a period ‘t’ is the same throughout the period then the demand is said to be ‘static’ which is different from the ‘dynamic’ demand where the demand in future periods may have different distributions.

When the demand is probabilistic or stochastic, it will be convenient to express the probability distribution of demand in a specific interval, which is rather different from dealing with demand rate that varies continuously over time. Some of the factors that cause uncertainty to the models are lead time, demand during lead time and the number of customers awaiting when the item is out of stock.

Inventory models for single-period planning are very important as they form groundwork for the multi-item and multi-period models. Inventories of paddy, wheat, petrol etc can be treated, as a single-product case where as the inventory of spare parts is an example of multi-product inventory. The literature of multi-product inventory control can be had from Silver[74] and Naddor [50].

1.10 INVENTORY POLICIES:

An inventory policy is a decision rule, which is designed to implement it at different time points. It specifies two entities namely 1) how much to order.
2) when to order. These policies aimed to obtain the "best" values of the parameters. There are two types of inventory policies. One is known as continuous review policy and the other being Periodic review policy.

1. CONTINUOUS REVIEW POLICY:

Here the level of inventory I(t) is observed at every time point t and hence called continuous review policy. The advantage of this policy is that the reorder point (ROP) will never be missed. But to adopt this policy it is quite expensive, as it requires additional staff for maintaining the records properly and also to record every transaction immediately and to update the balance of the stock. However, in certain situations it plays a vital role when the shortage cost of particular item is too high. Examples are inventory of oxygen cylinders in a hospital, inventory of cash in bank etc.
2. PERIODIC REVIEW POLICY:

This policy will be sometimes more convenient to review the stock position only at certain specified time say \( t_i \) for \( I = 1, 2, \ldots, n \). The level of the stock at the \( j \)th point can be denoted by \( I_j \). This is known as periodic review policy and decision regarding replenishment will be taken only at the end of the period. In this policy, the duration between two successive reviews is called review period and is denoted by \( T_r \), must be determined optimally. Several implications of these models are extensively discussed by Hadley and Whitin [32] and Naddor[50]. Keeping in view of these methods, standard notations are used for describing various periodic review policies. These are substituted below. The following are the inventory polices.

a) \((S,s, T_r)\) Policy:

This is a periodic review policy, the following is the modus operandi of the model. Order for \( Q_j \) units at the \( j \)th review, where

\[
Q_j = \begin{cases} 
0 & \text{if } I_j > s \\
(S - I_j) & \text{if } I_j \leq s
\end{cases}
\]

Here \( s \) is re-order point and \( S \) is the order level or target inventory.
b). \((s, T_r)\) Policy:

This is a particular case of \((S,s,T_r)\) policy when \(s = S\). The operation of this policy is order \(Q_j\) units at the \(j\)th review period, where

\[
Q_j = \begin{cases} 
0 & \text{if } I_j > S \\
(S - I_j) & \text{if } I_j \leq S 
\end{cases}
\]

This policy is also known as S-policy (or) single critical number policy. No re-order point appears in this policy and an order is placed at every review to raise the inventory up to \(S\).

(c). \((S,s)\) Policy:

This policy is an extension to that of S-policy. In this policy, an order for inventory is placed taking in to consideration the fixed ordering cost. This policy can be thought of as a special case of \((S,s, T_r)\) policy when \(T_r\) is very small. In this we order \(Q\) for \(Q = [S - I(t)]\) units, if \(I(t) < s\) and we do not place order when \(I(t) \geq s\). When the demand is not uniform over time and is subject to lot of fluctuations, then there is probability of over shoot of demand so that the ROP is missed when it is actually reached. This policy is also known as two-bin policy. If the rate of demand is fairly uniform and the demand arise in single units, then the ROP can never be missed when it is
reached so that one can precisely order for \( Q = (S - s) \) units always. This is also called as \((Q,s)\) policy by some authors (Johnson and Montgomery[40]).

(d). Base stock policy:

This is a continuous review policy with \( Q = 1 \) and \( S = s + 1 \) or \( s = S - 1 \), where \( s \) is the ROP and \( S \) is the target inventory. It is also denoted by \((S,S - 1)\) policy. It is found to be convenient policy in the field of queuing theory to determine \( S \). In this policy an order for replenishment is placed immediately after every withdrawal and the quantity order will be equal to quantity withdrawn. The base stock policy is characterised by a single parameter

1.11. THE DETERMINISTIC LOT SIZE MODELS:

Models in which the order quantity is determined along with the time between orders are commonly known as “lot size models”. The lot size is also called “Economic order quantity (EOQ)”. Every time \( Q \) units will be ordered and the time between the orders varies with \( Q \), which is a fixed order quantity. Ford Harris of the Westing house corporation 1931 first obtained the earliest derivation of simple lot size model formula.
R.H. Wilson has independently derived a model known as "Wilson’s economic order quantity" formula which is considered to be classic in the literature of scientific inventory management. In this basic model, the inventory reorder quantities are fixed and are placed whenever inventory on-hand comes down to a particular level known as the reorder point. Later for the first time F. Raymond dealt with a large variety of inventory systems and as a result of rapid development in the study of inventory systems led to numerous publications and Whitten’s book “theory of inventory management“. The classic formula and its analysis has become the governing principle for stock control. Johnson and Montgomery[40] has given a general form of lot size models with the following notation. Annual demand is deterministically known as D units, production rate P units per year. The fixed cost of ordering is A, unit variable cost is c, the holding cost is $h = ic$ per unit per year. $\pi$ is the shortage cost per unit independent of duration and $\tilde{\pi}$ is the shortage cost per unit. The objective is to determine the order quantity Q and the maximum back order level b, such that the sum of ordering, holding and shortage cost is minimized. It is assumed that all
shortages are backlogged and the lead time is zero. The cost function is then given by

\[ K(Q, b) = \frac{AD}{Q} + CD + \{iC[Q(1 - D/P) - B]\}^2 /2Q(1 - D/P) + \pi b^2 / (2Q(1 - D/P) + \pi bD/Q) \]  

The optimal values of \( Q \) and \( b \) are obtained by solving the following equations \( \partial k / \partial b = 0 \), which implies

\[ Q' = \frac{2AD}{iC(1 - D/P) - (\pi D)^2 / iC(iC + \pi)} \]  

\[ b' = (iC + \pi)^{-1}(iCQ' - \pi D)(1 - D/P) \]

Following are the several particular cases of the model derived.

**Case (i):** The shortage cost is infinite and there will be no back order, hence \( b = 0 \). Then the cost equation reduced to

\[ K(Q) = \frac{AD}{Q} + CD + \frac{iCQ(1 - D/P)}{2} \]

The value of \( Q^* \) is given by

\[ Q^* = \frac{2AD}{iC(1 - D/P)} \]

**Case (ii):** The rate of replenishment is instantaneous but shortages are allowed. The cost equation in this case becomes

\[ K(Q, b) = \frac{AD}{Q} + CD + \frac{iC(Q - b)^2}{2Q} + (2\pi Db + \pi b^2) / 2Q \]

The optimal values \( Q \) and \( b \) are given by
\[ Q^* = \left[ \frac{2AD}{iC - (\pi D)^2 / iC(iC + \pi)} \right]^{1/2} (1 + iC / \pi)^{1/2} \]  
...(1.11.7)

\[ b^* = \frac{(iCQ^* - \pi D) / (iC + \pi)} {iC(iC + \pi)} \]  
...(1.11.8)

**Case (iii)** : The replenishment rate is infinite and shortage cost is infinite.

The cost equation in this case is obtained from (1.9.4) by assuming \( P \to \infty \), which is given by

\[ K(Q) = \frac{AD}{Q} + CD + \frac{iCQ}{2} \]  
...(1.11.9)

The optimal value of \( Q \) becomes

\[ Q^* = \left[ \frac{2AD}{iC} \right]^{1/2} \]  
...(1.11.10)

The equation (1.11.10) is known as Wilson's lot size formula. The optimal duration between two successive orders is Economic order quantity (EOQ) given by

\[ t^* = \left[ \frac{2A}{DiC} \right]^{1/2} \]  
...(1.11.11)

The minimum cost with these values is obtained by substituting \( Q^* \) in (1.11.9), yielding

\[ K(Q^*) = \left[ 2ADiC \right]^{1/2} \]  
...(1.11.12)

The Wilson's formula given in (1.11.10) is not very sensitive to changes in the model parameters. In fact any deviation from the EOQ is given by
(1.11.10) does not result in a significant increase in the cost. This is because of the flatness of the total cost function near the optimal solution.

1.12. Deterministic Order Level Inventory Model:
In this order level systems there will be only one occasion to place an order and the next chance for procurement will take place only after a fixed time. Here the order quantity is a variable, while the time of ordering is fixed. The order level systems are deterministic systems with \((s, T_R)\) policy, where \(T_R\) is the fixed time interval between orders. \(T_R\) usually represents a single scheduling period or review period between orders. Naddor [50] calls it \((t_p, S)\) policy, where \(t_p = T_R\). It is obvious that the replenishment costs are not subject to control, since the scheduling period is a constant. According to Naddor [50] there can be numerous order level systems, depending on the type of costs involved. Following are the results of deterministic order level system from Naddor[50].

Assumptions:

1. Demand is deterministic at a constant rate of \(r\) quantity units per unit time.

2. The scheduling period is a prescribed constant \(t_p\).

3. The stock replenishes only at the beginning of each scheduling period up to order level \(s\). Shortages if, any, are back ordered.
4. The replenishment rate is infinite and lead time is zero.

5. The unit carrying and shortage costs are $C_1$ and $C_2$ respectively.

From the above assumptions the following things are noted.

(i). The number of replenishment $s$ per unit will be $I = l/t_p$, a constant. Hence $C_3$ (setup cost) a constant will not affect the decision about the optimal level $S^o$ of $s$.

(ii). From the first three assumptions we can note that the lot size $q$ is $q_p = rt_p$.

(iii) The inventory level of the entire system will depend on the values of $S$ and $q_p$. The average amount in inventory in the system can therefore be given by

$$I_1(S) = \begin{cases} 
0 & \text{if } S \leq 0 \\
S^2/2q_p & \text{if } 0 \leq S \leq q_p \\
S - q_p/2 & \text{if } S \geq q_p 
\end{cases} \quad \cdots(1.12.1)$$

and the average shortage $I_2(S)$ can be seen

$$I_2(S) = \begin{cases} 
q_p/2 & \text{if } S \leq 0 \\
(q_p - S)^2/2q_p & \text{if } 0 \leq S \leq q_p \\
0 & \text{if } S \leq q_p 
\end{cases} \quad \cdots(1.12.2)$$
Hence the total cost of order level system is given by

\[
C(S) = \begin{cases} 
  c_2 \left( \frac{q_p}{2} - S \right) & \text{if } S \leq 0 \\
  c_2 S^2 / 2 q_p + c_2 (q_p - S)^2 / 2 q_p & \text{if } 0 \leq S \leq q_p \\
  c_2 (S - q_p / 2) & \text{if } S \geq q_p 
\end{cases}
\]  

...(1.12.3)

The function of \( C(S) \) is linear when \( S \leq 0 \) and \( S \geq q_p \), and it is quadratic when \( 0 \leq S \leq q_p \). We can see that the minimum cost must be in \( 0 \leq S \leq q_p \).

Thus to find the solution of order level system, we have to find the minimum of the cost function \( C(S) \), which is given as follows with the indicated range.

\[
C(S) = c_2 S^2 / 2 q_p + c_2 (q_p - S)^2 / 2 q_p \quad \text{for } 0 \leq S \leq q_p .
\]  

...(1.12.4)

The optimal value of \( S \) is given by

\[
S^0 = q_p c_2 / (C_1 + C_2)
\]  

...(1.12.5)

Substituting \( S^0 \) in (1.12.4), we get minimum cost as

\[
C(S^0) = (1 / 2) q_p C_1 C_2 / (C_1 + C_2)
\]  

...(1.12.6)

In the lost sales case, the cost of shortage depends only on the quantity short and not on the duration of the shortage. Thus the total cost of the system for the given inventory system is as follows.
\[ C(S) = \begin{cases} 
\frac{C_2(q_p - S)}{t_p} & S \leq 0 \\
\frac{C_2S^2}{2q_p} + \frac{C_2(q_p - S)}{t_p} & 0 \leq S \leq q_p \\
\frac{C_2(S - q_p)}{2} & S \geq q_p 
\end{cases} \quad \text{...(1.12.7)} \]

It is evident that the optimal value \( S^0 \) of \( S \) should be in the range 0 to \( q_p \), hence the cost equation to be optimised is

\[ C(S) = \frac{C_1S^2}{2q_p} + \frac{C_2(q_p - S)}{t_p} \quad \text{...(1.12.8)} \]

The optimum solution is

\[ S^0 = rC_2 / C_1 \quad \text{...(1.12.9)} \]

The above equation holds good only if \( S^0 \leq q_p \). Otherwise the optimal level should be \( q_p \). The solution of the system can be summarized as follows.

\[ C(S^0) = \frac{C_2r}{C_1} \quad \text{if} \quad C_2 \leq C_1t_p \]

\[ q_p \quad \text{if} \quad C_2 \geq C_1t_p \quad \text{...(1.12.10)} \]

Substituting \( S^0 \) in equation (1.12.9) we get the minimal cost of the system in the lost sales case, as

\[ C(S^0) = \frac{C_2r}{C_1} - \frac{C_2^2r}{2C_1t_p} \quad \text{if} \quad C_2 \leq C_1t_p \]

\[ \frac{C_1q_p}{2} \quad \text{if} \quad C_2 \geq C_1t_p \]
1.13 PROBABILISTIC INVENTORY MODELS:

1.13.1. INTRODUCTION:

When the demand during a planning horizon happens to be a random variable then the associated inventory models are termed as probabilistic models. These models are broadly classified as (1) static or single period models (2) Dynamic or multi period models. In static models the demand variation known to behave as a random variable with specified probability distribution, with a fixed planning period. In these models, the demand over the period is considered to be the steady state demand. The primary problem encountered in many practical situations is taking the decision only once with regard to the purchase or production. For instance, the storage of a seasonal product like the yield of harvest, the production of special order items etc. When the parameters like demand, price level etc., vary from period to period then it will become impossible to describe the nature of the system based on the current situation. In such situations it will be more convenient to treat the entire problem as a single period problem. Karlin[42] discussed the one period models in detail. The analysis to static models in a way initiated to investigate dynamic models. The classical paper on static model was due to Arrow et al [4], Karlin [42] has given the extension work
on the one-stage models in the volume of research papers titled ‘studies in the mathematical theory of inventory and production’. The analysis of these models differs again with periodic or continuous review of stock. Arrow et al[4], Johnson and Montegomery[40], Naddor [50], Hadley & Whitin [32] discussed a number of probabilistic models. Advanced stochastic process is used to analyse such models including semi Markov processes and renewal process. (See Ravichandran[63], Cohen [16]). In the following sections some of the basic stochastic models with static demand are discussed.

1.13.2. STATIC ORDER LEVEL MODEL WITH INSTANTANEOUS DEMAND:

The model is developed basing on the following assumptions

1. The demand during the period is a random variable \( X \), with known continuous density function \( f(x) \) and this demand occur instantaneously instead of uniformly over time.

2. The cost of carrying one unit in inventory is \( H \) and the cost of shortages of one unit, which is backlogged, is \( n \) and the total cost of back logging is proportional to the quantity back logged.
3. S is the order level, which is the quantity that should be maintained at the beginning of each period.

4. The replenishment rate is infinite with zero lead time and there is no fixed ordering cost. Further an order has to be placed only at the beginning of the period.

The unit cost of unit is ‘c’. With these assumptions, it is required to determine optimum order level \( S^o \) of S such that the sum of holding and shortage cost is minimized. The model assumes that an instantaneous demand depletes entire inventory and may even result in a shortage in a small period of time. The sales of Crakers during Deevali festival, Raksha Bhandhan bands, Christmas trees & Stars during Christmas and cool drinks in summer are some examples of such situation. The stockists may keep large inventory of these items prior to the occasion, however the actual depletion of stocks will be over a small interval of time, say a day or two. Therefore the very problem is to decide the quantity to be had on hand by the time the demand arises. This problem is popularly known ‘news paper boy’s problem’ The total expected cost function in terms of ‘S’ is given by

\[
K(S) = CS + H \int_0^s (s - x) f(x)dx + \Pi \int_s^\alpha (x - s) f(x)dx ...
\]  

...(1.13.1)
Where the first and second integrals yields the expected surplus and expected shortage of quantities respectively. If \( I \) denotes the initial on-hand inventory at the beginning of the period, then a quantity of \((S-I)\) units are to be purchased, and the cost function of the system is given by

\[
K(S) = C(S-I) + G(S)
\]

Where \( G(S) = H \int_{0}^{s}(s-x)f(x)dx + \Pi \int_{s}^{a}(x-s)f(x)dx \ldots \) \( \ldots (1.13.2) \)

Minimizing the above equation with respect to \( S \) yields to

\[
F(S^0) = (\Pi - c)/(\Pi + H)
\]
\( \ldots (1.13.3) \)

where \( F(S^0) \) the cumulative distribution functions from which \( S^0 \) can be obtained. Replacing the appropriate integrals with summations can develop the discrete version of the above and other models.

The integral equation given in (1.13.3) has a basic concept which specifies that the order level is in the form of a quantile on the demand distribution corresponding to the ratio given on the R.H.S. The ratio \((\Pi - c)/(\Pi + H)\) lies between 0 & 1 for \( \Pi > c \) and acts as the probability of a shortage.
1.13.3. \((S, s)\) INVENTORY MODEL:

Before describing the above model it is better to describe \(S\) model and the assumptions are same for \((S, s)\) inventory model but for fixed ordering cost \(A\), which is just an extension of \(S\) model. The policy can be given as follows:

Order for \((S^0 - I)\) if \(S^0 > I\)

Do not order if \(S^0 \leq I\)

The quantity \(S^0\) is the optimal order level and is also called the critical number to be decided and therefore this policy is called single critical number policy or \(S\)-policy. Here the holding and penalty costs are assumed to be linear functions of the surplus and shortage quantities respectively. However, in general they can be treated as arbitrary functions of \(S\) and \(D\) depending on the situation. When \(I\) is the on-hand, one has to decide whether to place an order up to \(S\), or to continue for some more time with the existing inventory. In other words to take a decision either to order or not to order.

The expected cost function is given by

\[
K(S) = A + c(S - I) + G(S), \quad \text{if } S > I
\]

\[
G(S) \quad \text{if } S \leq I
\]

...(1.13.4)

The optimal value \(S^0\), of \(S\) can be obtained by minimizing \[CS + G(S)\]
and \( S^o \) is the smallest value of the order quantity which satisfies the equation

\[
C(S^o) + G(S^o) = A + C(S^o) + G(S^o)
\]  
\[
\text{(1.13.5)}
\]

The optimum ordering policy is

- Order \((S^o - 1)\) if \( I < S^o \)
- Do not order if \( I \geq S^o \)

The \((S, s)\) policy is the optimal operating doctrine for many types of inventory systems. The mathematical properties of \((S, s)\) policy under varying assumptions are discussed by Karlin [42].

1.13.4 \((Q,S)\) Inventory Model:

This is a continuous review model. When ever the on-hand inventory drops to \( S \), an order is placed for \( Q \) units. The advantage of this model is that the reorder point will never be missed when it is reached. This model also assumes that there is a positive lead time with the demand during lead time following a random variable with known probability distribution. The \((Q,S)\) inventory model in detail discussed by Hadley & Whitin [32], Johnson and Montegomery [40]. The expected value of demand in a unit of time, say a year is \( D \). The replenishment lead time is \( T \) and it is assumed to
be constant and sufficiently small. Here $X$ is the demand during a lead time and $f(x)$ denote its probability distribution. The fixed cost of procurement is $A$ and unit variable material cost $C$ and the cost of carrying a unit of inventory for one unit time is $H$. The shortages are to be backlogged at the cost of $n$ per unit shortage. With an intention to obtain an expression for average annual cost, compute the cost associated for a given cycle and multiply it with the number of cycles per year. The total average annual cost of the inventory system is given by

$$K(Q,s) = \frac{AD}{Q} + CD + H\left[\frac{Q}{2} + s - \mu\right] + \pi DB(s)/Q$$  \hfill (1.13.6)

Where $\mu$ is the expected demand during lead time and

$$B(S) = \int (x-s)f(x)dx$$  \hfill (1.13.7)

To minimize $K(Q,s)$ solve the equations $\frac{\partial K}{\partial s} = 0$ and $\frac{\partial K}{\partial Q} = 0$.

This yields

$$Q^* = \left[2D\{A + \pi B(s)\}/H\right]^{1/2}$$  \hfill (1.13.8)

and $s^*$ is the solution of $F'(s^*) = hQ^*/\pi D$ where $F'(s)$ is the complementary cumulative distribution of $x$ evaluated at $s$. Therefore the optimal value of solution of the $s^0$ of $s$ is the required solution of the following equation.
\[ F(s) = \frac{h(Q)}{\pi D} \quad \text{(1.13.9)} \]

From the two equations (1.13.8) and (1.13.9) it is obvious that the optimal value of \( s \) is a function of \( Q \) and the optimal value of \( Q \) is a function of \( s \). To find the optimal pair of \((Q^*, s^*)\) that minimizes \( K(Q, s) \), the following iterative method is adopted.

1. Assume \( B(s) = 0 \) and compute \( Q \) with equation (1.13.8) and this value is called as \( Q_i \).

2. By using \( Q = Q_i \) (1.13.9) the reorder point \( s \) is determined and this is called as \( s_i \).

3. Using equation (1.13.8) and equating \( s = s_i \) the new lot size \( Q_2 \) is obtained. \( B(s) \) first found from the equation (1.13.7).

4. Repeat step 2 and continue with \( q = Q_2 \). The procedure converges at the \( i^{th} \) iteration when \( Q_i = Q_{i-1} \) or \( s_i = s_{i-1} \). Usually the convergence is rapid. This model is similar to Wilson's fundamental deterministic model.

**1.13.5. BASE STOCK SYSTEMS:**

Base stock system is a particular case of the fixed ordering systems. In the base stock systems the quantity ordered is always one unit. An order is placed every time a unit is demanded so that the on-hand inventory and the on-order quantity are always equal to a constant say 'R' called base stock.
level. The problem is to determine the optimum value of \( R \) that minimizes the inventory cost is the solution of the following equation

\[
F(R^0) = \frac{\Pi}{(\Pi + H)} \quad \text{...(1.13.10)}
\]

The above equation is similar to a single critical number policy. The quantity on the R.H.S of (1.13.10) represents the fraction of customer's demand filled from the stock on time and hence it is a measure of the service level and is useful for managerial decision making problems. Some times the lead time will a random variable in many real world problems. The usual practice in this case is to place orders immediately whenever a specified quantity is consumed, so that the delay due to lead times can be controlled to some extent. For such cases the application of queuing theory can be used. Suppose, the lead time demand follows exponential distributions with density \( g(x) = \theta \exp(-\theta x) \), where 'x' is the demand size and the demand during a time interval of length follows Poisson distribution with parameter \( \theta \). The queuing analogy for the system will have customers as demands, service gives as lead time and the number of service channels as \( R \) in the lost sales case and unlimited in the backlog case. The system will then be \( M/M/R \) or \( M/M/\infty \) depending on the case. We are interested in a steady state probability distribution of the net inventory at any time, as this is essential
for calculating the expected cost of the system. Details of these type of
models are extensively discussed by Johnson and Montegomery [40].

1.13.6. DYNAMIC INVENTORY MODEL:
In contrast with static models, the dynamic models take in to consideration
the variable demand in each of the future periods of planning horizon. When
demand process changes over the time the associated inventory model is said
to be dynamic model. These models play an important role in the context of
periodic review system and the basic principle of analysis is the sequential
optimisation over time.

1.13.7. NOTATION FOR DYNAMIC INVENTORY MODEL:
The following notation is commonly used in modelling a dynamic inventory
situation
i. The planning horizon consists of N periods numbered as 1,2,3,.....N.
ii. $Q_t$ is the quantity to be replenished for the $t^{th}$ period, $I_t$ is the
inventory on-hand at the end of period and $D_t$ is the demand during
period $t$, so that $I_t = I_{t-1} + Q_t - D_t$. 
The cost of replenishment of $Q_t$ units can be described as a function $C_t(Q_t)$ and the cost of holding $I_t$ units by another function $h_t(I_t)$.

The cost associated with a set up or ordering during the periods $t$ and this cost is usually independent of $Q_t$.

Now the problem is to determine the value of $Q_t$ for $t = 1,2,\ldots,N$, such that the sum of the holding and ordering costs is minimum over the horizon. The two functions $C_t(Q_t)$ and $h_t(I_t)$ plays an important role in deriving useful algorithms for determining the optimal values of $Q_t$.

Several dynamic models are discussed in detail by Johnson and Montgomery [40]. The general dynamic programming model for the multi period inventory problem is developed as follows. In this case, shortages are not permitted. Following are defined for the $t^{th}$ period.

i. $Q_t =$ quantity is to be ordered.

ii. $D_t =$ expected demand

iii. $K_t(Q_t, I_t) =$ cost of producing $Q_t$ units and ending inventory of $I_t$ units.

iv. $f_t(I) =$ Minimum cost obtainable over the periods $t, t+1,\ldots,N$, where the net inventory at the start of the period $t$ is $I_t$. 

With these definitions the recursive equation of the system is given by

\[ f_t(I) = \min_{Q_t>0} [k_t(Q_t + I + x_t + D_t) + f_{t+1}(I + Q_t - D_t)], \]  

...(1.13.11)

for all \( t = 1, 2, ..., N \) and \( F_{N+1}(I) = 0 \). The total cost of inventory in the period \( t \) is given by

\[ K_t(Q_t, I_t) = C_t(Q_t) + H_t(I_t) \]  

...(1.13.12)

The cost function given above in equation will be convex if both the functions on the right hand side are convex and 'the convex cost algorithm' in this case is a very powerful method for decideing the optimal replenishment quantity for each period. This algorithm is based on the 'transportation-algorithm' for the case of linear holding and shortage costs with piece wise linear procurement costs. Johnson and Montgomery [40], Wagner [80] and Love [47] have discussed the use of this algorithm. A powerful algorithm based on a dynamic programming was developed by Wagner and Whitin [79] for the case of the function \( K_t(Q_t, I_t) \) is concave. This algorithm is popularly known as Wagner-Whitin algorithm.

1.13.8. WAGNER-WHITIN ALGORITHM:

This algorithm was first developed for the case of no backlogging but later developed also for the case of back-logging and this was carried out
by Zangwill [81]. This algorithm is simply an application of dynamic programming which is a mathematical procedure for solving sequential decisions problems. For the computation of this programme mainly two properties are required as given below.

i. Replenishment has to take place only when the level of the inventory is zero.

ii. To seek for an upper limit before a period $j$ so as to include its requirements, $D(j)$, in a replenishment quantity this can be shown that an optimal policy as the property $I_{t} \cdot Q_t = 0$ for $t = 1, \ldots, N$. This conveys the meaning that the requirement in a period are satisfied either from the procurement entirely in that period or from the procurement prior to that period and this can be worked out by using dynamic programming. With the purpose to minimize the replenishment plus carrying costs the Wagner-Whitin algorithm can be developed for a set of replenishment quantities. However, this algorithm has limited acceptance in practice and many practitioners consider this is not substantially sound from their standpoint. Many researchers like Aucamp and
Fogourty[5], De Matteis and Mendoza [24], Gorhan[29], Groff [31], Karni [45], and Silver and Meal [73] have developed dynamic optimisation of inventory problems and suggested various decision rules some of which have been extensively used in many practical situations. In particular Silver and Meal [73] have developed a heuristic algorithm, which is a simple and very popular algorithm for the dynamic economic lot size problems and has intuitively simple reasoning of its derivation and requires relatively less computational effort. It is widely accepted and used because of its computational efficiency in providing workable solutions that is near to optimal solution and so presented in the following section

1.13.9 SILVER-MEAL HEURISTIC:

The replenishment quantity \( Q \), associated with a particular value of \( T \), is \( Q = \sum_{j=1}^{T} D(j) \) is selected by the heuristic and to find \( T \) value that minimizes the relevant cost per unit time of replenishment and carrying inventory over the time period \( T \). The total relevant cost associated with a replenishment for \( T \) period be denoted by \( TRC(T) \). These costs include the fixed replenishment cost \( A \) and the inventory holding cost. With an aim to minimize the total relevant cost per unit time \( TRCUT(T) \), where

\[
TRCUT(T) = \frac{TRC(T)}{T} = \frac{[A + \text{carrying cost}]}{T}
\]  

...(1.13.13)
has to be evaluated for increasing values of $T$ until we get for the first time, $\text{TRCUT}(T+1) > \text{TRCUT}(T)$ ...(1.13.14)

when this takes place the corresponding $T$ is selected so that the number of periods that a replenishment order should cover and the replenishment quantity $Q$ is defined above. The calculations can be even simplified by normalizing total relevant cost per unit time by dividing by $v_r$. From equation (1.13.14) the general result for $T$ periods is obtained and given below

$$N_{\text{TRCUT}}(T) = \frac{\text{TRCUT}(T)}{v_r} = \frac{A}{v_r} + \sum_{j=1}^{T} (j-1)D(j)T^{-1} ...(1.13.15)$$

Further details on this heuristic and other dynamic working rule can be seen in Peterson and Silver [59], Love[47]. So far what all we have considered from the basic structure of inventory problems and their analysis which are helpful in providing some of the important results required for the present research and will be discussed in the next chapter.