Chapter 4

Semiannular Josephson junctions

A semiannular geometry is proposed for Josephson junction and analytical and numerical studies show that an external static magnetic field applied parallel to the dielectric barrier interacts through the interior of the junction and produce a tilted potential which pushes out trapped fluxons from the interior of the junction and flux-free state exists in the junction in the absence of an external bias. Due to the semiannular shape, the effective field at the ends of the junction has opposite polarities which supports penetration of opposite polarity fluxons into the junction in the presence of a forward biased $dc$ current. When the direction of the $dc$ current is reversed, flux penetration is not possible and flux-free state exists in the junction. Thus this geometry can be used in implementing fluxon based diodes. The rectification property of the junction is demonstrated using square wave signals and sinusoidal ac signals. It is found that the junction is extremely useful in rectifying rf magnetic fields. In the forward biased state, fluxons and antifluxons enter the junction and move in opposite directions. Using this property, we propose and demonstrate a novel bidirectional flux-flow oscillator.
4.1 Introduction

Fluxon dynamics in nonrectangular LJJ attracted much attention in recent years. The nonrectangular shape creates nonuniformity in the junction which can be advantageously employed in certain Josephson devices. In flux-flow oscillators (FFO), nonuniformity is used to reduce self-field effects and to facilitate unidirectional fluxon motion[85]. In Josephson trigger circuits, nonuniformity is employed to make a special dependance of the critical current upon the magnetic field[86]. Nonuniformity may mean unequal conditions for Josephson vortices in different parts of the junction. It may be due to nonuniform spatial distribution of critical and bias currents[3], temperature gradient effects[87], or due to many other reasons.

Recently fluxon dynamics in some unconventional structures like the multistacked junctions (both linear[36] and annular[88]), non-symmetric and nonuniform junctions[89] etc. are being carried out by a number of authors. A static magnetic field applied parallel to the barrier in a linear LJJ has no effect in the interior part of the junction and the small perturbation produced is through the open boundary[12, 49]. The effect of a spatially homogeneous static magnetic field on annular LJJ has undergone various theoretical and experimental studies[90, 91]. The external field produces periodic potential in the annular junction which can be used to trap the fluxons[92].

In the present work, we investigate the effects of an external homogeneous static magnetic field on the propagation of fluxons in a dissipative LJJ having a semiannular shape. Analytical and numerical studies show that the field interacts through the interior of the junction as well as through the boundary conditions and can exert a driving force supporting transitory motion (from one end to the other end) for any trapped static flux-quanta inside the junction. Thus under static conditions, flux-free state exists in the junction. The main advantage of this geometry is in the fact that it allows opposite polarity fluxons to enter the
junction from opposite ends only if the junction is biased in one direction (forward bias). If the direction of the bias is reversed (reverse bias), fluxons cannot enter the junction due to the repulsive Lorentz force and flux-free state exists in the junction. Thus the junction exhibits the basic properties of a diode. By controlling the strength of the magnetic field, it is possible to get a single fluxon and a single antifluxon configuration in the junction. Detailed analysis show that this single fluxon-antifluxon state \((\langle \uparrow \downarrow \rangle)\) is highly stable against fluctuations. The stable dynamics exhibited by this fluxon-antifluxon pair is utilized in constructing a fluxon based diode. It is found that even in the forward biased state, there is a threshold value of the current below which fluxons cannot enter the junction. The damping effects of an external magnetic field on the motion of a single trapped fluxon in the junction is also studied. Using the semiannular junction, rectification of alternating magnetic fields is demonstrated. A novel bidirectional flux-flow oscillator is also constructed using the device.

### 4.2 Derivation of the model equations

An overlap LJJ with a semiannular shape is considered as shown in Fig.4.1(a) with the discrete model shown in Fig.4.1(b). An external static magnetic field applied parallel to the dielectric barrier interacts nonuniformly and produces a spatially varying perturbation. The Kirchhoff equations for the Josephson phases in the cell and for the currents in one of the nodes are

\[
\varphi(X + dX) - \varphi(X) = \frac{2\pi}{\Phi_0} \left( d\Phi_e(X) - L_p I_L(X) \right) \quad (4.1)
\]

\[
I_L(X - dX) - I_L(X) = I(X) - I_e(X) \quad (4.2)
\]

where \(\varphi(X)\) is the Josephson phase at the point \(X\) of the junction, \(d\Phi_e(X)\) is the component of the external magnetic flux linked with the cell of length \(dX\), \(L_p\) is the inductance of the piece of the junction electrodes between \(X\) and
$X + dX$. $I_L(X)$ is the current through the inductance, $I_e(X)$ is the externally applied current, $I(X)$ is the current through the Josephson junction.

The external magnetic field $\hat{B}$ interacts with the interior of the junction and the component of the external flux in the plane of the junction over an infinitesimal interval $dX$ is calculated as [91, 90, 92, 93]

$$d\Phi_e(X) = \Delta \left( \hat{B} \cdot \hat{n} \right) dX = \Delta B \cos(KX) \ dX \quad (4.3)$$

where $\Delta$ is the coupling of the external magnetic field with the junction, $\hat{n}$ is the unit vector normal to the propagation direction and in the plane of the junction. Thus a homogeneous static field makes an effective nonhomogeneous field inside the junction. From Eq. (4.3), it is clear that for a linear junction (i.e., if $\hat{n}$ is independent of $X$) in a homogeneous magnetic field there will be no perturbation from the magnetic field to the interior of the junction. In this case there would only be interaction through the open boundary conditions. However, if the junction is in semiannular shape or in any other curved shape, $\hat{n}$ depends on $X$ and there is perturbation to the interior of the junction.

Using the relations

$$L' = L' \ dX; \ I(X) = j(X) \ dX; \ I_e(X) = -j_e(X) \ dX \quad (4.4)$$

and substituting Eqs. (4.3) into Eqs. (4.1) and (4.2), we write the later in the following form.

$$\frac{\partial I_L(X)}{\partial X} = -\frac{\Phi_0}{2\pi L'} \frac{\partial^2 \varphi}{\partial X^2} - \frac{\Delta B K}{L'} \sin(KX) \quad (4.5)$$

$$\frac{\partial I_L(X)}{\partial X} = -j_e(X) - j(X) \quad (4.6)$$

where $L'$ is the inductance per unit length of the junction, $K = \frac{\pi}{L}$ is the spatial periodicity of the field inside the junction and $L$ is the length of the junction.
We assume that the dielectric is spatially uniform so that \( \Delta \) and \( L' \) are independent of \( X \). In the case of simple resistively shunted junction (RSJ) model, the supercurrent density \( j(X) \) is the sum of the supercurrent, normal (quasi particle) current and displacement current densities,

\[
j(X) = j_0 \sin \varphi + \frac{\Phi_0}{2\pi R} \varphi_T + \frac{C \Phi_0}{2\pi} \varphi_{TT}
\]

(4.7)

here \( j_0, R, C \) are the critical current density, specific resistance and specific capacitance of the junction, respectively. Using Eqs.(4.5), (4.6) and (4.7) we get the sG equation

\[
\frac{C \Phi_0}{2\pi} \varphi_{TT} - \frac{\Phi_0}{2\pi L'} \frac{\partial^2 \varphi}{\partial X^2} + j_0 \sin \varphi = -\frac{\Phi_0}{2\pi R} \varphi_T + \frac{\Delta B K}{L'} \sin(KX) - j_e(X)
\]

(4.8)

The component of the external flux over an infinitesimal distance \( dX \) of the unit cell in terms of the quantized unit is

\[
d\varphi(X) = \frac{2\pi}{\Phi_0} d\varphi_e(X) = \frac{2\pi}{\Phi_0} \Delta B \cos(KX) dX
\]

(4.9)

The effect of an applied magnetic field on the junction is to induce currents in closed form across the junction. So the net current when integrated over the junction should be zero. Due to the semiannular shape, the external field induces a varying surface current along the junction. Since the spatial derivative of the superconducting phase is equivalent to the induced surface current, we get

\[
\frac{d\varphi(X)}{dX} = \frac{2\pi}{\Phi_0} \Delta B \cos(KX)
\]

(4.10)

This equation can be used to obtain the boundary conditions of the junction. Using the normalized quantities, \( T = \frac{\Phi}{\omega_0} \), \( X = x \lambda_J \), \( \lambda_J = (\frac{\Phi_0}{2\pi L j_0})^{\frac{1}{2}}, \omega_0 = (\frac{2\pi j_0}{C \Phi_0})^{\frac{1}{2}} \) in Eq. (4.8) we get the general perturbed sG partial differential equation

\[
\varphi_{tt} - \varphi_{xx} + \sin \varphi = -\alpha \varphi_t + b \sin(kx) - \gamma
\]

(4.11)
where \( \varphi(x, t) \) is the superconducting phase difference between the electrodes of the junction, \( k = \frac{\pi}{l} \) and \( b = \frac{2\pi \lambda_i \Delta B k}{\Phi_0} = 2k \frac{B}{B_{c1}} \). Where \( B_{c1} = \frac{\Phi_0}{\pi \Delta \lambda_j} \) is the first critical field of the superconductor.

Compared with the standard sine-Gordon model for Josephson junction, this equation has an extra term, \( b \sin(kx) \), which corresponds to a force driving fluxons towards left and antifluxons towards right. Therefore any static trapped fluxon present in the junction will be removed and flux-free state exists in the junction in the absence of an external bias. Thus the effect of the external field is to act like a bias current \( \gamma_b(x) = b \sin(kx) \), which has non-zero average in space. This bias current stops penetration of fluxon from the left end and penetration of antifluxons from the right end. So the junction does not support any fluxons in the static conditions. This non-zero average current induces a non-periodic field (potential) inside the junction.

From Eq. (4.10), we get the corresponding boundary conditions of the junction as

\[
\begin{align*}
\varphi_x(0, t) &= \frac{b}{k} \\
\varphi_x(l, t) &= -\frac{b}{k}
\end{align*}
\]

(4.12)

This boundary condition is consistent with the fact that effective field linked with the junction has opposite polarities at the ends of the junction. So only fluxons can enter from the left end \( (x = 0) \) and antifluxons from the right end \( (x = l) \) in a properly biased junction. From Eq. (4.12), we see that fluxons can enter from the left end and antifluxons can enter from the right end of the junction for positive values of \( \gamma \) (forward biased state). Negative values of \( \gamma \) drives fluxons towards left and antifluxons towards right and fluxon penetration becomes impossible (reverse biased state). Eq. (4.11) with boundary conditions Eq. (4.12) represent a semiannular LJJ in a homogeneous static magnetic field.
4.2.1 Lagrangian and Hamiltonian functions

Lagrangian density of Eq. (4.11) with \( \alpha = \gamma = 0 \) is

\[
L = \left\{ \frac{\varphi_t^2}{2} - \frac{1}{2} \left( \varphi_x - \frac{b}{k} \cos(kx) \right)^2 - (1 - \cos \varphi) \right\} \tag{4.13}
\]

Therefore the corresponding potential energy density is (second term of the above equation)

\[
U(x) = \frac{1}{2} \left\{ \varphi_x^2 - \frac{2b}{k} \cos(kx) \varphi_x + \left( \frac{b}{k} \cos(kx) \right)^2 \right\} \tag{4.14}
\]

The first term is independent of the applied field and the third term is a constant which is independent of the flux motion in the junction. Therefore the change in the potential due to the applied field can be determined from the second term as:

\[
U(x) = -\frac{b}{k} \int_{-\infty}^{+\infty} \varphi_x \cos(kx) \, dx \tag{4.15}
\]

Substituting Eq. (1.38) in (4.15) and integrating, we get

\[
U(x_0) = -2 b l \sec h \left( \frac{\pi^2}{2l} \sqrt{1 - u^2} \right) \cos(k \, x_0) \tag{4.16}
\]

For long junctions and at relativistic velocities, \( u \sim 1 \), Eq. (4.16) becomes

\[
U(x_0) = -C \cos(k \, x_0) \tag{4.17}
\]

where \( C = 2 b l \) is a constant. Eq. (4.17) shows that the potential is tilted by the applied field. The potential is plotted in Fig.4.2. Tilting is either to the left side or to the right side of the junction depending on the direction of the field. This tilt in the potential causes trapped static fluxons and antifluxons to move in the opposite directions and thus the junction remain flux-free under static conditions. Thus any trapped flux can be removed from the junction by applying a static magnetic field.

Energy of the unperturbed sG system is given by Eq. (1.37). Perturbational parameters modulate the velocity of the solitons and may cause to dissipate energy. The rate of dissipation is calculated by computing
\[
\frac{d}{dt}(H^P) = \left[ \varphi_x \varphi_t \right]_0^l + \int_0^l \left[ -\alpha \varphi_t^2 + (b \sin(kx) - \gamma) \varphi_t \right] dx 
\] (4.18)

where the first term on the right side account for the boundary conditions. From Eq. (1.38), we get \( \varphi_t = -u \varphi_x \) and from Eq. (4.12), we get \( \varphi_x^2(0,t) = \varphi_x^2(l,t) \) (symmetric boundary conditions). Substituting these expressions, we find that the first term in the right hand side of the above equation vanishes - a symmetric boundary condition does not change average energy value of a fluxon. Inserting Eq. (1.38) in Eqs. (4.18) and following perturbative analysis\[15\], we get

\[
(1 - u^2)^{-\frac{3}{2}} \frac{du}{dt} = -\alpha \frac{u}{\sqrt{1 - u^2}} - \frac{\pi}{4} \{ b \sec h \left[ \frac{\pi^2 \sqrt{1 - u^2}}{2l} \right] \sin(kx_0) - \gamma \} \] (4.19)

This expression describes the effect of perturbations on the fluxon velocity. In the absence of dc bias (i.e., \( \gamma = 0 \)), from Eq. (4.19), we get the threshold value of the magnetic field for producing equilibrium velocity (i.e., at \( \frac{du}{dt} = 0 \)) on a trapped fluxon as

\[
b = \frac{4 \alpha}{\pi} \frac{u_0}{\sqrt{1 - u_0^2}} \sec h \left[ \frac{\pi^2 \sqrt{1 - u_0^2}}{2l} \right] \sin(kx_0) \] (4.20)

For a long junction, \( \frac{\pi^2}{2l} \ll 1 \), we obtain the approximate equilibrium velocity of the fluxon when \( u_0 \sim 1 \) with \( x_0' = \frac{l}{2} \) as

\[
u_0 \simeq \pm \left[ 1 + \left( \frac{4\alpha}{\pi b} \right)^2 \right]^{-1/2} \] (4.21)

This equilibrium velocity is equivalent to that obtained in Ref.[15] with a dc bias. Thus it can be concluded that in semiannular LJJ, the magnetic field exerts a driving force on trapped fluxons and produces a transitory motion in the junction.

The effects of a dc current on the fluxon dynamics in the presence of the external field is studied using Eq. (4.11). Even in the forward biased state, ZVS
exists in the junction (flux-free state) when the driving force due to the field (\(\gamma_b(x)\)) and that of the dc current (\(\gamma\)) are nearly equal and is in opposite direction. By variation of the soliton position \(x_0\), from Eq. 4.21, we find the largest possible bias current of ZVS (\(u = 0\)) to be\[90\]

\[\gamma_1 = b \text{sech} \left( \frac{\pi^2}{2l} \right) \quad (4.22)\]

This is the threshold value of the applied bias, below which flux propagation is not possible in the junction. This threshold value depends on the magnetic field and is directly proportional to the field.

### 4.3 General properties of the junction

#### 4.3.1 Properties of the junction under a dc bias

Fig.4.3 shows the average velocity (equivalently average voltage) attained by the fluxon-antifluxon pair \(\langle \uparrow \downarrow \rangle\) as a function of the dc bias in the junction in the forward biased (positive values of \(\gamma\)) state and in the reverse biased state (negative values of \(\gamma\)). We have considered a junction of length \(l = 34\) and dissipation parameter \(\alpha = 0.1\). The field strength is fixed at \(b = 0.1\). The system is started with \(\varphi = 0\) and \(\frac{\partial \varphi}{\partial t} = 0\). For positive values of the sweeping dc current, flux penetration is possible in the junction. ZVS corresponding to flux-free state exists in the junction upto a bias value of \(\gamma = 0.32\). At this threshold value, a fluxon enter the junction from the left end and an antifluxon enter from the right end simultaneously and they move in opposite directions in the junction under the influence of the dc bias. This fluxon-antifluxon pair is found to be stable for sufficiently larger bias values. Dynamics of the pair \(\langle \uparrow \downarrow \rangle\) in the junction gives an average normalized maximum velocity of \(u \approx 2\). This pair executes highly stable motion upto a bias value of \(\gamma = 0.58\). On increasing the bias values further, large number of fluxons and antifluxons enter into the junction and they make successive reflections at the boundaries which result in a switching of the IVC to
high voltage states. We have not pursued the high voltage states of the junction as the number of fluxons taking part in the dynamics is not fixed. In the reverse sweep of the bias, i.e., on decreasing the bias uniformly in very small steps, we observed hysteresis in the dynamics and finite voltage is observed up to $\gamma = 0.22$.

In the inset of the figure the spatial derivative of the phase ($\varphi_x$) in the state $\langle \uparrow \downarrow \rangle$ along the junction is plotted. A fluxon on entering from the left end moves towards the right end and an antifluxon on entering from the right end moves towards the left end. For negative values of the $dc$ bias, flux penetration is not possible and ZVS (flux-free state) exists for all values of the negative $dc$ bias. In this region the junction behaves as a reverse biased diode.

### 4.3.2 Properties of the junction under a static field

It is important in practical applications to know the behavior of the junction under a static magnetic field especially the dependance of critical current ($I_c$) on the applied field ($H$)[3]. In weak static magnetic fields, LJJs behave like weak superconductors and show the Meissner effect. In this regime the critical current decreases linearly with the external field. This behavior exists up to a critical field $H_c$. At this critical field, magnetic flux in the form of fluxons can overcome the edge barrier effects and can penetrate the junction[94]. For LJJs the first critical field is $H_c = \frac{\Phi_0}{\pi \Lambda \lambda_J}$, where $\Lambda$ is the effective magnetic thickness of the junction. The dependance of $I_c$ (normalized to maximum Josephson current $I_0$) on a static magnetic field ($H / H_c$) applied to a semiannular LJJ of $l = 10$ is shown in Fig.4.4 (solid circles). For comparison, critical current versus magnetic field pattern of a standard rectangular LJJ is presented (open circles). In positive magnetic fields, $I_c(H)$ pattern in semiannular LJJ shows that static fluxons can exists in the junction and a minimum critical current is required to induce flux motion in the junction. In negative fields, the junction behaves differently and the critical current pattern is displaced and indicates that higher critical currents
are required to induce flux motion in the junction.

The threshold values of the dc bias ($\gamma_1$) allowing propagation of a single fluxon in the junction at various values of the field is shown in Fig.4.5. The threshold value increases on increasing the magnetic field. To determine the threshold values, we have considered the dynamics of a single trapped fluxon in the junction. Below the threshold value propagation is not possible in the junction and the trapped fluxon is annihilated. A small magnetic field applied to the junction can damp the motion of a trapped fluxon. On increasing the field, the fluxon slows down and finally annihilated in the junction. In Fig.4.6, the damping effects of a small field is shown for a trapped fluxon moving under different values of the dc bias.

### 4.4 Demonstration as a fluxon diode

Recently, fluxon based voltage rectifiers[95, 96, 89] have attracted much attention due to the fact that they can find important applications in Josephson digital devices[97]. Various geometries and external conditions are investigated towards this end. The influence of an artificially created ratchet potential on fluxon dynamics in nonuniform LJJ have been studied and voltage rectification properties of these LJJs are demonstrated in recent papers[95, 96]. The net unidirectional motion exhibited by a particle in a ratchet potential is the key factor which is also employed in magnetic flux cleaning applications[98] and in Abrikosov vortex diodes[99]. Ratchet voltage rectifiers based on three junction device[100], asymmetric SQUIDs[101] and on specially engineered arrays[102] have also been investigated in the past. However, working of all these voltage rectifiers critically depend on the ratchet potential and we cannot expect stable performance from these devices as ratchet potentials are highly sensitive to external perturbations. In addition, amplitude ranges of rectification is also limited in these devices and the rectified output does not have a linear relationship with the input.
Detailed analysis shows that the semiannular LJJ embedded with the static magnetic field has the characteristics of a diode. Thus fluxon based diodes can be implemented using this geometry. To demonstrate the rectification effects, we use the pair \( \langle \uparrow \downarrow \rangle \) dynamics in the junction. The IVC of the junction shows that fluxons and antifluxons can enter the junction only if the junction is forward biased and fluxon dynamics is not possible in the reverse biased state. Fig.4.3 demonstrate the forward biased state and reverse biased state of the junction. In the forward biased state (positive values of \( \gamma \) in Eq. (4.11), the pair \( \langle \uparrow \downarrow \rangle \) is highly stable against perturbations. This pair exists for sufficiently higher values of the \( dc \) bias. In the reverse biased state (negative values of \( \gamma \)), fluxons cannot exist in the junction and ZVS exists for all values of the negative bias. We have considered different parameters of the junction and found that the pair executes symmetric motion in the junction under the influence of the \( dc \) bias.

### 4.4.1 Rectification of a square wave

To demonstrate rectification effects of an \( ac \) current we used a square wave \( \gamma(t) = \begin{cases} A, & 0 \leq t < \frac{T}{2} \\ -A, & \frac{T}{2} \leq t < T \end{cases} \) in Eq. (4.11). The period of the square wave is taken much larger than the typical response time of the system (\( \sim 1 \) ns) so that it is in the adiabatic regime. The amplitude of the \( ac \) signal should be sufficiently large to induce flux motion in the junction. In the first half cycle of the square wave, fluxon penetration is not possible and zero voltage exists in the junction. In the second half cycle, one fluxon enter from the left end and one antifluxon enter from the right end and the pair \( \langle \uparrow \downarrow \rangle \) moves in opposite directions. The strength of the external field is adjusted in such a way that no more fluxons can enter into the junction. Rectification process is demonstrated in the time domain snapshots of Fig.4.7, where we plot the instantaneous voltage \( (V(t)) \) across the junction as a function of time. If the amplitude of the \( ac \) signal is below a threshold value, ZVS exists in the junction as it can be seen in the left panel of Fig.4.7. In Fig.4.8, we
plot the average voltage (averaged over a period of the input signal) as a function of the amplitude of the square wave. Average voltage increases on increasing the amplitude of the input signal. ZVS exists if the amplitude is below 0.56 (peak to peak, $A = 0.26$) and the output voltage increases linearly in the range 0.6 to 0.7 of the square wave amplitude. At higher values, additional fluxons enter into the junction so that the output is no longer proportional to the input current. We have considered different frequencies of the input signal and found that the pair $\langle \uparrow \downarrow \rangle$ gives stable and reliable results.

### 4.4.2 Rectification of a sine wave

To study the rectification properties of sinusoidal ac currents, we used a sine wave $\gamma(t) = -A \sin(\omega t)$ in Eq. (4.11). The period of the signal is taken much higher than the typical response time of the system. The frequency of the signal used is $\omega = 0.02$. The dynamics of the pair $\langle \uparrow \downarrow \rangle$ is studied under a magnetic field of strength $b = 0.21$ on a junction of length $l = 25$ and dissipation parameter $\alpha = 0.1$. Fig. 4.9 shows the time domain voltage pulses in the junction. In Fig. 4.10, we plot the average voltage as a function of the amplitude of the sine wave. Average voltage increases on increasing the amplitude of the input signal. We have considered different amplitudes and frequencies of the input signal and could get best results using the pair $\langle \uparrow \downarrow \rangle$.

### 4.5 Flux-flow state - demonstration as a bidirectional flux-flow oscillator

A FFO[103] is a LJJ in which an applied dc magnetic field and a uniformly distributed dc bias current drive a unidirectional motion of fluxons. The external static magnetic field required for the FFO operation is generated using a dc current in an external coil and is applied perpendicular to the FFO. The magnetic field penetrates the junction in the form of fluxons and their motion through
the junction leads to an electromagnetic radiation. According to the Josephson relation, a FFO biased at voltage $V$ oscillates with frequency $f = (2\pi / \Phi_0) V$ (at about 483.6 GHz/mV)[1]. Due to the losses in the superconducting electrodes, the maximum operational frequency is about $f = \Delta / (e \Phi_0)$ corresponding to the superconducting energy gap $\Delta$. Typically, for Niobium, the gap frequency $f$ is in the range of 650 - 700 GHz. The radiation frequency, which is also related to the fluxon velocity $u$, by $f = u / d_{fl}$, is determined by the spacing between the moving fluxons $d_{fl}$. The velocity and density of the fluxons, and thus the power and frequency of the emitted radiation can be controlled by controlling the bias current and the strength of the applied field. The wide-band tunability and narrow line-width of a Josephson FFO make them a perfect on-chip local oscillator for integrated submm-wave receivers [45]. Various geometries [37, 85, 104, 105, 106, 107] and superconducting materials are employed to make high performance oscillators. Using conventional superconducting junctions like $Nb – AlO_x – Nb$, FFOs have been successfully tested and these devices are found to be capable of delivering sufficient power ($\approx 1 \mu W$) in the frequency range 120 – 700 GHz.

To study the feasibility of making this device as a FFO[108, 109, 110], we have done a preliminary study and investigated the flux dynamics of a group of fluxons under a large magnetic field. In the proposed oscillator, fluxons enter the junction from the left end and move towards the right end due to the applied bias (in the forward biased state) and on reaching the right end, they are selectively terminated (a passive load of impedance $z$ in series with a diode is connected at the ends of the junction). In a similar way, antifluxons enter the junction from the right end and move towards the left end where they are terminated. In implementing the device, we used a special technique by which fluxons are absorbed selectively at the right end of the junction and antifluxons are absorbed at the left end of the junction. In experiment, this can be realized by using
a load resistor in series with a diode. At the right end, the diode should be placed in such a way that it allows the screening currents associated with the fluxons to go through the load (termination of fluxons) and disallows the screening currents associated with the antifluxons. Similarly at the left end, the diode should allow the screening currents of the antifluxons to go through the load (termination of antifluxons). Thus selective absorption of the fluxons can be achieved at the ends. Fig.4.11(a) shows the snapshots of the spatial profile \((\varphi_x)\) of a group of fluxons entering from left end and antifluxons from the right end in the junction. Fig.4.11(b) shows the snapshots \((\varphi_x)\) of the resonant motion of fluxons and antifluxons in the opposite directions in a coherent state. This resonant motion is highly stable and can be a mechanism for constructing the bidirectional oscillators. This resonant, coherent motion also helps to avoid any stray fluxons in the junction thus making the junction a highly tunable device. Fig.4.12(a) shows the corresponding time dependence of the voltage pulse form in the middle of the junction. All the voltage pulses are equally spaced showing spatial coherence in the junction. The calculated frequency spectrum using fast Fourier transform (FFT) of the voltage pulses is shown in Fig.4.12(b). The figure shows the dominant first harmonic of the oscillations at frequency \(f = 0.181\) (in normalized units) and the second harmonic at frequency at \(f = 0.362\). It is important for practical applications to know the influence of the load \((z)\) on the average output power of the device, in particular to see how it behaves at larger loads. Fig.4.13 shows the dependency of the average output power \((P = VI \equiv \frac{\varphi_z^2}{2})\) obtained from both ends as a function of the load for the values of the junction \(l = 20, \alpha = 0.1, \gamma = 0.4\) and \(b = 0.4\). The output power increases and becomes maximum at the impedance matching condition and then decreases slightly on increasing the load. At larger values of the load the output becomes practically independent of the load, which is a desirable feature for using these devices as oscillators.
The main characteristics of this flux-flow oscillator is that both fluxons and antifluxons take part in the dynamics and because of that output can be obtained from both ends. Only in the resonant state we get output from the junction and the resonant state avoids any stray fluxons inside the junction. The oscillator can be tuned by tuning the dissipative junction parameters, applied dc bias currents and the external magnetic field values.

4.6 $rf$ field rectification

In this section, a novel method for rectifying alternating magnetic fields is demonstrated using fluxons in semiannular LJJs. An external magnetic field applied parallel to the dielectric barrier of the semiannular junction has opposite polarities at the ends of the junction and supports penetration of opposite polarity fluxons into the junction in the presence of a constant dc bias. When the direction of the field is reversed, flux penetration is not possible and flux-free state exists in the junction. Thus effective rectification of an alternating magnetic field can be achieved in semiannular LJJs. This unique phenomenon is specific to this geometry and can be employed in $rf$ SQUID magnetometers.

4.6.1 Introduction

When a LJJ is irradiated with a microwave of frequency $f$, quantized voltages, $V_n = n hf / 2e$, are observed in the junction [1], where $n$ is an integer and $h$ is the Planck's constant. In IVC, this effects manifests itself as constant voltage steps crossing the zero current axis. The occurrence of these voltage steps is a direct consequence of the ac Josephson effect and the phase coherent pair tunneling in response to an external electromagnetic excitation. Since no voltage other than the quantized values $V_n$ are present, for zero current bias, Josephson tunnel junctions are ideal as voltage standards which require constant voltage output independent of any external perturbations.
In all the previous works on Josephson diodes, rectification properties are studied using alternating bias currents and effective means of rectification of alternating magnetic fields are not discussed. In this section, we demonstrate a novel method to construct fluxon based diodes for rectifying harmonically oscillating magnetic fields. Investigations on a dc biased semiannular LJJ placed in an alternating magnetic field applied parallel to the plane of the dielectric barrier shows that the junction supports flux-flow only in alternate half cycles of the field. The flux linked with the edges of the junction has opposite polarities and support penetration of fluxons and antifluxons simultaneously from opposite ends of the junction under a constant dc bias. When the direction of the field is reversed, flux penetration is not possible and flux-free state exists in the junction. Thus, with this geometry, effective rectification of oscillating fields can be achieved. This is a unique phenomenon associated with the semiannular junctions.

4.6.2 Theoretical model

A LJJ with a semiannular geometry is considered with an external harmonically varying magnetic field applied parallel to the dielectric barrier of uniform thickness[111]. The corresponding dynamical equation is

\[ \varphi_{tt} - \varphi_{xx} + \sin \varphi = -\alpha \varphi_t + b \sin(\omega t) \sin(kx) - \gamma \] (4.23)

The boundary conditions of the junction can be obtained from the induced current term \( \frac{d\varphi(x)}{dx} = \varepsilon H \sin(\omega t) \cos(kx) \) as

\[ \varphi_x(0, t) = \frac{b}{k} \sin(\omega t) ; \quad \varphi_x(l, t) = -\frac{b}{k} \sin(\omega t) \] (4.24)

These boundary conditions are consistent with the fact that the effective field linked with the junction has opposite polarities at the ends. For sufficiently higher positive values of \( \gamma \) in Eq. (4.23), fluxons can enter the junction from \( x = 0 \) and antifluxons can enter the junction from \( x = l \) (right-end) and they can move in opposite directions. As the boundary conditions are not reflective, after
a transitory motion, fluxons and antifluxons are exited from the junction. When
the direction of the field is reversed, fluxon (or antifluxon) penetration becomes
impossible and flux-free state exists in the junction. To get some information
on the fluxon dynamics, we first determine the potential induced by the external
field inside the junction and then find energy change associated with a moving
fluxon in the junction. Lagrangian density of Eq. (4.23) with $\alpha = \gamma = 0$ is

$$L = \left\{ \frac{\varphi_t^2}{2} - \frac{1}{2} \left( \varphi_x - \frac{b}{k} \sin(\omega t) \cos(kx) \right)^2 - (1 - \cos \varphi) \right\}$$  \hspace{1cm} (4.25)$$

Therefore the corresponding potential energy density is (second term of the above
equation)

$$U(x, t) = \frac{1}{2} \left\{ \varphi_x^2 - \frac{2b}{k} \sin(\omega t) \cos(kx) \varphi_x + \left( \frac{b}{k} \sin(\omega t) \cos(kx) \right)^2 \right\}$$  \hspace{1cm} (4.26)$$

The first term is independent of the applied field and the third term is independent
of the flux motion in the junction. Therefore the change in the potential due to
the combined effect of the applied field and the flux motion in the junction can
be determined from the second term as :

$$U(x, t) = - \frac{b}{k} \int_{-\infty}^{+\infty} \varphi_x \sin(\omega t) \cos(kx) dx$$  \hspace{1cm} (4.27)$$

Substituting Eq. (1.38) in (4.28) and integrating, we get

$$U(x_0, t) = -2b \frac{l}{\sec h} \left( \frac{\pi^2}{2l} \sqrt{1 - u^2} \right) \sin(\omega t) \cos(kx_0)$$  \hspace{1cm} (4.28)$$

For long junctions and at relativistic velocities, $u \sim 1$, Eq. (4.28) becomes

$$U(x_0, t) = -C \sin(\omega t) \cos(kx_0)$$  \hspace{1cm} (4.29)$$

where $C = 2b l$ is a constant. Eq. (4.29) shows that the potential is oscillating
at the frequency of the applied field. This oscillating potential controls the flux
flow inside the junction and helps in the rectification of the field.
Energy of the unperturbed sG system is given by Eq. (1.37). Perturbational parameters modulate the velocity of the solitons and may cause to dissipate energy. The rate of dissipation is calculated by computing

\[
\frac{d}{dt}(H^P) = [\varphi_x \varphi_t]_0^t + \int_0^t [-\alpha \varphi_t^2 + (b \sin(\omega t) \sin(kx) - \gamma) \varphi_t] \, dx \quad (4.30)
\]

where the first term on the right side account for the boundary conditions. From Eq. (1.38), we get \( \varphi_t = -u \varphi_x \) and from Eq. (4.25), we get \( \varphi_x^2(0, t) = \varphi_x^2(l, t) \) (symmetric boundary conditions). Substituting these expressions, we find that the first term in the right hand side of the above equation vanishes - a symmetric boundary condition does not change average energy value of a fluxon. Inserting Eq. (1.38) in Eq. (4.30) and following perturbative analysis[15], we get

\[
(1 - u^2)^{-3/2} \frac{du}{dt} = -\alpha \frac{u}{\sqrt{1 - u^2}} - \frac{\pi}{4} \left\{ b \sec h \left[ \frac{\pi^2 \sqrt{1 - u^2}}{2l} \right] \sin(\omega t) \sin(kx_0) - \gamma \right\} 
\]

(4.31)

This expression describes the effect of perturbations on the fluxon velocity. In the above equation, the first term in the right-hand side represents the energy dissipation due to internal damping, second term account for the energy change associated with the external field and the third term represents the input power from the bias current.

The effects of a dc current on the fluxon dynamics in the presence of the external field is studied using Eq. (4.31). ZVS exists in the junction (flux-free state) when the dc bias is below a threshold value. By variation of the soliton position \( x_0 \), from Eq. (4.31), we find the largest possible bias current of zero-voltage state \( u = 0 \) to be[90]

\[
\gamma_1 = b \sec h \left( \frac{\pi^2}{2l} \right) 
\]

(4.32)

This is the threshold value of the applied bias, below which flux propagation is
4.6.3 IVC in rf fields

An oscillating magnetic field is applied parallel to the dielectric barrier of the junction with a constant dc bias. In the positive half cycles of the applied field, flux penetration and propagation is possible and finite voltages are observed across the junction. In the negative half cycles of the field, fluxons (or antifluxons) cannot enter the junction due to the repulsive Lorentz force, and zero voltage exists in the junction. Simulations are started with $\varphi = 0$ on a junction of $l = 10$. Time period of the ac signals are taken much larger than the typical response time of the system. In the following simulations we assumed the dissipation parameter $\alpha = 0.1$. Fig. 4.14 shows the IVC of the junction for different values of the oscillating field amplitudes and at a constant frequency ($\omega = 0.1$). In the figure, applied magnetic field is increasing from the top to the bottom curve in the range 0.50 to 1.50 in steps of 0.1. At lower magnetic fields, critical currents for fluxon penetration is large and the critical current gradually decreases on increasing the field strength.

4.6.4 Rectification of alternating fields

To demonstrate the rectification properties of the junction, we show a series of plots showing the time domain snapshots of voltage pulse forms $v(t)$ as a function of time $t$. The magnitude of the field should be sufficiently large to introduce fluxons into the junction. At small magnetic fields, fluxons cannot enter the junction and zero voltage exists. For sufficiently higher amplitudes (e.g. $b = 1.0$), fluxon penetration is possible in the positive half cycles and we get finite voltage in the junction. This is shown in Fig. 4.15. Rectification takes place in the following way. In the first half (positive part) of the alternating
field, fluxons enter from the left-end and antifluxons enter from the right-end and they move in opposite directions under the influence of the dc bias. The motion of fluxons in opposite directions produces a finite voltage across the junction. During the second half (negative part) of the magnetic field, antifluxon (or fluxon) penetration is not possible due to the repulsive Lorentz force and zero voltage (flux-free state) exists in the junction. Thus effective rectification of the field can be achieved in semiannular Josephson junctions. The number of fluxons taking part in the dynamics (and therefore the output voltage) can be controlled by controlling the strength of the magnetic field.

In Fig.4.16 we plot the average velocity (averaged over a period of the field) as a function of the magnitude of the field for different length of the junctions. A constant dc bias is applied to the junction in order to maintain flux motion in alternate half cycles. In the figure average voltage increases from zero and then increases linearly at higher values of the external field. Thus this device gives output which is linearly proportional to the input.

By reversing the dc bias (i.e., $\gamma$ to $-\gamma$), positive part of the alternating field can be suppressed. Fig.4.17 represents this rectification and shows negative pulses. In this case, fluxons cannot enter the junction during positive half cycles of the field due to the repulsive Lorentz force while flux penetration and propagation is possible in the negative half cycles.

4.7 Conclusions

In conclusion, we have studied flux-quantum dynamics in a semiannular geometry and the results suggest that this geometry can be used for fabrication of fluxon based diodes for rectification of ac signals, rectification of alternating magnetic fields and for implementing bidirectional flux-flow oscillators. The magnetic field driven transit of a trapped flux quantum under static conditions can find applications in digital transmission lines and in flux cleaning in stacked junctions.
Using vertically stacked junctions, the power associated with the bidirectional flux-flow oscillator can be increased considerably. The $rf$ field rectification properties of this device may find important applications in sub-millimeter radio wave astronomy, SQUID magnetometers, SIS mixers, etc. The main advantages of the proposed diodes are (i) very simple to fabricate, (ii) output of the device is linearly proportional to the applied field, (iii) flux motion takes place only in alternate half cycles so that heating and energy losses associated with flux motion can be reduced and (iv) independent of external perturbations. In the proposed LJJ diode, velocity of a fluxon is proportional to the voltage and a nonzero average velocity over a period of the $rf$ field means rectification of the field. By properly selecting junction parameters and the $dc$ bias, it is possible to rectify fields in different amplitude and frequency ranges.
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Fig. 4.1a Geometry of the semiannular LJJ with the applied field $b$.

Fig. 4.1b Schematic representation of the junction using discrete elements.
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Fig. 4.2 Tilted potential $U(x_0)/C$ along the junction as a function of the fluxon coordinate $x_0$ (triangles) and the field induced term $\gamma_b(x)/b$ (squares) for a junction of $l=30$.

Fig. 4.3 Applied dc bias $\gamma$ versus the average velocity $u=V(l/2\pi)$ in the junction in the forward-biased state and in the reverse-biased state. Arrows indicate the direction of current sweep. The parameters are $l=34$, $\alpha = 0.1$ and $b=0.1$. Inset in the figure shows the spatial profile $\langle\phi_x\rangle$ of the fluxon-antifluxon pair $<\uparrow\downarrow>$ in the junction. Parameters are $l=20$, $\alpha = 0.1$ and $b=0.1$. 
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Fig. 4.4 Normalized critical current \( \frac{I_c}{I_0} \) vs. static magnetic field \( b \) of a semicircular (*) and rectangular (□) JJ.

Fig. 4.5 IVC of a single trapped fluxon showing threshold value of the bias current at different magnetic field values. ZVS exists below the threshold value. The parameters are \( l=20, \alpha = 0.05, b=0.0 \) (circles), \( b=0.1 \) (squares), \( b=0.12 \) (up triangles) and \( b=0.15 \) (down triangles).

Fig. 4.6 Damping effects of the magnetic field on a single fluxon trapped in the junction. Parameters are \( l=20, \alpha = 0.05, \gamma = 0.1, b=0.2 \) (circles), \( b=0.1 \) (squares), \( b=0.3 \) (up triangles) and \( b=0.4 \) (down triangles).

Fig. 4.7 Rectification of a square wave. Left panel shows zero output voltage for the input amplitude \( A=0.20 \). Right panel shows rectified voltage pulses for input amplitude \( A=0.32 \). Frequency of the signal is \( \omega = 0.02 \). Parameters are \( l=34, \alpha = 0.1, b=0.11 \).
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Fig. 4.8 Square wave amplitude vs. average velocity in the junction for different input signal frequency. Parameters are \( I=34, \alpha = 0.1, b=0.1, A=0.4, \omega = 0.02 \) (down triangle), \( \omega = 0.04 \) (up triangle) and \( \omega = 0.03 \) (circles).

Fig. 4.9 Voltage pulses across the junction as a function of time showing rectification of a sine wave. Parameters are \( I=25, \alpha = 0.1, b=0.21, A=0.27 \) and \( \omega = 0.02 \).

Fig. 4.10 Sine wave amplitude vs. average velocity in the junction for different input signal frequency. Parameters are \( I=30, \alpha = 0.1, b=0.1, A=0.4, \omega = 0.02 \) (circles), \( \omega = 0.03 \) (squares) and \( \omega = 0.04 \) (triangles).

Fig. 4.11 (a) Spatial profile \( (\phi_x) \) showing a train of fluxons entering the junction from the left end and a train of antifluxons entering from the right end. (b) Spatial profile showing resonant propagation of fluxons towards the right end and antifluxons towards the left end.
Fig. 4.12 (a) Voltage pulses in the middle of the junction. Parameters are $l=20, \alpha = 0.1, b=0.4, \gamma = 0.4$ and $z=0.02$.
(b) The corresponding Fourier power spectrum of the voltage pulses. Spectrum has been computed from 4250 data points.

Fig. 4.13 Average output power vs. load $z$ on the left end of the junction (circles) and on the right end of the junction (triangles). Parameters are same as in Fig. 4.12.
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Fig. 4.14 Applied dc bias vs. average velocity for different values of the applied rf field. The parameters are $l=10$ and $\omega = 0.1$. The field strength increases from the top to the bottom curve from 0.5 to 1.50 in steps of 0.1.

Fig. 4.15 Rectification of a rf field with $\gamma = 0.5$ on a junction of $l=10$. (a) Applied field of amplitude $b=1.0$ (pp) and frequency $\omega = 0.05$. (b) Output pulse form $v(t)$ as a function of time $t$.

Fig. 4.16 Magnetic field amplitude $b$ vs. average velocity for different junctions. The parameters are $\omega = 0.1$, $\gamma = 0.5$, $l=10$ (squares), $l=15$ (circles) and $l=20$ (triangles).

Fig. 4.17 Rectification on a junction of $l=10$ with $\gamma = -0.5$. (a) Applied field of amplitude $b=1.0$ (pp) and frequency $\omega = 0.05$. (b) Output pulse form $v(t)$ as a function of time $t$ showing negative pulses.