CHAPTER - I

INTRODUCTION
1.1. INTRODUCTION TO STOCHASTIC MODELS

The chief object of any scientific research is to suggest an adequate mathematical description of some natural or artificial phenomenon and to obtain solutions (to the model suggested) with help of the known tools and techniques. A model may be defined as a mathematical idealization used to approximate an observable phenomenon. In order to propose a model, the researcher has to consider certain assumptions. The success of the model is mainly based on the validity of these assumptions. In other words, if the assumptions are more realistic, the validity of the model is high. In recent times, there is a growing awareness to study Non-deterministic / Random / Stochastic models (S.M.s') rather than Deterministic or Probabilistic models. Briefly, a 'stochastic model' is defined as an idealized mathematical description of a random phenomenon. The word 'stochastic' derives from 'στoχαστικός', a target and a 'στoχαστικός' was a person who forecast future
event in the sense of aiming at the truth. In this sense, it entered English language in the sixteenth-century. Bernoulli in the 'Ars Conjectandi' (1719) refers to the 'ars Conjectandi sive stochastice'. The word passed out of usage in the twentieth century[7].

The most interesting system where S.M.s' can fruitfully be applied are those involving randomness in the systems themselves, rather than in their inputs. The systems of interest may be a large dynamical multi-input and multi-output systems such as national economy of any country, a medium such as atmosphere, ocean and so on [4]. As the economy of any nation is a multi-input and multi-output system, stochastic models are alone capable of solving problems relating to the system. The economy of any nation is mainly relied on an efficient banking system and its policies. For instance, changes in the interest rates will have direct influence on the saving patterns, which in turn influence the economy. Thus banking system in any nation has an important role to play in the development of any nation's economy. Development implies a progressive change from the existing situation to a new situation. Development in the banking reserves will give rise to the increasing scope of
potential investment which increase the national income. Thus potential investment policies are healthy veins for a mighty nation. In a developing economy like India, banking system occupies a prominent position in increasing the resources for investing in various sectors like agriculture, industry, research and so on.

In order to plan for prospective investments, first of all one must have the knowledge of resources available in the banking system. Thus the knowledge of the reserve level available with the system is essential for planning. Hence, in recent times, many researchers concentrate more on estimating the reserve level available with the system with the help of sophisticated tools and techniques.

Thus in this thesis, an attempt is made to know the money reserves available with the banking system by applying a S.M. suitable for the system. Since banking system is multi-input and multi-output one, stochastic models can fruitfully be applied to get the solutions for the problems relating to.
1.2. GAP, MOTIVATION AND OBJECTIVES:

Many researchers contributed their valuable works to obtain solutions to the problems arising in day today life in various fields like Biological and Physical sciences, Management, Cybernetics, Banking and so on through stochastic model's approach. In 1974, Venugopal [39] in his doctoral thesis proposed a stochastic storage model and obtained solutions relating to dam models. The model considered by him is briefly explained as follows:

Model I:

Let $X(t)$ denote the random variable (r.v.) representing the level of the dam at any arbitrary time, $t$. The reserve level, $X(t)$ is the result of the interaction (not in the sense of stochastic dependence) of two independent renewal processes namely (i) inputs and (ii) inter-input times, each governed by known probability laws. The rate of output/release is assumed to be uniform and decreases 'LINEARLY WITH A UNIT RATE' along with time, $t$. The inputs are assumed to increase the reserve level instantaneously at random epochs of time.
In the year 1983, Sarma [15] in his doctoral thesis, proposed a stochastic storage model by interchanging the assumptions of the dam model proposed by Venugopal [39] relating to inputs and outputs. In Venugopal's model, inputs are stochastic in nature whereas outputs are deterministic. In Sarma's model, the inputs are deterministic and the outputs are stochastic in nature. The model is more appropriate to the banking system because inputs can be considered deterministic, for instance, time deposits are deterministic and withdrawals are stochastic in nature. The model considered by Sarma [15] is briefly explained as follows:

Model II:

Let \( X(t) \) denotes the r.v. representing the money reserve available with the system at any arbitrary time, \( t \). The reserve level, \( X(t) \) is the inter-action (not in the sense of stochastic dependence) of two independent renewal processes namely i) the amount of withdrawals and ii) the inter-withdrawal times. The rate of input is assumed to be deterministic, that is, increases 'LINEARLY
WITH A UNIT RATE along with time, t. The outputs/withdrawals are assumed to decrease the reserve level instantaneously at random epochs of withdrawals.

It is important to note that in both the models explained above the outputs (in Model I) and inputs (in Model II) are assumed to be 'LINEAR AT A UNIT RATE'. This assumption makes the model applicable in limited situations. In particular, the stochastic banking model (Model II) is applicable for those systems where the input increases linearly at a unit rate along with the time, t. In fact, this situation is a rare one because in any banking system, the rate of input need not be strictly increasing along with the time t at a unit rate. In order to make the model more flexible one, suitable for the situations coming across in the live data collected from banks, the assumptions of 'Unit rate of increase' is to be relaxed. Thus there is a necessity to generalize the model II to widen the scope of the applicability of the model. This motivated us to work further, to generalize the model (Model II) by relaxing the assumption of 'Unit rate of input'.
The chief objective of the thesis is two-fold namely:

(i) To propose a more generalized and practically valid stochastic banking model and to obtain both analytic and explicit solutions for the related problems in banking system. In other words, an attempt is made to generalize Model II by relaxing the unit rate of input into the system. Here we have assumed that the input rate is a function of time \( t \), say \( f(t) \) and

(ii) to demonstrate the utility of the results obtained by us with the help of a live data collected at random from local banks. Thus an attempt is made in this thesis to make the expressions speak with a live example which are dumb otherwise.

1.3. CHapter Summaries:

As already mentioned in (1.2), the main objective of the thesis is to propose a generalized stochastic banking model and to obtain both analytic and explicit solutions and finally to demonstrate the utility of the results obtained by us with a live data. The thesis is divided into six chapters
of which the first chapter is an introductory one and the last chapter is completely devoted to the application of the results obtained in the fifth chapter. Finally, the thesis is appended with an extensive list of references. The contents of each chapter are briefly explained as follows:

The present chapter I is introductory in nature in which the basic concepts, tools and techniques used in the thesis are introduced. Finally, one section is completely devoted to review the earlier literature relevant to the work presented in this thesis.

In chapter II, we propose a stochastic banking model applicable to realistic situations which is more generalized one like the model proposed by Sarma [15]. The model is briefly explained as follows:

Let \( \{X(t)\} \) denote the general stochastic process (S.P.) describing the reserve level which is the conjoint effect of two independent renewal processes namely:

- (i) inter-withdrawal times
- (ii) the amount of withdrawals.
The rate of input fed into the system is assumed to be a function of $t$ (the time element), say $f(t)$. The random withdrawals spilt the reserve level of the system instantaneously.

Further, we have introduced two conditional probability density functions (c.p.d.f.s'), $M(x, y, t)$ and $M_1(x, y, t)$ as follows:

$$M(x, y, t) = \lim_{\Delta \to 0^+} \Pr \left[ 0 < x < X(t) \leq x + \Delta, X(u) > 0, u \in [0, t] \right] \frac{X(0^-) > X(0) = X(0^+)}{\Delta} \quad (1.3.1)$$

and

$$M_1(x, y, t) = \lim_{\Delta \to 0^+} \Pr \left[ 0 < x < X(t) \leq x + \Delta, X(u) > 0, u \in [0, t], \right] \frac{X(t^-) > X(t) = X(t^+)}{\Delta} \frac{X(0^-) > X(0) = X(0^+)}{\Delta} \quad (1.3.2)$$

These govern the reserve level, $X(t)$ at any given time, $t$ with the other intermediary and initial conditions as:

(i) the origin is a point of regeneration in $M(x, y, t)$ and $M_1(x, y, t)$.

(ii) both the origin and epoch 't' in $M_1(x, y, t)$ are points of regeneration.
Integral equations are formed for the above two c.p.d.f.s' namely $M(x,y,t)$ and $M_t(x,y,t)$ introduced in (1.3.1) and (1.3.2) and time-dependent solutions in terms of $M^*(x,s,p)$ and $M_t^*(x,s,p)$ are obtained with the help of double Laplace transform (d.l.t.) technique. Analytic solutions are obtained for $M/M/1$, $M/G/1$, $G/M/1$ and $G/G/1$ models with an infinite capacity. Earlier results obtained by Sarma [15] are derived as special cases.

Chapter III is completely devoted to obtain explicit solutions for $M(x,y,t)$ and $M_t(x,y,t)$ which are useful to solve certain problems relating to the banking system with an infinite capacity. For instance, explicit expressions can be obtained for the characteristic function (c.f.) governing the inter-critical states.

A critical state here means that the reserve level, $x(t)$ reaching to a pre-specified level, say $C \geq 0$ from above. Explicit solutions are obtained for $M/M/1$, $M/G/1$ and $G/M/1$ models. Explicit expression is also obtained for $G/G/1$ model with the help of a duality theorem introduced by Sarma [15]. Earlier results obtained by Sarma [15] are derived as special cases.
In chapter IV, a more practically valid assumption is introduced into the model, that is, here it is assumed that the reserve level, $X(t)$ cannot exceed a pre-specified level, $k < \infty$ with probability one. Time-dependent solutions are obtained for $M(x,y,t)$ and $M_1(x,y,t)$ after introducing integral equations and after incorporating the assumption relating to the capacity of the reserve level. Analytic solutions are obtained for $M/M/1$, $M/G/1$, $G/M/1$, and $G/G/1$ models. Earlier results obtained by Sarma [15] are derived as special cases.

Chapter V is completely devoted to obtain explicit solutions of $M(x,y,t)$ and $M_1(x,y,t)$, for the models $M/M/1$, $M/G/1$ and $G/M/1$ with finite capacity. With the help of the above explicit results, expressions are obtained for

(i) the c.f. governing the inter-critical states,
(ii) the Laplace Transform (L.T) of the p.d.f. of the first critical state when the initial level is $y$ and 
(iii) the L.T. of the p.d.f. of the duration between any two critical states.

Finally, chapter VI is an application-oriented one, where the utility of the results obtained in previous
chapters are demonstrated with the help of a live example. For this purpose, five banks are at random selected from the local town and the data is collected and analyzed with the help of the results obtained by us. A brief discussion on further scope of the work is given in the last section of this chapter.

1.4. METHODOLOGY:

The concepts, tools and techniques used in this thesis are:

(a) Stochastic Point Process,
(b) Imbedded Regenerative Process,
(c) Renewal Process,
(d) Laplace Transform and
(e) Integral Representation Technique.

First we proceed to explain some preliminaries of the S.P.s' with a particular stress on stochastic point process.

1.4. (a): Some basic concepts of stochastic processes and point processes:

Since the last century, there have been marked
changes in the approach to scientific enquiries. There has been great realization that stochastic (or non-deterministic) models are more realistic than probabilistic models in many situations. This can be explained more effectively with the following illustration. Let us consider the well known "Poisson distribution" with the probability density function (p.d.f.):

\[ f(x) = P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,\ldots, \infty, \quad \lambda > 0 \quad (1.4.1) \]

Where \( \lambda \) is the parameter of the distribution which may be estimated from the sample data as:

\[ \hat{\lambda} = \frac{\sum_{i=1}^{n} x_i f_i}{N} \quad (1.4.2) \]

Where \( \hat{\lambda} \) denotes the estimate of \( \lambda \) and \( f_i \) denotes number of days on which \( x_i \), the number of telephone calls received, \( i=1,2,\ldots,n \) and \( N = \sum_{i=1}^{n} f_i \) is the number of days on which the data is collected.

The above example is a typical case suitable for static theory which will be useful to answer such questions.
like the average number of telephone calls per day or to obtain confidence intervals to the average number of telephone calls and so on. While this information is useful in its own way, the other important and practical questions from the view point of management like the average number of telephone calls coming in a specified period of time, say between 10 A.M. to 11 A.M., the law governing the random interval between two telephone calls cannot be answered with the help of (1.4.1) and related theory. This can be answered only when the phenomenon of telephone calls fits into a stochastic model along with the following postulates.

Postulates of a Poisson process:

Consider a random event $E$ such as incoming telephone calls at a switch board, arrival of customers into the bank for service, occurrence of accidents at a certain place and so on. Let $N(t)$ denote the total number of events ‘$E$’, happened in an interval of duration, $t$, that is, if we start from an initial epoch $t = 0$, $N(t)$ represents the number of occurrences of such random events ‘$E$’ upto the epoch $t$ (more precisely upto $t+0$). Let $P(n,t)$ or $P_n(t)$ be the probability
that the r.v. $N(t)$ takes the value $n$, that is
\[ P(n, t) = P_n(t) = Pr \[N(t) = n\], \quad n=0,1,2,... \quad (1.4.3) \]
and
\[ \sum_{n=0}^{\infty} P(n, t) = 1 \quad (1.4.4) \]

Under the following three basic assumptions, it can be shown that the r.v., $N(t)$ follows the Poisson law with mean $\lambda t$, that is,
\[ P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n=0,1,2,3,... \quad (1.4.5) \]

Postulate 1: The r.v., $N(t)$ is independent of the number of events occurred prior to the interval $[0,t]$, that is, future changes will depend only on the present but not on the past changes.

Postulate 2: The p.m.f., $P_n(t)$ depends on the length of the interval $[0,t]$ and is independent where this interval is situated, that is, $P_n(t)$ is the same for all the intervals $[a_i, a_{i+1}]$ for all values of $a_i$, $i=1,2,...,n$. 
Postulate 3: \[ \text{In an infinitely small interval of length } h, \]
\[ \text{the probability of occurrence of one event is} \]
\[ \lambda h + o(h) \text{ and is } o(h) \text{ if the occurred events} \]
\[ \text{are more than one. Here } o(h) \text{ is used to}
\]
\[ \text{denote a function of } h \text{ which tends to zero}
\]
\[ \text{more rapidly than } h, \text{ that is, as } h \to 0, \]
\[ o(h)/h = 0. \] [10]

In the above postulates, first one incorporates the
Markovian nature (or memoryless) nature into the process and
the second one incorporates the time homogeneity into the
process. From the third postulate, it can be written that if
the interval \([t, t+h]\) is of a short duration,

\[ p_t(h) = \lambda h + o(h) \] (1.4.6)

since \[ \sum_{k=2}^{\infty} p_k(h) = o(h), \] (1.4.7)

\[ \sum_{n=0}^{\infty} p_n(h) = 1, \] (1.4.8)

it follows that

\[ p_0(h) = 1 - \lambda h + o(h). \] (1.4.9)

In other words, from the third postulate, it can be
seen that the probability of occurrence of an event in a small interval $h$ is proportional to the length of the interval and occurrence of more than one event in this interval is a rare one. From (1.4.5), it can be seen that the parameter of the distribution is not a constant $\lambda$, but it is a function of $t$, that is, $\lambda t$ which exactly introduces the dynamic or stochastic element into the model which is more practical and capable of answering more effectively the problems arising in day today life. A Stochastic Process (S.P) is defined as a family of random variables (r.v.s'), $X(\omega, t)$ of two arguments $\omega \in \Omega$ and $t \in I$. $\Omega$ is called the state space or sample space and $I$ is called the index set or Parameter set. In other words, a S.P. is defined as $X = (X_t, t \in I)$ is a collection of r.v.s'. That is, for each $t$ in the index set $I$, $X(t)$ is a r.v. We often interpret $t$ as time and call $X(t)$, the state of the process at time $t$. The expressions CHANCE PROCESS and RANDOM PROCESS are sometimes used as synonyms with the expression STOCHASTIC PROCESS.

More precisely, a S.P. is a family $X = (X_t, t \in I)$ where $I$ is an index set of r.v.s' defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking real values in the real line $\mathbb{R}$, or a
countable set; the S.P. is then said to have a continuous or a discrete state space, respectively. The term "Stochastic Process" is usually applied to an infinite family. When I is denumerable, the process is a discrete parameter process, for example, a random sequence \( (X_n, n = 0,1,...) \). When I is noncountable, the process is a continuous parameter process, for example, a random function.

For a fixed point \( \omega \in \Omega \), a function of \( t \) defined by \( X_t(\omega) = X(.\!,\omega) \) is a sample function or path, a trajectory, or realization of the process \( X \).

In physical applications, \( t \) usually represents time, and the empirical approach describes the S.P. as any process whose evolution (in time) depends on chance. Probabilistic conditions are imposed on r.v.s' \( X_t \) to characterize special cases of processes and to investigate behaviour of sample functions.

Classifications of stochastic Process:

Classifications of S.P.s' may be arranged according to diverse criteria, which basically involve:
(1) probabilistic structure,
(2) special process,
(3) methods and problems and
(4) applications.

The notorious division into pure and applied fields is predominant in S.P.s', although this distinction is somewhat blurred if only the mathematical techniques are considered but is clearly visible in the object of interest [42].

At present, we briefly discuss the classification of the S.P.s' according to the nature of their state space and index set. Various types of S.P.s' are given in the following table.

Table (1.4.10) : Classification of S.P.s'

<table>
<thead>
<tr>
<th>Type</th>
<th>State space 'Ω'</th>
<th>Index Set 'I'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Discrete</td>
<td>Discrete</td>
</tr>
<tr>
<td>2</td>
<td>Continuous</td>
<td>Discrete</td>
</tr>
<tr>
<td>3</td>
<td>Discrete</td>
<td>Continuous</td>
</tr>
<tr>
<td>4</td>
<td>Continuous</td>
<td>Continuous</td>
</tr>
</tbody>
</table>
The S.P.s' with discrete state space and index set are called Markov Chains whereas with continuous state space and discrete index set are known as Markov processes [27]. A special S.P.s' of type 3 have very many interesting properties especially if the sample paths $X.(\omega)$ are non-decreasing functions of $t$ with probability one. In such a case, by a relabelling the indices that characterize the set $\Omega$, the r.v., $N_t$, can be assumed to take only positive integral values. The r.v. $N_t$ can be then identified with the process in which the random points are realized on a one-dimensional continuum, so that $N_t$ represents the count of the random points upto $t$ starting from an appropriate origin, 0. That is, $X(t)$ represents total number of points where $X(\omega,t) = X.(\omega)$ distributed in the interval $[0,t]$. The properties of such processes were studied by Wold [41] in the year 1948 who called them "point processes". A brief introduction to the point processes is given as follows:

Point Processes:

Stochastic Processes concerned with the occurrence of events at points of time determined by some chance mechanism; as distinct from processes observed at fixed intervals and
either continuous in the time variable or determinate at such fixed intervals. Thus point process is obtained when one considers a sequence of events occurring in continuous time, individual events are distinguished only by their position in time. Let $T_n, n = 0, 1, 2, \ldots$ be a sequence of positive r.v.s' such that $T_0 = 0, T_n < T_{n+1}, T_n \rightarrow T_\infty \leq \infty, T_n$ representing the instant of occurrence of the $n^{th}$ event. The total number of events in the interval from 0 to $t$ is a r.v. $N_t$ that may be written in the following form:

$$N_t = \sum_{n=0}^{\infty} I(T_n \leq t), \quad (1.4.11)$$

Where $I_A$ is the indicator of a set $A$. The family $N = (N_t, 0 \leq t < \infty)$ is a Counting Process and

$$N_t = n \text{ if } T_n \leq t < T_{n+1}. \quad (1.4.12)$$

Note that $N$ is an increasing process and its sample functions are right-continuous step functions with upward jumps of magnitude $1$. Each of the random sequence $(T_n)$ and the counting processes $(N_t)$ is known as a POINT PROCESSES. Typical examples include the Poisson process, the birth processes, the renewal process, and a process with positive independent increments. Further, if we consider the examples...
in Poisson process, random walks, the space was the positive half-line and points represented times of events, for example, the arrivals of calls at a telephone exchange or customers at a queuing system. Hence point process is a random distribution of indistinguishable points in an underlying space such that only finitely many fall in any bounded (relatively compact) set [42].

A stochastic point process can be defined as follows:

Definition (1.4.13) : A 'Stochastic Point Process' is defined as a random stream of successive occurrences of points-events, point events being understood as the occurrence of a phenomena at a time point in the domain, I, where the ensemble of a point process is defined as collection of a discrete set of points which belong to I.

Approaches to point processes:

There exist two approaches to the analysis of point process: a random counting measure approach applicable in general spaces and a Counting process approach valid only for point process on \( R_+ \).
From the random counting measure viewpoint, a point process on a (locally compact, separable) space $E$ is a random integer-valued measure $N = (N(A) : A \subseteq E)$, that is, finite on compact sets; the r.v. $N(A)$ represents the number of points in the set $A$. A point process $N$ admits a representation as a sum of locally point masses, that is, $N = \sum \varepsilon_x$ where $X$ are random elements of $E$ and $\varepsilon_x$ denotes the point mass (Dirac measure) at $x$ ($\varepsilon_x(A) = 1$ or 0 according to $x \in A$ or $x \notin A$), the number of summands is also random. Thus for each set $A$,

$$N(A) = \sum \mathbb{I}(x_i \in A), \quad (1.4.14)$$

where $\mathbb{I}$ denotes an indicator function, and for $f$, a function on $E$,

$$N(f) = \int f dN = \sum f(x_i) \quad (1.4.15)$$

is the integral of $f$ w.r.t. $N$. In a simple point process, the $x_i$ are distinct (that is, there are no coincidence points). A point process $N$ for $\mathbb{R}_+ = [0, \infty]$ can be described by the (arrival) counting process $(N_t)$ is the number of points (often termed arrivals) in $[0,t]$, by the arrival times $T_k = \inf \{t : N_t = K\}$, the time of $k^{th}$ arrival or by the interarrival times $v_i = T_i - T_{i-1}$ ($T_0 = 0$ is not an arrival time). Structure may be imposed via $(N_t)$, as for
non-homogeneous Poisson processes, via \((T_k)\) which is rare, or via \((V_t)\) as for renewal* processes.

In the random counting measure approach, a key tool is converge in distribution. Given point processes, \(N, N_1, N_2, \ldots\) on \(E\), \((N_n)\) converges in distribution to \(N\) if probability distributions converge weakly. Many important classes of point processes, especially Cox processes and infinitely divisible point processes, admit characterizations as the class of all point processes arising as limits in distribution under some rarefaction mechanism [42].

A typical realization of point process and its counting and interval processes are clearly depicted in figure (1.4.15a). We finally observe that the basic assumptions underlying point processes are:

Assumption 1: All epochs are distinct.

* Renewal process is a class of Stochastic Processes in which times between events are independently and identically distributed.
Figure 1.4.15a: A typical realisation of a point process and its associates.
Assumption 2 : \( N_t \) is finite if the interval is finite. It is infinite if the interval is finite.

Assumption 3 : Occurrence of point events is stochastic in nature, that is, point events do not occur at predetermined times.

The analysis relating to stochastic point process is mainly based on the concept of product density. Hence an introduction to product density is given as follows:

**Product density :**

The concept of product density in its most general form was introduced by Ramakrishnan [13] during the course of his investigation of the fluctuation problem of electromagnetic cascades. The utility of the product densities in the other contexts has also been amply demonstrated by Ramakrishnan [14]. Consider stochastic processes with state space \( x \) and index set \( I \) as continuous. The central quality of interest in Ramakrishnan's theory is \( dN(x,t) \) which represents the number of entities having the parametric values in the interval \( (x, x+dx) \) at time \( t \). We next assume that the probability that there is an entity with a
parametric value in \((x, x+dx)\) is proportional to \(dx\), while the probability that there exists more than one with parametric values in \((x, x+dx)\) is of order smaller than \(dx\). Thus, if \(p(n)\) is the probability that \(n\) entities have parametric values in \((x, x+dx)\), then:

\[
p(n) = \begin{cases} 
  f_4(x,t)dx + o(dx), & \text{if } n = 1 \\
  o(dx), & \text{if } n > 1 \\
  1 - f_4(x,t) + o(dx), & \text{if } n = 0.
\end{cases}
\]

(1.4.16)

The \(m\)th moment of \(n\) is given by:

\[
E(n^m) = E \{[dN(x,t)]^m\} = \sum_n n^m p(n)
\]

\[
= E \{dN(x,t)\} + o(dx)
\]

\[
= f_4(x,t)dx + o(dx)
\]

(1.4.17)

From the above equation, it can be noted that for all values of \(m\), the moments of the stochastic variable \(dN(x,t)\) is equal to the probability that the stochastic variable takes the value one. The variable \(N(x,t)\) has an interesting interpretation which can be considered as the random variable representing the number of entities with the parametric values less than or equal to \(x\). Because of the interesting properties and assumptions of \(dN(x,t)\), the variable, \(N(x,t)\)
is a random function of points on the t-axis and of intervals or sets of intervals on the x-axis.

With the help of (1.4.17), it is easy to obtain a formula for the moments of the number of entities in a given interval \([a,b]\) of parametric space. Thus the mean number of entities is given by

\[
E \left\{ \left[ N(b,t) - N(a,t) \right] \right\} = E \left[ \int_a^b dN(x,t) \right]
\]

(1.4.18)

a relation which makes manifestly clear the idea that while \(f(x,t) \, dx\) has a probability density interpretation, the integral of \(f(x,t)\) over \(x\) yields only the mean number of entities in the range of integration, that is, \([a,b]\). An equivalent method of interpreting such a situation due to Bartlett [2] consists in choosing that only probability densities can be attached to particular values of \(x\) and not non-zero probabilities.

The mean square number of entities is formally given by:
\[ E \{ [N(b,t) - N(a,t)]^2 \} = \int_a^b \int_a^b E[dN(x_1,t)dN(x_2,t)]. \quad (1.4.19) \]

In view of the singular behaviour of r.v., \( dN(x,t) \), the line \( x_1 = x_2 \) in the domain of integration yields a non-negligible contribution to the integral. Accordingly we can split the domain of integration into two parts according as \( x_1 \neq x_2 \) and \( x_1 = x_2 \). Thus:

\[ E \{ [N(b,t) - N(a,t)]^2 \} = \int_a^b \int_a^b E[dN(x_1,t)dN(x_2,t)]_{x_1 \neq x_2} \]
\[ + \int_a^b \int_a^b E[dN(x_1,t)dN(x_2,t)]_0(x_1-x_2) \, (1.4.20) \]

Where \( \delta \) is the Dirac delta function. Ramakrishnan [14] has defined the product density of degree two by

\[ E[dN(x_1,t)dN(x_2,t)]_{x_1 \neq x_2} = f_2(x_1,x_2,t)dx_1dx_2. \quad (1.4.21) \]

The equation (1.4.21) can be interpreted as the simultaneous probability that an entity has a parametric value in \( (x_1,x_1+dx_1) \) and another in \( (x_2,x_2+dx_2) \) provided the two differential elements do not overlap, this being regardless of the entities assuming values in other ranges of \( x \). Thus (1.4.21) can be simplified using (1.4.17) as:

\[ E \{ [N(b,t) - N(a,t)]^2 \} = \int_a^b \int_a^b f_2(x_1,x_2,t)dx_1dx_2 + \int_a^b f_4(x)dx. \quad (1.4.22) \]
Equations (1.4.18) and (1.4.22) have also been obtained by Kendall [8] in his study of the stochastic fluctuation in the age distribution. Kendall [8] has defined the cumulant densities by:

\[ \alpha(x,t)dx = \text{var} \left\{ dN(x,t) \right\} = E\{dN(x,t)\} \]  
\[ \nu(x,y,t)dxdy = \text{cov} \left\{ dN(x,t) \right\} \left\{ dN(y,t) \right\}_x \neq y \]  

from which we obtain:

\[ \nu(x,y,t) = f_2(x,y,t) - f_1(x,t)f_1(y,t) \]  

The relations given in (1.4.16) have also been noticed by Kendall [8] who for this reason observes that \( n \) is asymptotically a Poisson variable, whose mean is equal to its variance (to first order in \( dx \)).

On similar lines, we can define higher order product densities as follows:

\[ f_n(x_1,x_2,\ldots,x_n,t) \, dx_1 \, dx_2 \ldots dx_n \]
\[ = E \{ dN(x_1,t) \, dN(x_2,t) \ldots dN(x_n,t) \}. \]  

Equation (1.4.26) represents the joint probability that there exists one entity having a parametric value in \((x_1,x_1+dx_1)\), one having a parametric value in \((x_2,x_2+dx_2)\),..., and one having a parametric value in \((x_n,x_n+dx_n)\) provided the intervals \( dx_1, dx_2, \ldots, dx_n \) do not overlap. If, however, two
of the dx's overlap, then right-hand side of (1.4.26) becomes:

$$E\{ \frac{dN(x_1,t)}{dN(x_2,t)} \cdots \frac{dN(x_{n-4},t)}{dN(x_{n-1},t)} \}^2$$

$$= E \{ \frac{dN(x_1,t)}{dN(x_2,t)} \cdots \frac{dN(x_{n-4},t)}{dN(x_{n-1},t)} \}$$

$$= f_{n-4}(x_1, x_2, \ldots, x_{n-1}, t) \, dx_1, dx_2, \ldots, dx_{n-1}. \tag{1.4.27}$$

Here it is assumed that $dx_{n-1} = dx_n$. The degeneracy to a lower order product density is due to the fact that every one of the moments of $dN(x,t)$ is equal to the probability that the r.v. $dN(x,t)$ takes the value 1.

To obtain the $r^{th}$ moment of the number of entities with parameters distributed over the interval $[a,b]$, we have proceed in a similar manner given for (1.4.18) through (1.4.20) and take into account the degeneracies that will arise due to the overlap of the differential elements of the parametric space. Hence we can write:

$$E \{ [N(b,t) - N(a,t)]^r \}$$

$$= \sum_{a}^{b} C_{r}^{s} \int_{a}^{b} \cdots \int_{a}^{b} f_s(x_1, x_2, \ldots, x_s; t) \, dx_1, dx_2, \ldots, dx_s \tag{1.4.28}$$

Where the coefficient $C_{r}^{s}$ denoting the number of $(r-s)$-fold degeneracy in an $r$-fold product is a function of $r$ and $s$.
only and in no way dependent on the product density $f$. To calculate $C^r$ we consider the special case when the total number of entities are fixed, say $N$. The product density of degree one can be written as:

$$f_1(x,t) \, dx = N f_1^o(x,t) \, dx$$  \hspace{1cm} (1.4.29)

with the condition

$$\int f_1^o(x,t) \, dx = 1$$  \hspace{1cm} (1.4.30)

where $f_1^o(x,t) \, dx$ represents the probability that a particular entity has a parametric value between $x$ and $x + dx$. Similarly we find $f_s(x_1,x_2,\ldots,x_s; t)$

$$= N(N-1)\ldots(N-s+1) f_1^o(x_1,t) f_1^o(x_2,t)\ldots f_1^o(x_s,t). (1.4.31)$$

$s = 2, 3, \ldots N$

By substituting (1.4.31) in (1.4.28) and integrating over the entire $x$-space, we get:

$$N^r = \Sigma C^r_s N(N-1)\ldots(N-s+1).$$  \hspace{1cm} (1.4.32)

The coefficients are obtained by taking $N = 1, 2, \ldots$ successively.

Further the characteristic and generating functionals can also be defined and their applications are given in an excellent manner by Srinivasan in the year 1969 [17].
1.4(b) Imbedded Regenerative Process:

Markov property is an important criterion to form integral or differential equations and to obtain results relating to stochastic storage models. In many situations, this Markov property holds good and obtaining results to those situations are easier. But sometimes we come across situations which are non-Markovian in nature. For instance, in the queuing systems, $M/E_k/1$ and $E_k/M/1$ if we want to obtain the solution for the queue length, $L(t)$, it becomes difficult because of the non-Markovian nature of the distribution $E_k$ ($E_k$ represents Erlangian distribution with $k$ states). In order to obtain solutions to these models, Markovian nature is to be introduced into the model. The task of introducing Markovian nature into the model can be achieved through imbedded regenerative process technique. The concept of 'point of regeneration' was first introduced by Palm in the year 1943 [12] in his studies relating to the flow of telephone traffic. In his words, a point of regeneration is defined as follows:

Definition (1.4.33): If a S.P. described by a random function $x(t)$ is such that for some particular epoch $T$ and for all $t > T$, 

\[ x(t) \]

\[ \text{such that } \]
dist [x(t) | x(T)] = dist [x(t) | x(\tau) for all \tau < T],
\hspace{1cm} (1.4.34)

then the epoch T or the event by which it is identified is said to be a point of regeneration for the process [27].

In brief, we can say that at the epochs of regeneration the process forgets its past history and starts afresh from scratch which is exactly the Markovian property to be introduced into the model. Those processes which admit such regenerative points are called regenerative processes. For instance, in single server queuing system, the set of points at which the counter is free are regenerative points and hence the queuing system is regenerative. Such regenerative processes are abundant in the fields of queuing, storage, reliability, replacement and in the theories of Markov and Semi-Markov processes. During the classical period, before 1965, researchers concentrated on the fields like reliability and replacement. Recent research is concentrated more with cases like Brownian motion, and the interest is on the structure of the random set of all regeneration times and on the excursions of the process between any two regenerations [42].
Let us consider a non-Markovian S.P., $X(\omega,t)$ in continuous time. It is often possible to pick out a series of regenerative points in discrete time. Let these points be denoted by $t_1, t_2, \ldots, t_n$ such that $t_1 < t_2 < \ldots < t_n$. Since the process $X(\omega,t)$ has regenerative points, it is called regenerative process. Let us define a new process at the points of regeneration namely, $t_1, t_2, \ldots, t_n$, such that the transition probabilities associated with $\{X(t_{n+1})/X(t_n)\}$ can be calculated in a simpler manner. This type of analysis through a redefined process at regenerative points hidden in the general S.P., $X(\omega,t)$ is called 'imbedded regenerative process' technique. This newly defined process in discrete time which is Markovian is said to be embedded in the old process in continuous time which is a non-Markovian nature. Thus with the help of imbedded regenerative process technique, a non-Markovian process can be viewed as a Markovian process and solutions can be obtained. To fix the ideas clearly, the technique of imbedded regenerative process is explained with an example as follows:

Example (1.4.35) : In telephone traffic theory, the holding times have a negative exponential density of the form $e^{-\lambda \tau} \, d\tau \quad (0 < \tau < \infty)$ and the time intervals between
consecutive incoming calls are statistically independent. Then the epoch of the arrival of the new call is a point of regeneration for the process. We assume permissibly that the time intervals between consecutive incoming calls have a negative exponential distribution, and the process will then be Markovian. Even, as Palm [12] pointed out, the secondary process of 'blocked cells' (which may constitute the input process for some other time of the system) will not be Markovian but it will still have the regenerative property, the epoch of each blocked cell being a point of regeneration. Hence even when the most drastic simplicity assumptions have been made the consideration of a regenerative non-Markovian process cannot be avoided if the traffic on a number of independent lines is to be considered [19 and 20].

1.4 (c) Renewal Process:

The classic Poisson process is mainly based on the fundamental assumption that the intervals between successive occurrences of events $E$ are independently and identically distributed with a negative exponential distribution. That is, if $t_0, t_1, \ldots, t_n$ are the epochs of occurrence of an event,
the r.v.s', $x_n = t_n - t_{n-1}$, $n = 1, 2, 3, \ldots$ are independent and they follow a negative exponential distribution. This can be generalized by assuming a general distribution in the place of negative exponential distribution. This idea gave birth to a new process called renewal process which is defined as follows:

Definition (1.4.36) : Let $\{ x_n, n = 1, 2, \ldots \}$ be a sequence of non-negative independent random variables. Assume that $\Pr \{ x_n = 0 \} < 1$, and that the r.v.s' are identically distributed with a distribution function $F(.)$. Since $X_n$ is non-negative, it follows that $E \{ X_n \}$ exists and let us denote:

$$E (X_n) = \int_0^\infty x \, dF(x) = \mu, \quad (1.4.37)$$

Where $\mu$ may be infinite. Whether $\mu = \infty$, $1/\mu$ shall be interpreted as zero. Let $S_0 = 0$, $S_n = X_1 + X_2 + \ldots + X_n$; $n \geq 1$ and let

$$F_n(x) = \Pr \{ S_n \leq x \} \quad (1.4.38)$$

be the distribution function of $S_n$. Let the r.v.

$$N(t) = \text{Sup} \{ n : S_n \leq t \} \quad (1.4.39)$$

The process $\{ N(t), t \geq 0 \}$ is called a renewal process with distribution $F$. 
It is important to note that every epoch of renewal, that is \( t_i \)'s are points of regeneration (as explained in 1.4(b)). There exist many situations in industry especially while dealing with life-testing problems of electrical equipment, the production of components is done in batches and so on where renewal processes can be fruitfully applied to obtain solutions. In this thesis, we assumed that the inter-withdrawal times follow a renewal process where the event of interest \( E \) is a withdrawal. Similarly it is also assumed that the amount of withdrawals forms a renewal process. The function,

\[
M(t) = E \left\{ \sum_{n=1}^{\infty} F_n \right\} = \sum_{n=1}^{\infty} F_n \tag{1.4.40}
\]

is known as the renewal function of the process with distribution \( F \). The renewal function plays a vital role in the renewal theory. The following are some of the interesting results relating to renewal processes which are given (without proofs) for a ready reference.

**Result (1.4.41)**: The renewal process \( N(t) \) satisfies the following two properties namely:

\[
N(t) \geq n \Leftrightarrow S_n \leq t, \tag{1.4.42}
\]

\[
N(t) \geq n \text{ if and only if } S_n \leq t. \tag{1.4.43}
\]
Result (1.4.44) : The probability mass function of the renewal process, \( N(t) \) denoted by \( p_n(t) \) is given by:
\[
p_n(t) = \Pr \{ N(t) = n \} = F_n(t) - F_{n+1}(t), \quad n = 0, 1, 2, \ldots
\]  
(1.4.45)

Result (1.4.46) : The mean of the renewal process \( N(t) \) denoted by \( M(t) \) is given by:
\[
M(t) = \sum_{n=1}^{\infty} F_n(t).
\]  
(1.4.47)

Result (1.4.48) : The renewal function \( M(t) \) satisfies the equation:
\[
M(t) = F(t) + \int_0^t M(t-x) \, dF(x).
\]  
(1.4.49)

The equation (1.4.49) is known as 'the integral equation of renewal theory' or 'renewal equation' and the argument involved in the derivation is known as 'renewal argument' which is explained as follows:

Let the renewal take place at \( x \)
units of time from the point of observation, that is 0. This implies that the r.v., \( X_4 = x \) forms the renewal function (by conditioning the duration of first withdrawal \( X_4 \)):

\[
M(t) = E\{N(t)\}
\]

\[
= \int E\{N(t) / X_4=x\} dF(x). \quad (1.4.50)
\]

Consider \( x > t \), that is, no renewal occur in \([0,t]\). Thus:

\[
E\{N(t) / X_4=x\} = 0. \quad (1.4.51)
\]

Similarly consider \( 0 \leq x \leq t \), that is, the first renewal occurs at \( x \leq t \). Then the process again starts at epoch \( x \). Thus the expected number of renewals in the remaining interval of length \((t-x)\) is

\[
E\{N(t-x)\} \text{ which is equal to } M(t-x).
\]

Therefore:

\[
E\{N(t) / X_4=x\} = 1 + E\{N(t-x)\}
\]

\[
= 1 + M(t-x). \quad (1.4.52)
\]

Thus considering the equations (1.4.50) and (1.4.52), we get:

\[
M(t) = F(t) + \int_0^t M(t-x) dF(x)
\]
Result (1.4.53) : The equation of the type:
\[ \gamma(t) = g(t) + \int_{0}^{t} \gamma(t-x) dF(x), \quad t \geq 0 \]  
(1.4.64)

Where \( g, F \) are known and \( \gamma \) is unknown is called "Renewal type equation" and there exists a unique solution.

Result (1.4.55) : The average number of renewals per unit time for large \( t \) converges to \( 1/\mu \), that is, with probability one,
\[ M(t)/t \to (1/\mu) \quad \text{as} \quad t \to \infty \]  
(1.4.56)

Where \( \mu = E(X) \leq \infty \)  
(1.4.57)

This is usually known as 'the elementary renewal theorem'.

Result (1.4.58) : If \( X_1 \) is a lattice r.v. with period \( d \), then we have:
\[ \lim_{n \to \infty} \Pr \{ \text{renewal at } nd \} = (d/\mu). \]  
(1.4.59)

* LATTICE : If \( x \) is a discrete random variable and the numbers \( x_i \) in
\[ P(X=x_i) = p_i = F(x_i) - F(x_i^-) \]  
(I)
from an arithmetic progression:
\[ x_i = a + b_i, \quad i = 1, 2, \ldots \]  
(II)
then \( x \) is said to be lattice type. The condition (II) is not restrictive in the sense that if we allow some of \( p_i \) in (I) equal to zero which result gaps in the progression (II), then \( x \) is of lattice type if the numbers \( (x_i - a) \) are integral multiples of the same number [1].
If $X_i$ is a non-lattice r.v., then for fixed $h \geq 0$, we have:

$$[M(t) - M(t-h)] \rightarrow (h/\mu) \text{ as } t \rightarrow \infty.$$  \hspace{1cm} (1.4.60)

**Result (1.4.61)**: Let $h(t)$ be a non-negative, non-increasing function of $t \geq 0$ such that:

$$\int_0^\infty h(t) \, dt < \infty$$  \hspace{1cm} (1.4.62)

and $X_i$ be non-lattice, then

$$\int_0^t h(t-x) \, dM(x) \rightarrow (1/\mu) \int_0^\infty h(t) \, dt$$

as $t \rightarrow \infty$,  \hspace{1cm} (1.4.63)

the limit is taken as 0 when $\mu = \infty$.

(1.4.61) is popularly known as "Key Renewal Theorem" or "Smith's theorem" from which many important results can be derived. For instance,

$$h(t) = (1/h), \quad 0 < t \leq h \quad = 0, \text{ otherwise.}$$  \hspace{1cm} (1.4.64)

**Remark (1.4.64a)**: (1.4.58) Can be derived as a particular case of (1.4.61) as follows:
Laplace Transform (L.T) is a generalization of generating function. L.T.s serve as a powerful tool in many fields of applied research as an indirect method, effectively to obtain solutions for functions defined in \((0, \infty)\), especially given through differential or integral equations [4]. In most of the studies of probability distributions, it is easier to find the L.T. of a probability distribution rather than the distribution itself.

L.T. is a mathematical device. If a function is specified for all positive values of the time, we can write down a relative quantity called its L.T. Conversely if the L.T. is known, the value of the function can be found from it (Inverse Laplace Transform). The importance of the method arises from the fact that, for the types of problems we are interested in, it is easy to write down the L.T. of the solution and from this situation itself is then easily found by a routine process. Many of the functions required for the solution of problems in the mathematics, statistics, physics,
electronics and engineering are discontinuous at \( t = 0 \) whereas the resulting L.T. is a continuous function and this makes the manipulation easier and more a matter of routine. The overall effect of the use of the L.T. is to reduce the order of difficulty of the problem [5].

The L.T. is a straightforward algebraic technique for solving linear differential equations. The L.T. is used strictly as an operational calculus. An operational calculus is a technique by which the solution of differential equations is obtained by transforming the equations to algebraic equations. The usefulness of the method is based on the premise that algebra is easier than calculus [5].

Let us now introduce certain familiar facts concerning the theory of L.T.

Let \( f(t) \) be a function which is Riemann integrable¹,

1. A bounded function \( f \) is said to be Riemann integrable over \([a, b]\) if

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx
\]

where \( \int_{a}^{b} f(x) \, dx \) = the infimum of the set of upper sums

and \( \int_{a}^{b} f(x) \, dx \) = the supremum of the set of lower sums.
with its absolute value on any finite interval \([a, b]\) \((0 \leq a < b < \infty)\) and is given on the right half of the \(x\)-axis, that is, \(0 \leq t < \infty\). Let \(p = \sigma + i\tau\) be the complex variable and \(F^*(p)\) be the L.T. of the function \(F(t)\) which is defined as follows:

\[
F^*(p) \overset{\text{def.}}{=} \int_0^\infty F(t) \ e^{-pt} \ dt. \quad (1.4.66)
\]

Here, the value of the improper integral over the half axis \([0, \infty)\) will be taken to mean the limit to which the integral over the finite interval \([0, B]\) tends as \(B \to \infty\), so that

\[
\int_0^\infty F(t) \ e^{-pt} \ dt = \lim_{B \to \infty} \int_0^B F(t) \ e^{-pt} \ dt. \quad (1.4.67)
\]

Here, the L.T. is applicable to the function \(F\) for the value \(p\) of the parameter if for that value of \(p\), the integral (1.4.66) converges. If the transformation (1.4.66) is applicable to \(F\) for \(p = p_o = \sigma_o + i\tau_o\), it is applicable to \(F\) for any value \(p = \sigma + i\tau\) for which \(\text{Re} \ (p-p_o) = \sigma - \sigma_o > 0\).

Consider the function :

\[
\phi(t) = \int_0^t F(u) \ e^{-p_0 u} \ du. \quad (1.4.68)
\]

If the integral (1.4.66) converges for \(p = p_o\), \(\phi(t)\) has a finite limit, \(\lim_{t \to \infty} \phi(t)\) and consequently bounded on the half
The following two theorems from complex analysis are more frequently used in this thesis to obtain time-dependent solutions.

**Theorem (1.4.70):** If a function \( f(z) \) is analytic in a simply connected domain \( D \), the integral of the function along any closed contour \( c \) lying in \( D \) is equal to zero:

\[
\int_{c} F(z) \, dz = 0. \tag{1.4.71}
\]

(1.4.70) is known as 'Cauchy's theorem'.

**Theorem (1.4.72):** Let a single valued function \( F(z) \) be continuous on the boundary \( c \) of domain \( D \) and be everywhere analytic inside this domain, except for a finite number of singular points \( a_1, a_2, \ldots, a_n \).

Then

\[
\int_{c} F(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{res } F(a_k) \tag{1.4.73}
\]

Where \( \text{res } F(a_k) \) is the residue of the function \( F \) of order \( m \) at the singular point \( a_k \) which is defined as follows:

---

2. A singular point of a function is a value of \( z \) at which \( f(z) \) fails to be analytic.
$$\text{res } F (a_k) = \left\{ \frac{1}{(m-1)!} \right\} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a_k)^m F(z)].$$  

(1.4.74)

For poles of the first order, (that is, for $m = 1$) the formula (1.4.74) becomes:

$$\text{res } F (a) = \lim_{z \to a} [(z - a) F(z)].$$  

(1.4.75)

Here, if we define the function $F(z)$ in the neighbourhood of the point 'a' as the quotient of two analytic functions namely, $\phi$ and $\psi$ at this point,

$$F(z) = \{\phi(z)/\psi(z)\}$$  

(1.4.76)

such that $\phi (a) = 0$ while $\psi (z)$ has a zero of the first order at 'a' (that is, $\psi (a) = 0$, $\psi '(a) \neq 0$) then (1.4.75) is reduced to:

$$\text{res } F (a) = \{\phi (a)/\psi '(a)\}.$$  

(1.4.77)

We find the original function $F(t)$ from L.T. via the inversion formula or inverse L.T. which is defined as:

Definition (1.4.78): Let $F(t)$ be the original function and $F^*(p)$ be the L.T. The original function, $F(t)$ can be computed from the L.T. By making use of the complex integral:

$$L^{-1} (F^*(p)) = \{ F(t) (1/2\pi i) \int_{C-i\infty}^{C+i\infty} F^*(p) e^{pt} \, dt \}$$  

(1.4.79)
Where \( c \) is the abscissa in the half-plane of the absolute convergence of the Laplace integral. It is difficult to compute \( F(t) \) from (1.4.79) since it requires a knowledge of the function \( F^*(p) \) for complex value \( p = c + iy \), \(-\infty < y < \infty\) and the integral is improper with an oscillating kernel. But (1.4.79) is an integral of an analytic function taken along a contour in the complex plane. As such it can be transformed by applying familiar methods of the theory of functions of a complex variable, for instance, by changing the path of integration, computing residues and the like. These transformations help in obtaining an expression (of practical convenience) for the original function from which we can extract important properties of the function defined by the complex integral in certain cases [9]. Further a double Laplace transform (d.l.t) \( F^*(s,p) \) of a function \( F(y,t) \) is defined as:

\[
F^*(s,p) \overset{\text{def}}{=} \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} F(y,t) \, dt \\
\text{Re. } p > 0, \text{ Re } s > 0 \tag{1.4.80}
\]

and \( \tilde{F}(y,p) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F^*(s,p) e^{sp} \, ds \). \( \tag{1.4.81} \)

Finally, 'The convolution property' of L.T. is given after explaining 'convolution'.
If $L^{-1} \{ F^*(p) \} = F(t)$ \hspace{1cm} (1.4.82)

and $L^{-1} \{ G^*(p) \} = G(s)$, \hspace{1cm} (1.4.83)

then $L^{-1} \{ F^*(p) G^*(p) \} = \int_{0}^{t} F(u) G(t-u) = F \ast G(t)$.

(1.4.84)

In (1.4.84) $F \ast G(t)$ is known as the 'Convolution' or 'faltung' of $F$ and $G$. Now we give 'the Convolution property' that the L.T. of convolution is the product of the L.T.'s of the colvoluting functions, that is,

$$F \ast G(p) = F(p) G(p) \quad [11]. \quad (1.4.85)$$

1.4 (e) Integral Representation Technique (I.R.T):

Integral Representation Technique is mainly used to derive explicit results of stochastic storage models. After obtaining d.l.t. for the integral equations, inverse L.T. is to be taken for obtaining results. But the expressions of d.l.t. consist of some unknown constants, without determining those unknown constants, inverse L.T. cannot be taken. Hence these unknown constants are to be evaluated first. Earlier Srinivasan and others during the years 1969-74 [18 to 38] used
to provide arguments to evaluate these unknown constants and then inverse L.T. is taken after determining these constants. In 1974 Venugopal [39] in his doctoral thesis provided a technique with which these unknown constants present in the expressions of d.l.t.s' can be determined which is known as 'Integral Representation Technique'(I.R.T). The fundamental idea involved in this technique is explained with a stochastic banking model considered by Sarma [15] which also serves as a ready reference to derive the results of him from our results obtained in the forthcoming chapters.

Venugopal and Sarma [40] in the year 1979 in their Stochastic Banking Model introduced the c.p.d.f., $\hat{M}(x,y,t)$ which is defined as:

$$\hat{M}(x,y,t) = \lim_{\Delta \to 0^+} \Pr \left[ 0 < x < X(t) \leq x + \Delta, X(u) > 0, \sum_{u \in [0,t]} X(0-) > X(0) = y \right] \Delta$$

(1.4.86)

Where $X(t)$ denotes the r.v. describing the general 'reserve level' of the system.

They obtained an expression for $\hat{H}(x,y,t), \hat{H}^*(x,s,p)$ as:

$$\hat{H}^*(x,s,p) \equiv \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} \hat{M}(x,y,t) \, dt, \Re s > 0, \Re p > 0$$

(1.4.87)
Where from specializing the case of $M/G/1$ systems with capacity $k = \infty$, (1.4.87) reduces to

$$
\hat{M}^*(x,s,p) = \left( \frac{e^{-sx} - e^{-\lambda p - \lambda G(\lambda p) p}}{\lambda + p - s - \lambda G(s)} \right).
$$

(1.4.88)

Here (i) the inter-demand times are governed by the probability law, $h(.)$ which is stated as:

$$
h(u) = \lambda e^{-\lambda u}, \quad \lambda > 0, \quad u \geq 0.
$$

(1.4.89)

(ii) the demand quantum follows the known probability law, $g(.)$ and its L.T. is defined as:

$$
G(s) = \int_0^\infty e^{-sv} g(v) dv, \quad \text{Re.} s > 0
$$

(1.4.90)

and (iii) $G(s)$ is assumed to be a rational function of the form:

$$
G(s) = \frac{A(s)}{B(s)}
$$

(1.4.91)

Where $B(s)$ has $m$ simple poles at $\zeta_i$, $i = 1, 2, \ldots, m$.

One can easily show that denominator in (1.4.88) has $(m + 1)$ zeroes out of which only one zero has a positive real part, while the remaining $m$ zeroes have negative real parts. Let $\xi$ denote the zero with positive real part.

**Fundamental Idea Underlying the technique:**

The argument is that since $\hat{M}^*(x,s,p)$ is analytic in
the right-half plane \((s > 0)\), the constant \(\hat{M}^*(x, \lambda + p, p)\) appearing in the numerator of (1.4.88) can be evaluated by using the consequent property (that is, analyticity of \(\hat{M}^*(x, s, p)\) for \(s > 0\)) that the numerator also vanishes at \(s = \xi\), as such, they obtained:

\[
\hat{M}^*(x, \lambda + p, p) = \frac{e^{-\xi x} - e^{-(\lambda + p)x}}{\lambda G(\lambda + p)}.
\]  

(1.4.92)

By substituting (1.4.92) in (1.4.88), \(\hat{M}^*(x, s, p)\) is solved theoretically. The necessity of obtaining explicit results arises from the fact that even though an expression for the constant is provided in (1.4.92), the presence of \(\xi\) therein renders the result not explicit enough and also in particular one would like to have solutions entirely in terms of the (originally assumed) functions \(h(.)\), \(g(.)\) or their L.T.s'. Thus the gap is intelligently closed by Venugopal in the year 1974 [39] by evolving an I.R.T, eliminating the need for solving \(\xi\). This technique is explained in the present situation with the help of the following lemma.

**Lemma (1.4.93):** \(e^{-\xi x}\) is uniquely determined by the integral equation:
\[ e^{-\xi x} = \lim_{s \to \xi} \frac{\lambda + p - s - \lambda G(s)}{(s - \xi) 2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} e^{-\eta x} \frac{d\eta}{[\lambda + p - \eta - G(\eta)]} \]  

(1.4.94)

Proof: By closing the contour (with a semi-circular arc of finite radius in the limit) to the right, the line integral on the right hand side (r.h.s.) of (1.4.94) will be evaluated. Multiplying this value with the other factor on the r.h.s. of (1.4.94), the Lemma (1.4.93) is established. (*)

Theorem (1.4.95): The characteristic function (c.f.), that is, the L.T. of the (random) time, \( t \) in \( M(x,y,t) \) is given by:

\[ \hat{M}(x,y,p) \overset{\text{def.}}{=} \int_0^\infty e^{-pt} \hat{M}(x,y,t) \, dt, \quad \text{Re.} \, p > 0 \]

\[ = (1/2\pi i) \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{-s(x-y)}}{\lambda + p - s - \lambda G(s)} \, ds. \quad (1.4.96) \]

Proof: By substituting (1.4.92) in (1.4.88), they obtain:

\[ \hat{M}^x(x,s,p) = \frac{e^{-sx} - e^{-\xi x}}{[\lambda + p - s - \lambda G(s)]}, \quad (1.4.97) \]

By making use of (1.4.94) and inverting d.l.t in
(1.4.96) with respect to (w.r.t) 's', they obtain:

$$M(x,y,p) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{sy} \hat{M}(x,s,p) \, ds$$, (1.4.98)

Now by shifting the line ($\beta-i\omega$ to $\beta+i\omega$) to the left of the line ($\sigma-i\omega$ to $\sigma+i\omega$, $\text{Re.}\zeta > \sigma > 0$) and evaluating (1.4.98), taking into account this shift and after some calculations, they obtain:

$$M(x,y,p) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\eta x}}{\frac{[\lambda + p - \eta - \lambda G(\eta)]}{\beta-i\omega \Gamma(s-\zeta)}} \, ds$$, (1.4.100)

establishing the theorem (1.4.95) [15].

Now it is clear that $\hat{M}(x,s,p)$ in terms of the c.f., $\hat{M}(x,y,p)$ is explicitly determined through known parameter $\lambda$ and $G(.)$. Thus I.R.T. is well demonstrated.