CHAPTER 1

INTRODUCTION

1.1 Frame theory

The first mathematician to take the notion of open set as basic to the study of continuity properties was Hausdorff in 1914. Using the lattice of open sets, Marshall stone [ST], was able to give topological representation of Boolean algebras and distributive lattices and H. Wallman(1938) [WA] used lattice theoretic constructs to obtain the wallman compactification. In the 1940's McKinsey and Tarski [M; T] studied the "algebra of topology" that is topology studied from a lattice theoretical viewpoint. But a fundamental change in the outlook came in late fifties; Charles Ehresmann [EH] in 1959 first articulated the view that a complete lattice with an appropriate distributivity property deserved to be studied in their own right rather than simply as a means to study topological spaces. He called the lattice a local lattice. Dowker and Strauss([D; P], [D; P], [D; P]) introduced the term frame for a local lattice and extended many results of topology to frame theory. It was with the publication of John Isbell's "Atomless parts of spaces" [IS], in 1972 that the real importance of the subject emerged. Since then Frame theory is studied extensively by many authors.

1.2 Fuzzy set theory

Among the various paradigmatic changes in science and mathematics in this century, one such change concerns the concept of uncertainty. According to the traditional view, science should strive for certainty in all its manifestations hence,
uncertainty (vagueness) is regarded as unscientific. According to modern view, uncertainty is considered essential to science; it is not only an unavoidable plague, but it has, in fact, a great utility. L.A. Zadeh in 1965 introduced the notion of fuzzy sets to describe vagueness mathematically in its very abstractness and tried to solve such problems by giving a certain grade of membership to each member of a given set. This in fact laid the foundations of fuzzy set theory. Zadeh has defined a fuzzy set as a generalisation of the characteristic function of a subset. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse, a value representing its grade of membership in the fuzzy set. The membership grades are very often represented by real numbers in the closed interval between 0 and 1. The nearer the value of an element to unity, the higher the grade of its membership. The fuzzy set theory has a wider scope of applicability than classical set theory in solving various problems. Fuzzy set theory in the last three decades has developed along two lines:

1. as a formal theory which got formalised by generalizing the original ideas and concepts in classical mathematical areas.

2. as a very powerful modeling language, that can cope with a large fraction of uncertainties of real life situations.

1.3 Intuitionistic fuzzy set theory

In 1983, K. Atanassov proposed a generalization of the notion of fuzzy set, known as Intuitionistic Fuzzy sets. He introduced a new component degree of non membership in addition to the degree of membership in the case of fuzzy sets with the requirement that their sum be less than or equal to one. The complement of the two
degrees to one is regarded as a degree of uncertainty. Since then a great number of theoretical and practical results appeared in the area of Intuitionistic Fuzzy sets.

1.4 Summary of the Thesis

The main objective of this thesis is to study frames in Fuzzy and Intuitionistic Fuzzy contexts. The whole work is divided into six chapters. A brief chapter wise description of the thesis is given below.

Chapter 1

This is devoted to the basic definitions and results concerning Frames, Fuzzy sets and Intuitionistic Fuzzy sets which are required in the succeeding sections. All results here are quoted from existing literature.

Chapter 2

In this chapter we introduce the notion of fuzzy frames and we prove some results, which include

- If $\mu$ is a fuzzy subset of a frame $F$, then $\mu$ is a fuzzy frame of $F$ iff each non-empty level subset $\mu_1$ of $\mu$ is a subframe of $F$.
- The category FuzzFrm of fuzzy frames has products.
- The category FuzzFrm of fuzzy frames is complete.

Chapter 3

In this chapter we introduce the notion of fuzzy quotient frames. The operation of binary meet and arbitrary join on a frame $F$ induces, through Zadeh's extension principle new operations on the partially ordered set $1^F$. Here we define a fuzzy-quotient frame of $F$ to be a fuzzy partition of $F$, that is, a subset of $1^F$ and having a frame structure with
respect to new operations. We also define and study fuzzy ideals over $F$. The results proved in this chapter include

- If $\mu$ and $\gamma$ are fuzzy frames of a frame $F$ having supremum property with respect to $\land$ and $\lor$ then $\mu \land \gamma$ and $\mu \lor \gamma$ are fuzzy frames of $F$.
- If $R$ is an invariant fuzzy binary relation on a frame $F$ then its fuzzy partition $P_R$ is a fuzzy quotient frame of $F$.
- The set $I_F$ of all fuzzy ideals of the frame $F$ is a frame.

Chapter 4

In this chapter we define and study the notion of intuitionistic fuzzy frames and obtain some results, which include

- If $A$ is an intuitionistic fuzzy set in $F$ then $A$ is an intuitionistic fuzzy frame of $F$ iff $\Box A$ and $\Diamond A$ (‘necessity’ and ‘possibility’ operators) are intuitionistic fuzzy frames of $F$.
- If $A$ is an intuitionistic fuzzy set on $F$ then $A$ is an intuitionistic fuzzy frame on $F$ iff every non-empty level set $A_t$, $t \in [0,1]$ of $A$ is a subframe of the frame $F$.
- The category $IFFrm$ of intuitionistic fuzzy frames has products.
- The category $IFFrm$ of intuitionistic fuzzy frames is complete.

Chapter 5

In this chapter we introduce the concept of Intuitionistic fuzzy Quotient frames and has obtained the result:
If $R$ is an invariant intuitionistic fuzzy similarity relation on a frame $F$ then its fuzzy partition $P_R$ is an intuitionistic fuzzy quotient frame of $F$.

Chapter 6

Here we establish the categorical link between frames and intuitionistic fuzzy topologies. The main results include the following:

- $U$ is a contravariant functor from the category IFTOP of intuitionistic fuzzy topological space to the category FRM of frames.
- $\Sigma$ is a contravariant functor from the category FRM of frames to the category IFTOP of intuitionistic fuzzy topological spaces.
- $\Sigma$ and $\Omega$ are adjoint on the right.

1.5 Basic Definitions and Results

1.5(a) Frames and Topological spaces

In the same way as the notion of Boolean algebra appears as an abstraction of the power set $P(X)$ of a set $X$, the notion of frame arises as an abstraction from the topology $\tau$ of the topological space $(X, \tau)$.

The following definitions are adapted from [BA], [BA], [BA], [D; P], [D; P], [J], [P], [V].

**Definition 1.5.1.** A frame is a complete lattice $L$ satisfying the distributive law $x \land (\lor S) = \lor \{x \land s | s \in S\}$ for all $x \in L$ and $S \subseteq L$, where $\land$ denotes binary meet and $\lor$ denotes arbitrary join.
Definition 1.5.2. A subset $M$ of a frame $L$ is a subframe of $L$ if $O_L, e_L \in M$ where $O_L$ and $e_L$ are respectively bottom and top element of $L$, and $M$ is closed under finite meets and arbitrary joins.

Note 1.5.3. Given $a, b \in L$ a frame, with $a \leq b$ then $[a, b] = \{x \in L \mid a \leq x \leq b\}$ is a frame but not a subframe of $L$.

Definition 1.5.4. For frames $L, M$ a map $h : L \rightarrow M$ is a frame homomorphism if $h$ preserves finite meets (including top or unit element) and arbitrary joins (including bottom or zero element). That is $h(a \land b) = h(a) \land h(b)$ and $h(\lor X) = \lor h(x)$ for all $a, b \in L$ and $X \subseteq L$.

Definition 1.5.5. For a family of frames $\{L_i \mid i \in I\}$, its product $L$ is the Cartesian product of underlying sets with $\leq$ defined as $(a_i)_{i \in A} \leq (b_i)_{i \in A}$ iff $a_i \leq b_i$ for all $i \in I$.

Definition 1.5.6. For any frame $F$, a subset $J \subset F$ is an ideal if, $J$ is a downset that is if $(a \in J, b \leq a) \Rightarrow b \in J$ and $J$ is closed under finite joins.

Proposition 1.5.7. The set $\mathcal{IF}$ of all ideals of a frame $F$ is a frame, under inclusion order.

There is an important relation between frames and topological spaces which we describe below. The category of frames and frame homomorphisms will be denoted by $\mathcal{Frm}$. The category of topological spaces and continuous maps will be denoted by $\mathcal{Top}$.

Definition 1.5.8. The contravariant functor $\Omega : \mathcal{Top} \rightarrow \mathcal{Frm}$ which assigns to each topological space $(X, \tau)$ its frame $\tau$ of open sets and to each continuous function
\( f: (X, \tau \to (X', \tau') \) the frame map \( \Omega(f): \tau' \to \tau \) given by \( \Omega(f)(u) = f^{-1}(u) \), where \( u \in \tau' \) is called the open functor from Top to Frm.

**Definition 1.5.9.** Let \( L \) be a frame. The spectrum of \( L \) is the set \( \text{pt}_L \) of all frame homomorphisms \( p: L \to \{0, 1\} \) with the spectral topology \( \mathcal{T}_{\text{pt}_L} = \{ \Sigma_x | x \in L \} \) where \( \Sigma_x = \{ p \in \text{pt}_L | p(x) = 1 \} \). The contravariant functor \( \Sigma: \text{Frm} \to \text{Top} \) which assigns to each frame its spectrum \( \Sigma(L) = (\text{pt}_L, \mathcal{T}_{\text{pt}_L}) \) and to each frame map \( f: L \to L' \) the continuous map \( \Sigma(f): \Sigma(L') \to \Sigma(L) \) given by \( \Sigma(f)(p) = p \circ f \), where \( p \) is a point of \( L' \) is called the spectrum functor from \( \text{Frm} \) to \( \text{Top} \).

**Theorem 1.5.10.** \( \Sigma \) and \( \Omega \) are adjoint on the right with adjunctions \( \eta_L: L \to \Omega \Sigma L \) given by \( a \mapsto \Sigma a \) and \( \varepsilon_x: X \to \Sigma \Omega X \) given by \( x \mapsto \bar{x} \) where \( \bar{x}(U) = \text{card}(U \cap \{x\}) \).

**1.5(b) Fuzzy Sets**

The following definitions are adapted from [DU; P], [K; Y], [MO; M], [OV], [ZA]_1, [ZI].

**Definition 1.5.11.** A fuzzy set \( \mu \) of a set \( X \) is a function from \( X \) to \( I = [0, 1] \).

**Definition 1.5.12.** The set all fuzzy sets of \( X \), denoted by \( I^X \) is the set of all functions from \( X \) to \( [0, 1] \).

**Definition 1.5.13.** Let \( \mu \) and \( \gamma \) be fuzzy sets of a non empty set \( X \). Then

\[ \mu = \gamma \iff \mu(x) = \gamma(x) \text{ for all } x \in X \]
\[ \mu \subseteq \gamma \iff \mu(x) \leq \gamma(x) \text{ for all } x \in X \]

\[ \mu \lor \gamma = \delta \iff \delta(x) = \max \{ \mu(x), \gamma(x) \} \text{ for all } x \in X \]

\[ \mu \land \gamma = \delta \iff \delta(x) = \min \{ \mu(x), \gamma(x) \} \text{ for all } x \in X \]

**Definition 1.5.14.** Let \( \{ \mu_{\alpha} | \alpha \in \Lambda \} \subseteq I^X \). Then define \( \bigcap_{i} \mu_{i}(a) = \inf \{ \mu_{\alpha}(a) | \alpha \in \Lambda \} \)

and \( \bigcup_{i} \mu_{i}(a) = \sup \{ \mu_{\alpha}(a) | \alpha \in \Lambda \} \).

**Definition 1.5.15.** If \( \mu \) is fuzzy set of \( X \), for any \( t \in I \) the set \( \mu_t = \{ a \in X | \mu(a) \geq t \} \) and 

\[ \hat{\mu}_t = \{ a \in X | \mu(a) > t \} \] are respectively called level subset and strong level subset of \( \mu \).

**Definition 1.5.16.** If \( \mu \) is fuzzy set of \( X \) then the height of \( \mu \) is defined by 

\[ \text{hgt}(\mu) = \sup_{x \in X} \mu(x). \]

**Proposition 1.5.17.** Let \( \mu \) and \( \gamma \) be fuzzy sets of a non empty set \( X \). Then 

\[ (\mu \lor \gamma)_t = \mu_t \lor \gamma_t. \]

**Definition 1.5.18.** Let \( X \) and \( Y \) be two non empty sets and \( \mu \) any fuzzy set of \( X \). Let \( f \) a function from \( X \) into \( Y \). Then \( \mu \) is said to be \( f \)-invariant if for all \( x, y \in X \), \( f(x) = f(y) \Rightarrow \mu(x) = \mu(y). \)

**Proposition 1.5.19.** Let \( f \) be a mapping from a set \( S \) to a set \( M \) and let \( \{ \mu_{\alpha} | \alpha \in \Lambda_1 \} \)

and \( \{ \lambda_{\alpha} | \alpha \in \Lambda_2 \} \) be families of fuzzy sets in \( S \) and \( M \) respectively. Then we have,
i) \( f( \bigcup_{a \in \Lambda_1} \mu_a) = \bigcup_{a \in \Lambda_1} f(\mu_a) \) ii) \( f^{-1}( \bigcup_{a \in \Lambda_2} \lambda_a) = \bigcup_{a \in \Lambda_2} f^{-1}(\lambda_a) \) iii) \( f f^{-1}(\lambda_a) = \lambda_a \) if \( f \) is surjective iv) \( f^{-1}(\mu_a) = \mu_a \) if \( \mu_a \) is \( f \)-invariant.

**Definition 1.5.20.** Let \( \otimes \) be any arithmetic operation and \( A, B \) any two fuzzy numbers then by Zadeh's extension principle \( A \otimes B \) is a fuzzy set given by

\[
A \otimes B(z) = \sup_{x \otimes y = z} \min\{A(x), B(y)\}
\]

**Definition 1.5.21.** A fuzzy binary relation \( R \) of a set \( X \) is a function from \( X \times X \) to \( I \) where \( I = [0, 1] \).

**Definition 1.5.22.** A fuzzy binary relation \( R \) on a set \( X \) \((R \in [1]^X \times X)\) is said to be a fuzzy similarity relation if it satisfies for all \( x, y, z \in X \)

1. \( R(x, x) = 1 \) (reflexive)
2. \( R(x, y) = R(y, x) \) (symmetric)
3. \( R(x, y) \land R(y, z) \leq R(x, z) \) (transitive)

### 1.5(c) Intuitionistic Fuzzy Sets

The following definitions are adapted from [AT]$_1$, [AT]$_2$, [B;B]$_1$, [CO]$_1$, [CO]$_2$, [D;K]$_1$

**Definition 1.5.23.** An intuitionistic fuzzy set \( A \) in a nonempty set \( X \) is an object having the form \( A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\} \) where the functions \( \mu_A : X \rightarrow [0,1] \) and \( \gamma_A : X \rightarrow [0,1] \) denote the degree of membership and degree of nonmembership respectively and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for all \( x \in X \).
Definition 1.5.24. Let $X$ be a non empty set and let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in X\}$ be intuitionistic fuzzy sets in $X$. Then,

i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$

ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

iii) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) \mid x \in X\}$

iv) $A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) \mid x \in X\}$

v) $A \cup B = \{(x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x)) \mid x \in X\}$

vi) $\Box A = \{(x, \mu_A(x), 1-\mu_A(x)) \mid x \in X\}$

vii) $\Diamond A = \{(x, 1-\gamma_A(x), \gamma_A(x)) \mid x \in X\}$

Remark 1.5.25. Operators $\Box$ and $\Diamond$ are called [AT] respectively ‘necessity’ and ‘possibility’ which will transform every intuitionistic fuzzy set in to a fuzzy set.

Definition 1.5.26. Let $\{A_i \mid i \in \Lambda\}$ be an arbitrary family of intuitionistic fuzzy sets in $X$ then,

i) $\bigcap_{i \in \Lambda} A_i = \{(x, \land \mu_{A_i}(x), \lor \gamma_{A_i}(x)) \mid x \in X\}$

ii) $\bigcup_{i \in \Lambda} A_i = \{(x, \lor \mu_{A_i}(x), \land \gamma_{A_i}(x)) \mid x \in X\}$

Definition 1.5.27. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy set in $X$. For any $t \in [0,1]$, $A_t = \{x \in X \mid \gamma_A(x) \leq t \leq \mu_A(x)\}$ is called a level subset of the intuitionistic fuzzy set $A$. 

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Result 1.5.28. Let A and B be intuitionistic fuzzy sets of a non empty set X. Then
\[(A \cup B)_t = A_t \cup B_t.\]

Definition 1.5.29. Let X and Y be two non empty sets. An intuitionistic fuzzy relation R is an intuitionistic fuzzy set of X×Y given by,
\[R = \{((x, y), \mu_R(x, y), \gamma_R(x, y)) | x \in X, y \in Y\}\]
where \[\mu_R : X \times Y \rightarrow [0, 1]\]
and \[\gamma_R : X \times Y \rightarrow [0, 1]\] satisfy the condition, \[0 \leq \mu_R(x, y) + \gamma_R(x, y) \leq 1\] for every \((x, y) \in X \times Y.\)

IFR( X× Y ) denote the set of all intuitionistic fuzzy subsets of X×Y.

1.5(d) Category Theory

The following definitions are adapted from [A; H; S], [BO], [JO], [MA]

Definition 1.5.30. A category C consists of three things:

(b) A class of object, ob C denoted by capital letters

(c) For each ordered pair of objects (A, B), a set hom(A, B) whose elements are called morphisms with domain A and codomain B.

(d) For every ordered triple of objects (A, B, C) a map \((f, g) \mapsto g \circ f\) of the product set hom(A, B)× hom(B, C) into hom(A, C).

Also the objects and morphisms satisfy the following conditions

1. If (A, B) \neq (C, D) then hom(A, B) and hom(C, D) are disjoint.

2. If \(f \in \text{hom}(A, B), g \in \text{hom}(B, C)\) and \(h \in \text{hom}(C, D)\) then \((hg)f = h(gf)\).
3. For every object $A$ we have an element $I_A \in \text{hom}(A, A)$ such that $f \circ I_A = f$ for every $f \in \text{hom}(A, B)$ and $I_A \circ g = g$ for every $g \in \text{hom}(B, A)$

**Definition 1.5.31.** Let $C$ be a category then the dual category of $C$ is denoted by $C^{\text{op}}$ and is defined as,

(a) $\text{ob } C^{\text{op}} = \text{ob } C$

(b) $\text{hom}_{C^{\text{op}}}(A, B) = \text{hom}_C(B, A)$

(c) If $f \in \text{hom}_{C^{\text{op}}}(A, B)$ and $g \in \text{hom}_{C^{\text{op}}}(B, D)$ then $g \circ f$ (in $C^{\text{op}}$) = $f \circ g$ (as given in $C$)

(d) $I_A$ is as in $C$.

**Definition 1.5.32.** Let $C$ and $D$ be two categories, then a covariant functor $F : C \to D$ consists of,

(a) A map $A \mapsto F(A)$ of $\text{ob } C$ into $\text{ob } D$

(b) For every pair of objects $(A, B)$ of $C$ a map $f \mapsto F(f)$ of $\text{hom}_C(A, B)$ into $\text{hom}_D(F(A), F(B))$.

Also these satisfy the following conditions:

1. If $g \circ f$ is defined in $C$ then $F(g \circ f) = F(g) \circ F(f)$

2. $F(I_A) = I_{F(A)}$

**Definition 1.5.33.** A contravariant functor from $C$ to $D$ is defined to be a covariant functor from $C^{\text{op}}$ to $D$. 

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Definition 1.5.34. Let \( f \) and \( g \) be \( C \)-morphisms from \( A \) to \( B \). A pair \((E, e)\) is called an equalizer in \( C \) of \( f \) and \( g \) if (1) \( e: E \to A \) is a \( C \)-morphism (2) \( f \circ e = g \circ e \) and (3) for any \( C \)-morphism \( e': E' \to A \) such that \( f \circ e' = g \circ e' \), there exist a unique \( C \)-morphism \( \overline{e}: E' \to E \) such that \( e' = e \circ \overline{e} \).

Definition 1.5.35. Let \( \{A_\alpha | \alpha \in \Lambda \} \) be an indexed set of objects in a category \( C \) we define a product \( \prod A_\alpha \) of the \( A_\alpha \) to be a set \( \{A, P_\alpha | \alpha \in \Lambda \} \) where \( A \in \text{ob } C \), \( P_\alpha \in \text{hom}_C (A, A_\alpha) \) such that if \( B \in \text{ob } C \) and \( f_\alpha \in \text{hom}_C (B, A_\alpha), \alpha \in \Lambda \) then there exist a unique \( f \in \text{hom}_C (B, A) \) such that \( P_\alpha \circ f = f_\alpha \).

Result 1.5.36. A category \( C \) is complete if and only if it has equalizers and products over arbitrary sets of objects.

Definition 1.5.37. Let \( C \) and \( D \) be two categories and \( F \) and \( G \) be two functors from \( C \) to \( D \). Then a natural transformation \( \eta \) from \( F \) to \( G \) is a map that assigns to each object \( A \) in \( C \) a morphism \( \eta_A \in \text{hom}_D (F A, G A) \) such that for any object \( A, B \) of \( C \) and any \( f \in \text{hom}_C (A, B) \) we have \( G(f) \circ \eta_A = \eta_B \circ F(f) \).

Definition 1.5.38. Let \( A \) and \( X \) be categories. An adjunction from \( X \) to \( A \) is a triple \((F, G, \varphi) : X \to A\), where \( F \) and \( G \) are functors \( X \xrightarrow{G} A \) while \( \varphi \) is a function which assigns to each pair of objects \( x \in X, a \in A \) a bijection \( \varphi = \varphi_{x,a} : A(Fx, a) \cong X(x, Ga) \).
Here \( A( Fx, a) \) is a bifunctor \( X^{\text{op}} \times A \xrightarrow{F \times \text{id}} A^{\text{op}} \times A \xrightarrow{\text{hom}} \text{set} \) which sends each pair of objects \((x, a)\) to the hom-set \( A( Fx, a) \) and \( X(x, Ga) \) is a similar bifunctor \( X^{\text{op}} \times A \rightarrow \text{set} \). The naturality of the bijection \( \varphi \) means that for all \( k: a \rightarrow a' \) and \( h: x' \rightarrow x \) both the diagrams:

\[
\begin{array}{ccc}
A( F x, a) \xrightarrow{\varphi} X( x, G a) & & A( F x, a) \xrightarrow{\varphi} X( x, G a) \\
(\text{Gk})^* & & (\text{Fh})^* \\
\downarrow & & \downarrow \\
A( F x', a') \xrightarrow{\varphi} X( x', G a') & & A( F x', a') \xrightarrow{\varphi} X( x', G a')
\end{array}
\]

commute. Here \( k^* = A( F x, k) \) and \( h^* = X(h, G a) \)

Remark 1.5.39. Adjunction may also be described as bijections which assigns to each arrow \( f: F x \rightarrow a \) an arrow \( \varphi f: x \rightarrow G a \) the right adjunct of \( f \), such that the condition of (1) \( \varphi ( f \circ F h ) = \varphi f \circ h , \varphi ( k \circ f ) = G k \circ \varphi f \) hold for all \( f \) and all arrows \( h: x' \rightarrow x \) and \( k: a \rightarrow a' \). Given such an adjunction, the functor \( F \) is said to be a left adjoint for \( G \), while \( G \) is called a right adjoint for \( F \).