Chapter 1

Fundamentals of Josephson junctions

Josephson junction based devices making use of many closely coupled junctions are being considered for making reliable high-$T_c$ superconducting electronic devices. Coupled Josephson junctions are used in the fabrication of Josephson voltage standards, in the microwave generators based on the coherent action of many junctions, and in Josephson computer elements. Large networks of Josephson junctions also received much attention as model systems for phase transition studies. The layered high-$T_c$ superconducting materials show properties of many layered closely coupled vertically stacked junctions. Therefore studies of the dynamical properties of coupled Josephson junctions can help to understand the properties of the layered superconducting materials.

In this chapter the Josephson effect is introduced and the dynamics of the charges and the electromagnetic fields in short and long Josephson junctions are related to the phase difference between the order parameter describing the Cooper pairs in each superconducting electrodes. The fundamental nonlinear properties of Josephson junctions are briefly reviewed giving emphasis to the basic equations governing fluxon dynamics in single long junctions and in coupled junctions. The basic equations governing fluxon dynamics in two-coupled junctions are derived. Various types of electromagnetic excitations in Josephson junctions are reviewed
and various regimes of fluxon dynamics is presented.

1.1 The Josephson junction

Josephson junctions are systems in which two superconductors are weakly coupled to one another as shown in Fig.1.1 [1, 2]. In each of the two superconductors the conduction electrons are interacting with phonons of the crystal lattice. At low temperatures this effect gives rise to an effective interaction between the electrons which then forms pairs of opposite spin and angular momentum. Such pairs are called Cooper pairs and are the carriers of the charge in the superconductor. Due to the anti-parallel spin and the angular momenta of the electrons in each pair, the total angular momentum vanishes and the Cooper pairs have Boson character. At zero temperature, all Cooper pairs are Bose-condensed into the electronic ground state of the superconductor. All excited quasiparticles states are separated by an energy gap $\Delta$, which is proportional to the effective binding energy of the Cooper pair, from the superconducting ground state. The superconducting state can be described by an effective macroscopic wave function with an amplitude proportional to the density of Cooper pairs $\rho_i$ and a phase $\theta_i$

$$\Psi_i = \sqrt{\rho_i} \exp(i\theta_i) \quad (1.1)$$

where $\Psi$ is the superconducting order parameter.

The two superconductors are weakly coupled with one another due to small overlap of the macroscopic wave functions. The overlapping of the wave functions is shown in Fig.1.1b. Different types of weak links are discussed in literature[3, 4, 5, 6, 7]. Coupling of two superconductors via a thin insulating barrier is a common type junction and such a system is called a superconductor-insulator-superconductor (SIS) tunnel junction.

The typical tunneling current-voltage characteristics of an SIS Josephson tunnel junction is depicted in Fig.1.2a. Four different tunneling regimes as shown in
Fig.1.2a-e can be observed in this characteristics. At zero voltage, Cooper pairs tunnel through the barrier \((S \rightarrow S)\), giving rise to a non-dissipative current. At voltages \(0 < V < 2\Delta/e\), quasiparticles tunnel through the barrier giving rise to the quasiparticle subgap current \((Q \rightarrow Q)\). The voltage \(V_g = 2\Delta/e\) is called the gap voltage. At voltages \(V \geq 2\Delta/e\), Cooper pairs are broken up and quasiparticles tunnel \((S \rightarrow Q)\) through the barrier. All the three processes follow the linear branch of normal electron tunneling \((n \rightarrow n)\) at voltages \(V > V_g\).

1.2 · The Josephson effect

The tunneling of Cooper pairs through the insulating barrier of an SIS type junction was predicted by Josephson in 1962[1] and experimentally observed for the first time by Anderson and Rowell in 1963[7]. Solving the quantum mechanical problem of the tunneling of Cooper pairs across a potential barrier in a point like junction, Josephson found that the local superconducting tunnel current density at zero voltage is given by

\[
j = j_0 \sin \phi
\]

where \(\phi = \theta_1 - \theta_2\) is the difference in phase between the order parameters of the two superconducting junctions. This equation describes the \(dc\) Josephson effect, i.e., a nonlinear current flow across the junction in the absence of an applied voltage across the junction. The maximum supercurrent current density \(j_0\) of the junction, calculated from macroscopic theory by Ambegaoker and Baratoff [8] is given by

\[
j_0 = \frac{\pi}{4} \frac{2\Delta(T)}{\rho e} \tanh\left(\frac{\Delta(T)}{2k_b T}\right)
\]

where \(\Delta(T)\) is the temperature dependent energy gap of the superconductor and \(\rho\) is the normal tunnel resistance of the junction per unit area. The electron charge is denoted by \(e\) and \(k_b\) is the Boltzmann constant. Applying a constant \(dc\) voltage across the junction, the phase difference \(\phi\) evolves in time according
to the *ac* Josephson equation

\[ V = \frac{\Phi_0}{2\pi} \frac{d\phi}{dt} \]  

(1.4)

where \( \Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-15} \text{Wb} \) is the flux quantum. At the constant voltage \( V \), the supercurrent through the junction oscillates with the characteristic frequency

\[ \frac{d\phi}{dt} \frac{1}{2\pi V} = \frac{1}{\Phi_0} = 483.6 \text{ MHz/\mu V} \]  

(1.5)

Thus Josephson junctions acts as a frequency to voltage standard.

### 1.3 Shapiro steps

Shapiro steps are constant voltage steps on the current-voltage characteristics of the junction when the junction is irradiated with an electromagnetic radiation. It was first observed by Shapiro in 1963[9]. In a voltage biased junction, due to the influence of the external field of frequency \( f_1 \), the effective voltage becomes

\[ V = V_0 + V_1 \cos(2\pi f_1 t) \]

Therefore the phase changes as \( \varphi = \int \frac{2\pi}{\Phi_0} V dt \). Substituting in Eq. (1.2), we get the expression for the supercurrent as

\[ I_s = I_c \sin[\varphi_0 + \frac{2\pi}{\Phi_0} V_0 t + \frac{V_1}{\Phi_0 f_1} \sin(2\pi f_1 t)] \]

simplifying the above expression, we see that, a time-independent (dc) current distribution occurs at \( \frac{2\pi}{\Phi_0} V_0 = 2\pi n f_1 \) at the dc voltage \( V_0 = n f_1 \Phi_0, n = 0, \pm 1, \pm 2, \ldots \). Typical applications of this effect need large number of junctions in series to get 1V.

### 1.4 Magnetic field effects

When a magnetic field is applied to a short Josephson junction, the corresponding phase difference across the junction can be shown to be[3]

\[ \phi = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0} \int A d\bar{l} \]  

(1.6)
where $\vec{A}$ is the electromagnetic vector potential. Considering a junction as shown in Fig. 1.3a, the difference in phase $\phi$ between the two coordinates $P$ and $Q$ chosen at different points along the junction is given by

$$\phi(Q) - \phi(P) = \frac{2\pi}{\Phi_0} \left[ \int_{P_1}^{P_2} \vec{A}(P) \, d\vec{l} - \int_{Q_1}^{Q_2} \vec{A}(Q) \, d\vec{l} \right] \quad (1.7)$$

If an external magnetic field $\vec{H}$ is applied in the plane of the junction, the flux enclosed in the contour is given by

$$\Phi = \int_s \mu_0 \vec{H} \, d\vec{s} = \oint \vec{A} \, d\vec{l} \quad (1.8)$$

$$= \int_{Q_1}^{Q_2} \vec{A} \, d\vec{l} + \int_{P_1}^{P_2} \vec{A} \, d\vec{l} + \int_{P_1}^{P_2} \vec{A} \, d\vec{l} + \int_{P_2}^{Q_2} \vec{A} \, d\vec{l} \quad (1.9)$$

The second and fourth terms in above equation vanish if the closed path is chosen considerably deeper in the superconductor than the London penetration depth $\lambda_J$, which is the characteristic screening length of the magnetic field in a superconductor. Thus, equating Eqs. (1.9) and (1.7) and considering the flux enclosed in the differential small section $d\vec{x}$ of the junction, we get

$$\frac{\phi(Q) - \phi(P)}{d\vec{x}} = \frac{2\pi}{\Phi_0} \Lambda \mu_0 H \quad (1.10)$$

where $\mu_0$ is the permeability of free space, $\Lambda = t_j + 2\lambda_L$ is the magnetic thickness of the junction and $t_j$ is the thickness of the tunnel barrier. $\Lambda \mu_0 H$ is the magnetic flux per unit length penetrating into a junction taking into account the screening of the magnetic field due to the superconductors (Fig. 1.3b). Thus the gradient of $\phi$ can be expressed as

$$\nabla \phi = \frac{2\pi}{\Phi_0} \Lambda \mu_0 \vec{H} \times \hat{z} \quad (1.11)$$

where $\hat{z}$ is the unit vector normal to the plane.

### 1.5 Static phase distribution in a small junction

The total supercurrent carried by a Josephson junction depends on the applied external magnetic field $H$. According to Eq. (1.11), the field induces a constant
gradient of the phase difference across the junction. Thus, the local Josephson current oscillates sinusoidally with the coordinate perpendicular to the field. The total supercurrent is given by\[3, 6\]

\[ I_c = \int_A j_c \sin(2\pi \mu_0 \Lambda H x) \, dA \]  

(1.12)

over the junction area \(A\), where we assume a spatially homogeneous critical-current density \(j_c\). If a rectangular junction is considered the integral can be solved explicitly as

\[ I_c(H) = I_c(0) \frac{\sin(\pi \Phi/\Phi_0)}{\pi \Phi/\Phi_0} \]  

(1.13)

where \(\Phi = \mu_0 \Lambda H w\) is the total flux threading the junction length. This expression is called the critical-current diffraction pattern of a rectangular junction and is shown in Fig.1.3c.

### 1.6 Dynamics of a small junction

If the length of the junction is smaller than \(\lambda_J\), the electrodynamics of the junction can be described by neglecting the variation of the phase difference across the junction area. In this case, the junction looks like as in Fig.1.4a and can be described by the equivalent electrical circuit shown in Fig.1.4b. This model is called the resistively and capacitively shunted junction (RCSJ) model\[3, 6\]. Using Kirchoff’s laws, the total current through the junction is given by

\[ I = I_c \sin \phi + \frac{V}{R} + C \frac{dV}{dt} \]  

(1.14)

Introducing the superconducting phase difference across the barrier, the above equation forms

\[ I = I_c \sin \phi + \frac{\Phi_0}{2\pi R} \frac{d\phi}{dt} + \frac{\Phi_0 C}{2\pi} \frac{d^2\phi}{dt^2} \]  

(1.15)

This equation is equivalent to a driven and damped pendulum or equivalently the viscous motion of a particle in a tilted potential (washboard potential).
1.7 The long Josephson junction (LJJ)

LJJs possess an extremely rich spectrum of linear and nonlinear electromagnetic excitations[10]. In large area Josephson junction, the phase difference $\phi$ between the top and bottom electrodes may vary in space. The spatial extension of the junction gives rise to the existence of solitons (fluxons) [4, 11], breathers and other nonlinear and linear excitations. In such junctions, the characteristic length scale of the spatial variation of $\phi$ is called the Josephson length $\lambda_J$. If the length of the junction is much larger then the Josephson length ($l \gg \lambda_J$), then the junction is called a long Josephson junction. Flux dynamics in LJJ can be described by the well-known sine-Gordon equation and Josephson junction forms one of the outstanding physical systems in which nonlinear properties can be studied experimentally.

1.8 The sine-Gordon equation

The sine-Gordon equation describing flux dynamics in a LJJ can be derived from the equivalent electrical circuit describing the junction. A LJJ and its equivalent discrete model in an external homogeneous magnetic field $H$ applied parallel to the dielectric barrier is given in Fig.1.5. In this model the junction is described by a parallel connection of small RCSJ like Josephson junctions interconnected by a parallel connection of an inductance and a resistance[12, 13]. An external bias current $I_k$ is injected in each node $k$ and the external flux $\Phi_{ext}$ threading each cell is taken into account. In this model, the wave equation is derived considering the flux quantization

$$\phi_{k+1} - \phi_k = \frac{2\pi}{\Phi_0} (\Phi_{ext} - LI_k^L)$$

where the flux threading the loop $k$ due to an externally applied field can be expressed as $d\Phi_{ext} = \mu_0 \Lambda H \Delta x$. The Kirchoff law at the node $k + 1$ is given by

$$I_k^{Rs} + I_k^L + I_{k+1} = I_{k+1}^L + I_{k+1}^{Rs} + I_{k+1}^{RCSJ}$$

(1.17)
Thus considering a small section $\Delta x$ of the long junction we can write down the continuous limit of the above equations as

$$\frac{\phi_{k+1} - \phi_k}{\Delta x} = \frac{\partial \phi}{\partial x} = \frac{2\pi}{\Phi_0} \left( \mu_0 H - L^* I^L \right)$$

(1.18)

$$\frac{\partial I^L}{\partial x} = j - j^{RCSJ} - \frac{\partial I^{R*}}{\partial x}$$

(1.19)

with $L^* = L/\Delta x$, $j = I/\Delta x$ and $j^{RCSJ} = I^{RCSJ}/\Delta x$. Differentiating Eq. (1.18) with respect to space we find

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{2\pi}{\Phi_0} \left( \mu_0 H - L^* \frac{\partial I^L}{\partial x} \right)$$

(1.20)

substituting Eq. (1.19) with $I^{R*} = -1/\rho_s \partial V/\partial x$ and the RCSJ current density (1.15) into Eq. (1.20) and considering a homogeneous external magnetic field ($\partial H/\partial x = 0$), we get the above equation as

$$\frac{\Phi_0}{2\pi L^*} \frac{\partial^2 \phi}{\partial x^2} = -j + j_c \sin \phi + \frac{V}{\rho} + C^* \frac{\partial V}{\partial t} - \frac{1}{\rho_s} \frac{\partial^2 V}{\partial x^2}$$

(1.21)

where $C^* = C/\Delta x$, $\rho = R\Delta x$ and $\rho_s = R_s \Delta x$. Expressing the voltages using the equation $V = (\Phi_0/2\pi) \partial \phi/\partial t$ and using the ac Josephson relation, we get the perturbed one-dimensional wave equation for the superconducting phase difference $\phi(x,t)$ called the perturbed sine-Gordon equation

$$\frac{\Phi_0}{2\pi L^*} \frac{\partial^2 \phi}{\partial x^2} - \frac{\Phi_0 C^*}{2\pi} \frac{\partial^2 \phi}{\partial t^2} - j_c \sin \phi = -j + \frac{\Phi_0}{2\pi \rho} \frac{\partial \phi}{\partial t} - \frac{\Phi_0}{2\pi \rho_s} \frac{\partial^3 \phi}{\partial x^2 \partial t}$$

(1.22)

where $L^*$ is the specific inductance of the junction, $C^*$ is the specific capacitance of the junction, $\rho$ is the quasiparticle resistance per unit length and $\rho_s$ is the surface resistance of the superconducting electrodes per unit length. The electric and magnetic fields are related to the phase difference $\phi$ in the following way:

$$E = \frac{V}{t_j} = \frac{1}{t_j} \frac{\Phi_0}{2\pi} \frac{\partial \phi}{\partial t}$$

(1.23)

$$H = \frac{1}{L^*} \frac{\Phi_0}{2\pi} \frac{\partial^2 \phi}{\partial x^2}$$

(1.24)
The specific inductance and capacitance of the junction are given by \( L^* = \mu_0 d' \) and \( C^* = \frac{\varepsilon_0 \varepsilon_j}{t_j} \), where \( \varepsilon_j \) is the relative dielectric constant of the junction barrier, \( t_j \) is the thickness and \( d' \) is the magnetic thickness. In the limit of the thick electrodes \( (d > \lambda_L) \), \( d' \) is given by \( d' = 2\lambda_L + t_j \). Dividing Eq. (1.22) by \( j_c \) and introducing the Josephson length \( \lambda_J \) and the plasma frequency \( \omega_p \)

\[
\lambda_J = \sqrt{\frac{\Phi_0}{2\pi L^* j_c}} \quad (1.25)
\]

\[
\omega_p = \sqrt{\frac{2\pi j_c}{\Phi_0 C^*}} \quad (1.26)
\]

Eq. (1.22) can be expressed as

\[
\lambda_J^2 \phi_{xx} - \frac{1}{\omega_p^2} \phi_{tt} - \sin \phi = -\frac{j}{j_c} + \frac{1}{\omega_p^2 C^* \rho} \phi_t - \frac{\lambda_J^2 L^*}{\rho_s} \phi_{xxt} \quad (1.27)
\]

From the above equation, the phase velocity of linear waves in the system is given by

\[
c_0 = \omega_p \lambda_J = c \sqrt{\frac{t_j}{\varepsilon_j d'}} \quad (1.28)
\]

where \( c_0 \) is termed as the Swihart velocity[14] and \( c \) is the velocity of light in vacuum. In long junctions, the Swihart velocity is typically only a few percent of \( c \) because the magnetic field penetrates into the superconductor on a length scale \( d' \), while the electric field is localized only in the junction barrier of thickness \( t_j \ll d' \). Normalizing the time with plasma frequency and space with the Josephson penetration depth, \( \tilde{t} = \omega_p t \) and \( \tilde{x} = x / \lambda_J \), the perturbed equation becomes

\[
\phi_{\tilde{t}} - \phi_{\tilde{x}} + \sin \phi = -\alpha \phi_{\tilde{t}} + \beta \phi_{\tilde{x}} + \gamma \quad (1.29)
\]

The perturbation terms in the right hand side of the above equation are defined as

\[
\alpha = \sqrt{\frac{\Phi_0}{2\pi j_c \rho^2 C^*}} = \frac{1}{\rho C^* \omega_p}, \quad \beta = \sqrt{\frac{2\pi j_c L^*}{\Phi_0 C^* \rho^2 s}} = \frac{\omega_p L^*}{\rho_s} \quad (1.30)
\]
where the first term is the normalized bias current, the second term is the damping term due to quasiparticle resistance and the third term corresponds to the damping due to the surface impedance of the superconducting electrodes. The terms $\alpha \phi_i$ and $\beta \phi_{xxi}$ represent normal electron current flow across and along the junction respectively (shunt and longitudinal losses).

1.8.1 Boundary conditions

The boundary conditions of a long overlap junction of normalized length $l$ in the absence of an external magnetic field is given by $\phi_\pm(0, \bar{t}) = 0 = \phi_\pm(l, \bar{t})$. In this case, any trapped fluxons in the junction executes oscillatory motion in the junction and they cannot escape from the junction due to the impedance mismatch. When an external magnetic field is applied parallel to the dielectric barrier of the junction, then the corresponding boundary conditions become $\phi_\pm(0, \bar{t}) = \bar{H} = \phi_\pm(l, \bar{t})$. Where $\bar{H} = \frac{2\pi}{\Phi_0} \mu_0 \Lambda H \lambda_J$ is the normalized magnetic field. In this case, fluxons are nucleated at one end of the junction and they are driven to the opposite end by the bias current. When the fluxons reach the opposite end of the junction they are pushed out from the junction.

1.9 Lagrangian and Hamiltonian functions

To calculate the energy of the system it is useful to introduce the Lagrangian and Hamiltonian of the system. To determine the Lagrangian, the energies of the electromagnetic fields and the Josephson coupling are to be considered. Combining the kinetic energy $T_{kin}$ associated with the energy density of the electric field and the potential energy $U_{pot}$ associated with the energy density of the magnetic field and the Josephson coupling, we obtain the Lagrangian $L = T_{kin} - U_{pot}$ by integrating over the junction volume $V$

$$L = \int_V \left[ \frac{1}{2} \varepsilon_0 \varepsilon_j E^2 - \frac{1}{2} \mu_0 \mu_r H^2 - \delta(z) \frac{\Phi_0}{2\pi} J_c (1 - \cos \phi) \right] dV \quad (1.32)$$
Expressing the electromagnetic field by the phase difference $\phi$ according to Eq. (1.23) and (1.24) and rewriting their coefficients in terms of $\lambda_J$ and $\omega_p$ we find

$$L = \int_0^l \int_0^w \left\{ \int_{-l/2}^{l/2} \left[ \frac{1}{2} \omega_p^2 \frac{j_e^2}{\varepsilon_0} \left( \frac{\phi_t}{\omega_p^2} \right)^2 \right] dz - \int_{-d/2}^{d/2} \left[ \frac{1}{2} \mu_0 \mu_r j_c^2 (\lambda_J^2 \phi_x)^2 \right] dz \right\} dy dx$$

(1.33)

upon rearranging the coefficients and performing the integration over the width of the junction $w$ and perpendicular to the junction plane and considering the different penetration depths of the electric and magnetic fields into the junction barrier, we find the Lagrangian

$$L = \frac{1}{2\pi} j_c w \int_0^l \left[ \frac{1}{2} \omega_p^2 \phi_t^2 - \frac{1}{2} \lambda_J^2 \phi_x^2 - (1 - \cos \phi) \right] dx$$

(1.34)

The normalized Lagrangian is

$$L = \frac{L}{\Xi_0} = \int_0^l \left[ \frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - (1 - \cos \phi) \right] dx$$

(1.35)

with the characteristic energy scale of the junction $\Xi_0 = \frac{\phi_0}{2\pi} j_c \omega p \lambda J$. Here $l = L/\lambda_J$ is the normalized junction length. Making use of the Lagrangian formalism, the sine-Gordon equation is obtained by calculating the equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \phi_t} + \frac{d}{dx} \frac{\partial L}{\partial \phi_x} - \frac{\partial L}{\partial \phi}$$

(1.36)

The Hamiltonian, determining the total energy of a LJJ is given by $H = H^{sG} + H^P$. Where $H^{sG}$ is Hamiltonian of the unperturbed sG equation given by[15]

$$H^{sG} = \int_0^l \left( \frac{1}{2} \phi_t^2 + \frac{1}{2} \phi_x^2 + 1 - \cos \phi \right) d\bar{x}$$

(1.37)

It contain the magnetic energy ($\propto \phi_t^2$), the electric energy ($\propto \phi_x^2$) and the Josephson coupling energy ($\propto 1 - \cos \phi$). $H^P$ is the contribution to the total energy due to the perturbation terms.
1.10 Excitations of the sine-Gordon system

In a sG system, a large variety of linear and in particular nonlinear excitations like solitons, anti-solitons, breathers and plasmons etc. do exist. The unperturbed sG equation is known to posses the Painlevé property[16] and is completely integrable. However perturbation terms make it nonintegrable. There are several different approach to the analytical description of soliton dynamics in nonintegrable systems. The most powerful perturbative technique is based on the inverse scattering transform (IST). IST was introduced by Gardner et al.[17]. Lax[18], Zakharov and Shabat[19] and Ablowvitz et al. [20, 21, 22] developed it further. The method is well explained and details of the method can be found in a number of books[23, 24, 25]. Equations exactly integrable by the IST posses many remarkable properties such as Backlund transforms[26], the Painlevé property, the possibility of representation in the Hirota bilinear form[27] and so on.

1.10.1 Soliton solutions

Neglecting all terms in the right hand side of the perturbed equation, Eq. (1.29), the unperturbed sG equation is given by

\[ \phi_{tt} - \phi_{xx} + \sin \phi = 0 \]

This equation represents a dispersive nonlinear wave equation which can be solved exactly giving the soliton solution[28, 29]

\[ \phi (\tilde{x}, \tilde{t}) = 4 \arctan \left[ \exp \left( \sigma \frac{\tilde{x} - u\tilde{t} - x_0}{\sqrt{1 - u^2}} \right) \right] \]  

(1.38)

Depending on the polarity \( \sigma \), \( \phi \) describes a kink (for \( \sigma = +1 \)) or an antikink (for \( \sigma = -1 \)) in the phase difference \( \phi \) moving at a normalized velocity \( 0 \leq u \leq 1 \). The kink corresponds to a jump of \( \phi \) from 0 to \( 2\pi \) (or \( 2\pi \) to 0 for an antikink). The supercurrent distribution (\( j \propto \sin \phi \)) associated with this excitation changes sign around the center of the kink. Solitary waves exist in systems in which
dispersion, which leads to the spreading of the energy of the wave form in space, and the nonlinear effects compensate each other. As a result, a stable solitary wave may propagate in a nonlinear medium while its energy remains localized in space. The kink in a sine-Gordon system is a topological soliton and there is no dynamical restriction on its existence.

1.10.2 Fluxons and Antifluxons

In the superconducting state only quantized flux can enter the junction. A quantum of flux with the magnetic field value \( \Phi_0 = \frac{\hbar}{2e} = 2.07 \times 10^{-15}\text{Wb} \) has the properties of a particle and behaves as a soliton in the junction. The solution of the unperturbed sG equation (with \( \sigma = +1 \) in Eq. 1.38) represents a fluxon if the total phase difference \( \phi \) along the junction varies from 0 to \( 2\pi \) as \( x \) varies from \(-\infty\) to \(+\infty\). Fig.1.6a shows this phase variation and represents a kink soliton or a fluxon. Thus a quantum of flux which produces a phase variation from 0 to \( 2\pi \) along the junction is called a fluxon.

If the flux quantum makes a phase variation from \( 2\pi \) to 0 along the junction as \( x \) varies from \(-\infty\) to \(+\infty\), then it is called an antifluxon (antikink). The phase variation (Eq. 1.38 with \( \sigma = -1 \)) corresponding to an antikink is shown in Fig.1.6b. Thus fluxons and antifluxons have the same magnetic field value and differs in polarity[30].

The supercurrent associated with the fluxon \( (j \propto \sin \phi) \) flows in closed form across the junction. The supercurrent flows horizontally within a penetration depth \( \lambda \) inside the superconductor[4]. These current loops encircle the flux and the resulting configuration is called a Josephson vortex. Since the supercurrent density is zero at the center, there is no core for the Josephson vortex[31]. The supercurrent direction associated with the antifluxon is in opposite direction to that of a fluxon as shown in Fig.1.6c. Fluxons of the same polarity repels each other while fluxons of opposite polarity attracts each other.
1.10.3 Breather solution

Under certain conditions, a kink and an antikink may form a bound pair called breather[32]. Thus a breather corresponds to a bound state of a soliton and an antisoliton which oscillates around the center of mass. The solution can be written in the form

$$\phi_{br}(\bar{x}, \bar{t}) = 4 \tan \theta \left[ \frac{\sin(\bar{t} \cos \theta)}{\cosh(\bar{x} \sin \theta)} \right]$$

Breathers are unstable with respect to perturbations and decay after some transient time.

1.10.4 Plasmons

In a LJJ, linear small amplitude excitations of $\varphi$ do exist. These can be modelled by the linearized sG equation

$$\varphi_{\bar{t}} - \varphi_{\bar{x}\bar{x}} + \varphi = 0$$

which has linear wave solution of the form

$$\varphi(\bar{x}, \bar{t}) = \varphi_0 \exp(ik\bar{x} - i\omega\bar{t})$$

with a spectrum $\omega(k) = \sqrt{1 + k^2}[32]$, where $k$ is the wave number of the mode and $\omega$ is the frequency. There is a gap of $\Delta \omega = 1$ in the excitation spectrum. These linear excitations of the LJJ are called plasmons.

1.11 Perturbative analysis

In LJJ, fluxons can be driven by external forces, i.e., using a current bias applied to the junction. The bias current gives rise to a Lorentz-Magnus force acting on the charge carriers of the vortex, resulting in the propagation of the fluxon along the junction. Due to the presence of dissipation, driving forces and damping forces are balanced for a certain fluxon velocity, leading to a steady motion of
the fluxon. McLaughlin and Scott [15] showed that the dynamics of the fluxon can be described by the lowest order perturbation theory. In this approximation, the effect of the perturbations is assumed to influence only the dynamics of the center of mass coordinate of the fluxon but not its shape. Substituting the soliton solution, Eq. (1.38), into the unperturbed sine-Gordon Hamiltonian, the normalized energy of the fluxon moving with the velocity \( u \) can be obtained as

\[
H_{sG} = \frac{8}{\sqrt{1 - u^2}}
\]

thus we can see that the rest energy of the soliton is 8, which is equal to the normalized rest mass of the fluxon. The change of the fluxon energy with time is given by

\[
\frac{dH_{sG}}{dt} = \frac{8u}{(1 - u^2)^{3/2}} \frac{du}{dt}
\]

The perturbational parameters modulate the velocity of the solitons and may dissipate energy. The rate of dissipation is calculated from the expression

\[
\frac{dH^P}{dt} = -\int_{-\infty}^{\infty} \left( \alpha\phi_t^2 + \beta\phi_{\xi t}^2 + \gamma\phi_t \right) dx
\]

where \( H^P \) is the Hamiltonian of the perturbation terms. The first and second terms represent the dissipation due to quasiparticle tunneling and due to the surface impedance while the third term represents the power supplied to the junction from the bias current. Substituting the soliton solution, Eq. (1.38), to the above equation and integrating, we get

\[
\frac{dH^P}{dt} = -8\alpha \frac{u^2}{\sqrt{1 - u^2}} - \frac{8\beta}{3} \frac{u^2}{(1 - u^2)^{3/2}} + 2\pi\gamma u
\]

At equilibrium condition at which the energy supplied to the system is equal to the energy dissipated, we get

\[
8\frac{du}{dt} + 8\alpha u (1 - u^2) + \frac{8\beta u}{3} + 2\pi\gamma (1 - u^2)^{3/2} = 0
\]

neglecting the surface damping term the equilibrium velocity can be obtained as

\[
u = \pm \left[ 1 + \left( \frac{4\alpha}{\pi\gamma} \right)^2 \right]^{-1/2}
\]
In Fig.1.7a typical normalized current-voltage characteristics is plotted. On increasing the bias current, fluxon velocity approaches the maximum velocity. At the maximum velocity relativistic effects are observed. The unperturbed sG equation is invariant with respect to Lorentz transformations. Thus solitons undergo Lorentz contraction. Therefore the field profile changes with the velocity. Fig.1.7b shows the variation of the field profile of a fluxon.

1.12 Coupled Josephson junctions

There has recently been considerable interests in coupled LJJ due to a variety of applications[33, 34]. Using low-$T_c$ superconductors stacks can be formed by layers of (Nb/AlO$_x$)$_x$Nb. For anisotropic layered high-$T_c$ superconductors, such as Bi$_2$Sr$_2$CaCu$_2$O$_y$ and Tl$_2$Ba$_2$Ca$_2$Cu$_3$O$_y$, it has been demonstrated that the crystal itself shows the features of stacked LJJJs. An important case occurs when the thickness of the superconducting layer is comparable to or less than the magnetic penetration depth of the superconducting layer. In such cases strong inductive coupling can be expected among the LJJJs making the stack. In the case of high-$T_c$ intrinsic Josephson junction stacks, inductive coupling is extremely strong. In multilayers, due to the close spacing of the superconductor-insulator lattice, the superconducting screening currents range across many layers and induce a coupling between individual junctions. The coupling of the junction can be adjusted by varying the thickness of the superconducting films. An external field parallel to the layers penetrates stacked junction in the form of fluxons. In multilayers, the magnetic field associated with the fluxons spread over many layers. Josephson junction multilayers are good candidates for high power flux-flow oscillators at THz frequencies.

A coupled junction consists of multiple thin films of the superconductor (eg. Nb) which are weakly linked in the vertical direction through insulating (eg. Al/AlO$_x$) layers. Fig.1.8 shows a stack of overlap Josephson junctions. To
bias the junction stack a vertical current is applied across the junction. The width of the system is made much smaller than the length of the junction. A schematic representation of the various layers and dimensions of the stack is shown in Fig.1.8a. The film thickness plays an essential role, as it determines the strength of the coupling between the stacked junctions[35, 36, 37, 38, 39, 40].

The mathematical model used to describe the system was first proposed by Sakai, Bodin and Pedersen[41]. The importance of the model is that all parameters such as characteristic lengths, frequencies and coupling parameters can be calculated from the system’s physical properties such as the critical current density, junction conductance and capacitance.

When an external magnetic field is applied to stacked Josephson junctions in the direction of the \( y \)-axis or when currents flow in the system, magnetic flux penetrates into the Josephson layers and gives rise to the field distribution shown in Fig.1.9. The straight forward approach to model the coupling between the superconducting layers is thus based on the vector potential and the currents associated with it. Consider the flux \( \Phi \) enclosed in the path \( P_1 P_2 Q_2 Q_1 \)

\[
\Phi = \oint \vec{B}_i \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l}
\]  

(1.45)

Using the quantum mechanical definition of the current density

\[
\vec{j} = -\frac{1}{2e\mu_0\lambda^2} (\hbar \nabla \theta + 2e\vec{A})
\]

and integrating along the paths parallel to the layers

\[
\int_{Q_2}^{P_2} \vec{j}_L \cdot d\vec{l} = -\frac{\hbar}{2e\mu_0\lambda^2} (\theta_{P_1} - \theta_{Q_1}) - \frac{1}{\mu_0\lambda^2} \int_{Q_1}^{P_1} \vec{A} \cdot d\vec{l}
\]  

(1.46)

\[
\int_{Q_1}^{P_1} \vec{j}_U \cdot d\vec{l} = -\frac{\hbar}{2e\mu_0\lambda_{l-1}^2} (\theta_{P_2} - \theta_{Q_2}) - \frac{1}{\mu_0\lambda_{l-1}^2} \int_{Q_2}^{P_2} \vec{A} \cdot d\vec{l}
\]  

(1.47)

The superindices \( U \) and \( L \) indicate the currents flowing in the top and bottom of a superconducting film. Assuming the density of the surface currents \( \vec{j}_L \) and \( \vec{j}_{U-1} \)
and the magnetic field $\vec{B}$ constant over the short distance $dx$ and then adding
the above equations and using the phase difference expression

$$\phi = \theta_{Q_2} - \theta_{Q_1} + \frac{2e}{\hbar} \int_{Q_1}^{Q_2} \vec{A} \cdot d\vec{l}$$

(1.48)

along with Eq. (1.45), we get

$$\frac{\hbar}{2e} \partial_{x} \phi_l = \mu_0 \lambda^2_l j_{l-1} - \mu_0 \lambda^2_l j_{l} - d_l B_l$$

(1.49)

To calculate the surface currents we rewrite the second London equation, $\partial_{zz} \vec{B} = \frac{1}{\lambda^2} \vec{B}$, and solve it for the superconducting layer $l$ with the appropriate magnetic fields as boundary conditions

$$B(z) = \frac{B_{l+1} \sinh(z/\lambda_l) - B_l \sinh((z - t_l)/\lambda_l)}{\sinh(t_l/\lambda_l)}$$

(1.50)

Using Ampere’s law $\nabla \times \vec{B} = \mu_0 \vec{j}$, for the geometry we get

$$j^L_l = \frac{B_l \cosh(t_l/\lambda_l) - B_{l+1}}{\mu_0 \lambda_l \sinh(t_l/\lambda_l)}$$

$$j^U_{l-1} = \frac{B_{l-1} - B_l \cosh(t_{l-1}/\lambda_{l-1})}{\mu_0 \lambda_{l-1} \sinh(t_{l-1}/\lambda_{l-1})}$$

Inserting the result into Eq. (1.49), it becomes

$$-\frac{\hbar}{2e} \partial_{x} \phi_l = s_{l-1} B_{l-1} + s_l B_{l+1} + d_l B_l$$

(1.51)

where the effective magnetic thickness $d'_l$ and the coupling parameter $s_l$ are defined by

$$d'_l = d_l + \lambda_l \coth\left(\frac{t_l}{\lambda_l}\right) + \lambda_{l-1} \coth\left(\frac{t_{l-1}}{\lambda_{l-1}}\right)$$

(1.52)

$$s_l = -\frac{\lambda_l}{\sinh\left(\frac{t_l}{\lambda_l}\right)}$$

Inserting Ampere’s law for the $z$-component of the current $\mu_0 j^z = \partial_x B(x)$ and taking the current densities that are described by the resistively and capacitively shunted Josephson junction

$$j^z_l = \frac{\hbar}{2e} C_l \partial_{tt} \phi_l + \frac{\hbar}{2e} G_l \partial_t \phi_l + J_l \sin \phi_l - I_l$$

(1.53)
we can write the Eq. (1.51) in the matrix form
\[
\frac{\hbar}{2e\mu_0} \partial_{xx} \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\cdots \\
\phi_N
\end{pmatrix} = \begin{pmatrix}
d'_1 & s_1 & \cdots & 0 \\
s_1 & d'_2 & s_2 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & s_{N-1} & d'_N & s_N
\end{pmatrix} \begin{pmatrix}
j_1^x \\
j_2^x \\
\cdots \\
j_N^x
\end{pmatrix}
\tag{1.54}
\]

The effective Josephson penetration depth becomes
\[
\lambda^{(2)}_J = \left( \frac{\hbar}{2e\mu_0 (d' + s)} \right)^{1/2}
\]
and the velocity of light in the barrier becomes
\[
\tilde{c}^{(2)} = \frac{1}{\sqrt{\epsilon \mu_0}} \left( \frac{d}{d' + s} \right)^{1/2}
\]
compared to a single-junction soliton case we note that
\[
\frac{\lambda^{(2)}_J}{\lambda^{(1)}_J} = \frac{\tilde{c}^{(2)}}{\tilde{c}^{(1)}} = \left( \frac{d}{d' + s} \right)^{1/2}
\]
since \( s < 0 \), \( \tilde{c}^{(2)} \) is larger than \( \tilde{c}^{(1)} \). Thus in stacked junctions, velocity of light exceed the velocity of light in single junction case. In normalized units the above equation becomes
\[
\partial_{xx} \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\cdots \\
\phi_{N-1} \\
\phi_N
\end{pmatrix} = \begin{pmatrix}
1 & \sigma & \cdots & 0 \\
\sigma & 1 & \sigma & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \sigma & 1 & \sigma
\end{pmatrix} \begin{pmatrix}
\partial_{tt}\phi_1 + \alpha \partial_t \phi_1 + \sin \phi_1 - \gamma \\
\partial_{tt}\phi_2 + \alpha \partial_t \phi_2 + \sin \phi_2 - \gamma \\
\cdots \\
\partial_{tt}\phi_{N-1} + \alpha \partial_t \phi_{N-1} + \sin \phi_{N-1} - \gamma \\
\partial_{tt}\phi_N + \alpha \partial_t \phi_N + \sin \phi_N - \gamma
\end{pmatrix}
\tag{1.55}
\]
where \( \sigma = \frac{s}{d'} \). The external magnetic field does not influence the dynamics of the stack if the top and bottom electrodes are thicker than \( \lambda \). The boundary conditions when an external magnetic field is applied to the junction are
\[
\frac{\hbar}{2e\mu_0} \partial_x \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\cdots \\
\phi_N
\end{pmatrix} = B_{ext} \begin{pmatrix}
d^z_1 + s_{1+s_0} \\
\cdots \\
d^z_l + s_{l+s_l-1} \\
\cdots \\
d^z_N + s_{N+s_N-1}
\end{pmatrix}
\tag{1.56}
\]
The derivative of the phase difference at the edges of the stack is called the open boundary conditions. In this case, flux can enter end exit the junctions.
1.12.1 Two coupled junctions

A two coupled stack is an important configuration for both theoretical and experimental studies. In this case, fluxon dynamics can be described using the system of equations[41]

\[
\begin{align*}
\phi_{tt} - \frac{1}{1-S^2} \phi_{xx} + \sin \phi &= -\alpha \phi_t - \gamma - \frac{S}{1-S^2} \psi_{xx} \\
\psi_{tt} - \frac{1}{1-S^2} \psi_{xx} + \sin \psi &= -\alpha \psi_t - \gamma - \frac{S}{1-S^2} \phi_{xx}
\end{align*}
\]  

(1.57)

where \( \phi \) is the phase difference of the eigen functions of the first junction and \( \psi \) is the phase difference of the eigen functions of the second junction. \( S (S < 0) \) is the normalized coupling constant. A two coupled stack supports two types of fluxon motion in it. Both in-phase and out-of-phase locked modes of fluxon motion can be observed. It has been predicted that the in-phase flux-flow mode multiplies the power of flux-flow oscillator whereas the out-of-phase mode doubles the main radiation frequency of the oscillator. The out-of-phase flow of fluxons in a two-fold stack is shown in Fig.1.10a and in-phase flow of fluxons is shown in Fig.1.10b.

1.13 Regimes of fluxon dynamics

Josephson junction with open boundary condition will interact with the environment not only through the bias current but also through the external magnetic field at the boundaries. Therefore the dynamics in LJJ with open boundary conditions is complex. Three major regimes of fluxon motion can be observed in single long junctions as well as in stacks.

1.13.1 Zero Field Steps (ZFS)

In the absence of magnetic field the current voltage characteristics (IVC) of a long junction shows a family of so-called zero field steps. In this state one or more fluxons or antifluxons propagate in the junction, driven by the bias current.
At the junction boundaries they are reflected with the opposite polarity\[42, 43\]. The reflection at the boundaries gives rise to microwave emission. The maximum voltage of these steps is then calculated as $V_{\text{max}} = \Phi_0 \frac{n\xi}{L}$.

### 1.13.2 Fiske Steps (FS)

When a magnetic field is applied to the junction, the field penetrates partially into the junction and will decrease the fluxon energy at one side and increase it at the other side of the junction. At magnetic fields larger than a certain threshold value, fluxons are nucleated at one end of a current biased junction and is annihilated at the other end. In the process of annihilation, plasmons are emitted, which resonate with the junction cavity. In this case the IVC shows steps called the Fiske steps. The maximum voltage of the Fiske steps can be calculated as $V_{\text{max}} = \Phi_0 \frac{n\xi}{2L}$. This is valid only in a limited range of magnetic field values. Fiske steps are cavity resonances in LJJ\[44\].

### 1.13.3 Flux-Flow Steps (FFS)

In the high field limit of the Fiske modes, dynamics is dominated by the flow of fluxons. These are nucleated at one junction edge and viscously flow in a dense chain through the junction to exit at the other end. This effect is effectively utilized in the flux-flow oscillator. The maximum voltage of the flux-flow step is $V_{\text{max}} = H \Lambda \bar{c}$, where $H$ is the applied field, $\Lambda$ is the magnetic thickness and $\bar{c}$ is the Swihart velocity. This relation is valid for superconducting electrodes thicker than the London penetration depth\[45\].

### 1.14 Annular junctions

Different geometries are proposed for LJJ to study the fluxon dynamics and among them, annular geometries offer the advantage of reflectionless motion of fluxons and is studied extensively theoretically and experimentally\[11, 46, 47, 21\].
The annular geometry is of particular importance for the experimental and theoretical investigation of non-linear properties of LJJ. An annular LJJ tunnel junction is formed by two ring shaped superconducting electrodes separated by a thin tunnel barrier as shown in Fig. 1.1. The electrodynamics of a junction of length \( l \) is described by the perturbed sG equation with the periodic boundary conditions

\[
\varphi(x = 0) = \varphi(x = l) - 2\pi n
\]

\[
\frac{\partial \varphi}{\partial x}(x = 0) = \frac{\partial \varphi}{\partial x}(x = l)
\]

The number of kinks initially present in the annular junction is conserved due to the closed topology. Experimentally, annular junctions are prepared in states with \( n \) topologically trapped Josephson vortices by cooling the junction from the normal to the superconducting state in a small applied field. Alternately, vortices may be trapped in the junction by locally heating up one of the electrodes in an external field using an electron or laser beam in a low temperature scanning microscope. The number of the flux-quanta trapped in the junction can be determined from the IVC of the junction.

### 1.15 Conclusions

In short, LJJ's offer the possibility of studying solitons that account for the magnetic flux-quanta (fluxon) moving along the tunnel barrier. A fluxon is basically a quantum of magnetic field which can be used for transmission of information or can be an object based on which certain novel Josephson devices such as flux-flow oscillators, voltage rectifiers, logic gates, magnetic field rectifiers, field sensors, etc. can be implemented. Fluxons can be trapped in the junction either during the normal-superconducting transition or by applying an external magnetic field parallel to the junction. In the superconducting state only fluxons or antifluxons can exist in the junction and they are driven by the Lorentz force associated with a \( dc \) current. In the absence of an external magnetic field, trapped
fluxons cannot escape from a linear junction and they make successive reflections at the edges of the junction. Progressive fluxon motion in LJJ is associated with a $dc$ voltage which can be detected across the junction.
Chapter 1. Figures

Fig. 1.1 (a) Two superconductors weakly coupled to one another. (b) Amplitude of the macroscopic wave function of the two superconductors.

Fig. 1.2 (a) Current voltage characteristics of a Josephson tunnel junction. (b) Bose representation of the electron density of states of the superconductor. (c) Josephson tunneling process. (d) Quasiparticle tunneling process. (e) Cooper pair dissociation and tunneling into quasiparticle states.
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Fig. 1.3 (a) Closed path across the barrier of a Josephson junction. (b) Magnetic field penetration into the superconductor. (c) Critical current diffraction pattern of a small rectangular junction.

Fig. 1.4 (a) Sketch of a small Josephson junction. (b) Discrete circuit model of a small junction.

Fig. 1.5 (a) Sketch of a LJJ. (b) The equivalent discrete model of the LJJ. The phase difference across the junction at node $k$ is given by $\phi_k$. The current through the RCSJ junction is $i_{RCSJ}$. 

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Fig. 1.6a The kink solution of the sG equation.

Fig. 1.6b The antikink solution of the sG equation.

Fig. 1.6c The supercurrent encircling a fluxon and an antifluxon in a rectangular junction. The applied field $H$ parallel to the dielectric barrier induces a screening current and the field penetrates the junction over a distance $\lambda_j$.

Fig. 1.7a Normalized bias current vs. fluxon velocity.
Fig. 1.8 A stack of inductively coupled overlap Josephson junctions in an applied magnetic field $H$. A bias current is applied from the top electrode to the bottom electrode.

Fig. 1.7b Gradient of the phase proportional to the magnetic field threading the junction. Total flux associated with the kink in the phase is $\Phi_0$.

Fig. 1.8a Schematic representation of the various layers and dimensions of the stack.

Fig. 1.9 (a) Field penetration inside a stack. (b) Closed integration path across the barrier.
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**out-of-phase**

Fig.1.10a Out-of-phase flow of fluxons in a two coupled junction

**in-phase**

Fig.1.10b In-phase flow of fluxons in a two coupled junction

Fig.1.11 An annular long LJJ. The current distribution associated with a fluxon is indicated with the closed arrow mark.