CHAPTER - 2

Peristaltic Transport Of
An Incompressible Power-Law
Fluid Through a Channel.
2.1. INTRODUCTION:
Peristaltic action is a means of transportation of fluids through organs of human and other living beings, such as ureter, ductus difference of the male reproductive system, gastro-intestinal tract, bile duct etc. A number of biomedical instruments used to pump peristaltic motion of blood and other fluids. The experimental and theoretical fluid mechanics of peristaltic transport was first examined by Latham (10). A review of much early literature presented by Jaffrin and Shafiro (6). A summary of the theoretical investigations reported so far has been in a paper by Srivastava and Srivastava (20). Fung and Yi (3) analysed the role of Reynolds number and wave length in peristaltic motion of moderate amplitude, making use of perturbation method with an amplitude ratio (i.e., the ratio between the wave length and width of the channel) as perturbation parameter. They obtain a critical pressure gradient and observed that the backward flow in the central region of the stream occur when the time average pressure gradient is greater than the critical value. The occurrence of the back flow throws light on an important pathological condition is known as vasicoureteral reflux condition, when a backward flow of urine is set up from bladder to ureter and thus causing bacteria to pass from bladder to the kidney.
Burns and Perks (2) has discussed the peristaltic flow produced by sinusoidal peristaltic wave along a flexible wall of the channel under the pressure gradient. Following Manton (11-13), Rudraiah et al. (16-18), Krishna and Manohar Rao (9) discussed the peristaltic motion of incompressible fluid through a porous medium through a flexible wavy channel. Raju and Devanathan (14,15), Girija Devi and Devanathan (5), Kaimal (7) have studied the Non-Newtonian effects in peristaltic flow taking into consideration that the flow through the ureter exhibits Non-Newtonian characters. Krishna et al.(8) discuss the peristaltic pumping of a viscous conducting fluid through a flexible wavy channel under the influence of transverse magnetic field.

Keeping the above facts in view, in this chapter we discuss the peristaltic transport of an incompressible visco-elastic second order Rivlin-Erickson fluid through a flexible channel, making use of long wave length approximation. The perturbation analysis is carried out to obtain the velocity field, the stream lines, the pressure rise per wave length and the stress. The phenomena of the reflux and trapping and the variation of the time average flux with pressure rise are discussed analytically in detail. The computational analysis has been carried out for drawing streamlines, velocity profiles, the profiles indicating the pressure rise with reference to average flux etc., are plotted for different sets of governing parameters.

2.2. FORMULATION AND SOLUTION:

We consider the peristaltic flow of an incompressible power law fluid through a two-dimensional channel of width 2a. The channel is symmetric with respect to its axis and the deformation of its flexible walls is due to propagation of an infinite train of travelling peristaltic wave imposed on the wall. Choosing Cartesian co-ordinate system \( o(x,y) \) the flexible walls are represented by
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\[ y = s(x,t) = a + b \sin \left( \frac{2\pi}{\lambda} (x - ct) \right) \]  

(2.2.1)

Where \( b \) is amplitude of the peristaltic wave, \( c \) is the wave velocity, \( \lambda \) is wave length. With reference to the laboratory frame \((x,y)\) the flow is unsteady. However, with reference to the frame moving with the same speed \( c \) the flow is steady.

We choose the "Ostwald - Dewaele" power law model in which stress and strain relation is given by

\[ \tau_{ij} = -m(\Phi)^{n-1} \dot{\gamma}_{ij} \]  

(2.2.2)

Where, \( \Phi \) the dissipation function is given by

\[
\Phi = \left\{ \sum \sum \left( \frac{\dot{\gamma}_{ij} \dot{\gamma}_{ji}}{2} \right) \right\}^{\frac{1}{2}} = \left[ \frac{1}{2} \left\{ \frac{\partial u}{\partial x} \right\}^2 + \frac{\partial v}{\partial y} \right] \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\}^{\frac{1}{2}}
\]

Where \( m \) is the consistency and \( n \) is the power law index parameter.

\[ \tau_{xx} = -m(\Phi)^{n-1} \dot{\gamma}_{xx} = -2m(\Phi)^{n-1} \frac{\partial u}{\partial x} \]

\[ \tau_{yy} = -m(\Phi)^{n-1} \dot{\gamma}_{yy} = -2m(\Phi)^{n-1} \frac{\partial v}{\partial y} \]

\[ \tau_{xy} = -m(\Phi)^{n-1} \dot{\gamma}_{xy} = -m(\Phi)^{n-1} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]

Where \( u, v \) are velocity components along \( o(x,y) \) directions respectively.

Making use of above stress and strain relations the equations governing the two dimensional flow of power law fluid are

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0
\]  

(2.2.3)

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = - \frac{\partial p}{\partial x} + 2m \frac{\partial}{\partial x} \left( \phi_f \frac{\partial u}{\partial x} \right) + m \frac{\partial}{\partial y} \left( \phi_f \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right)
\]  

(2.2.4)
\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\frac{\partial p}{\partial y} + 2m \frac{\partial}{\partial y} \left( \phi_f \frac{\partial \mathbf{v}}{\partial y} \right) + m \frac{\partial}{\partial x} \left( \phi_f \left( \frac{\partial \mathbf{u}}{\partial y} + \frac{\partial \mathbf{v}}{\partial x} \right) \right) \]

\[ \text{(2.2.5)} \]

Where \( \phi_f = \Phi^{n-1} \) \hspace{1cm} \[ \text{(2.2.6)} \]

and \( p \) is the fluid pressure, \( \rho \) is the density. The fluid is shear thinning, Newtonian or shear thickening accordingly as \( n < 1, n = 1, n > 1 \) respectively.

In view of the incompressibility of the fluid and two dimensionality of the flow we introduce the stream function \( \psi_f \) such that \( \mathbf{u} = \frac{\partial \psi_f}{\partial y} \) and \( \mathbf{v} = -\frac{\partial \psi_f}{\partial x} \).

On eliminating \( p \) from equations (2.2.4) and (2.2.5) the governing equations in terms of \( \psi_f \) reduces to

\[ \rho \left[ \frac{\partial}{\partial t} \left( \nabla^2 \psi_f \right) + \frac{\partial}{\partial x} \left( \frac{\partial \psi_f}{\partial y} \nabla^2 \psi_f \right) - \frac{\partial}{\partial y} \left( \frac{\partial \psi_f}{\partial x} \nabla^2 \psi_f \right) \right] \]

\[ = m\phi_f \nabla^4 \psi_f + 2m \nabla^2 \psi_f \frac{\partial^2 \phi_f}{\partial x \partial y} + m \left( \frac{\partial^2 \psi_f}{\partial y^2} - \frac{\partial^2 \psi_f}{\partial x^2} \right) \left( \frac{\partial^2 \phi_f}{\partial y^2} - \frac{\partial^2 \phi_f}{\partial x^2} \right) + \]

\[ + 2m \left( \frac{\partial \phi_f}{\partial y} \frac{\partial^3 \psi_f}{\partial y^3} + \frac{\partial \phi_f}{\partial x} \frac{\partial^3 \psi_f}{\partial x^3} \right) \]

\[ \text{(2.2.7)} \]

where \( \nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) and \( \nabla^4 = \left( \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} \right) \)

The relevant boundary conditions on \( \psi_f \) are

\( \psi_f = 0 ; \quad (\psi_f)_{yy} = 0 \) on \( y = 0 \) \hspace{1cm} \[ \text{(2.2.8)} \]

\( (\psi_f)_y = 0 \) on \( y = S(x) \) \hspace{1cm} \[ \text{(2.2.9)} \]
\[ \psi_f(s) - \psi_f(0) = \frac{q}{2} \] (2.2.10)

The condition (2.2.8) guarantees the symmetry of the flow with respect to mid axis of the channel and (2.2.9) corresponds to the no slip of the axial velocity on the flexible wall. (2.2.10) is the imposed flux condition across the channel.

We now make use of transformation between fixed frame (laboratory frame) \( o(x,y) \) and the wave frame \( O(X,Y) \) as follows:

\[
X = x - ct; \quad Y = y
\]

\( U \) and \( V \) are the velocity components in wave frame are related to the velocity components \( u, v \) in the fixed frame as follows

\[
U = u(x - ct, y) - c \\
V = v(x - ct, y)
\]

If \( \psi_w \) is the stream function in wave frame given by \( \psi_w = \psi_f - cy \) If the length of the channel is finite and is equal to integral number of wave lengths and the pressure difference between the ends of the channel is constant, flow is steady in the wave frame. We assume that these conditions are meet and hence solve the steady problem in the wave frame.

The instantaneous wall frame \( q(x,t) \) across the mid channel between the axis and the boundary wall is

\[
\frac{q}{2} = \int_0^s u \, dy
\]

If \( Q \) is the rate of volume flow independent of \( x \) and \( t \) in the wave frame then

\[
\frac{Q}{2} = \int_0^s U \, dY \quad \text{it follows that} \quad \frac{q}{2} = \frac{Q}{2} + cs
\]

The time mean volume flow at each cross section which measures the mean discharge rate is obtained by integrating over the period of wave

\[
\left( \tau = \frac{\lambda}{c} \right)
\]
\[ q = \frac{1}{\tau} \int_{\tau_0}^{\tau} q \, dt = Q + \frac{2c}{\tau} \int_{\tau_0}^{\tau} s \, dt \]

This integral which is to be evaluated at a constant axial distance \( x \) along the channel reduces to \( \frac{a\lambda}{c} \) in case of sinusoidal boundary and thus we obtain \( \frac{q}{2ac} = \frac{Q}{2ac} + 1 \) \hspace{1cm} (2.2.11)

We now introduce the following non-dimensional quantities:

\[
\begin{align*}
    x^* &= \frac{X + ct}{\lambda}; \quad y^* = \frac{Y}{a}; \quad Q^* = \frac{Q}{2ac}; \quad q^* = \frac{q}{2ac} \\
    \theta^* &= \frac{q}{2bc} \; ; \quad (u^* v^*) = \left( \frac{U}{c}, \frac{V}{c} \right) \epsilon \; ; \quad \phi = \frac{b}{a} \; ; \quad p^* = \frac{\rho a^{n+1} p}{m \lambda c^n} \\
    \Psi^* &= \frac{\Psi_w}{ac} = \frac{\Psi_f - cy}{ac} = \frac{\Psi_f}{ac} - y^* \; ; \quad s^*(x^*) = \frac{s(x, t)}{a} \hspace{1cm} (2.2.12) \\
\end{align*}
\]

\( \epsilon = \frac{a}{\lambda} \) \hspace{1cm} (long wave length approximation parameter)

The boundary wall in non-dimensional form is given by

\( y^* = 1 + \phi \sin(2\pi x^*) \)

The governing equations in non-dimensional form reduces to (on dropping asterisks)

\[ \epsilon \left[ \frac{\partial}{\partial x} \left( \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \left( \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \left( \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \right] \]
\[
R_p \left[ \zeta \left( \frac{\partial^4 \psi}{\partial y^4} + \epsilon^4 \frac{\partial^4 \psi}{\partial x^4} + 2 \epsilon^2 \frac{\partial^2 \psi}{\partial x^2 \partial y^2} \right) + \left( \frac{\partial^2 \zeta}{\partial y^2} - \epsilon^2 \frac{\partial^2 \zeta}{\partial x^2} \right) \left( \frac{\partial^2 \psi}{\partial y^2} - \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} \right) \right] + 2 \epsilon \frac{\partial^2 \zeta}{\partial x \partial y} \left( \frac{\partial^2 \psi}{\partial y^2} + \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} \right) + 2 \left( \frac{\partial^2 \psi}{\partial y \partial \psi} + \epsilon^4 \frac{\partial^2 \psi}{\partial x \partial \psi} \right) \right]
\]

where \( \zeta = \left[ \left( \frac{\partial^2 \psi}{\partial y^2} + \epsilon^2 \frac{\partial^2 \psi}{\partial x^2} \right)^2 + 8 \epsilon^2 \left( \frac{\partial^2 \psi}{\partial x \partial \psi} \right)^2 \right]^{\frac{n-1}{2}} \) is the non-dimensional dissipation function.

\[
R_p = \frac{m}{2^{n-1} \rho c^2 \alpha} \left( \frac{c}{a} \right)^n
\]
is power law fluid parameter. which reduces to the Reynolds number in case of Newtonian fluids \((n=1)\)

The non-dimensional boundary conditions in terms of \( \psi \) are

\[
\psi = 0; \quad \psi_y = 0 \quad \text{on} \quad y = 0
\]
\[
\psi_y = -1 \quad \text{on} \quad y = s
\]
\[
\psi(s) - \psi(0) = Q \quad \text{on} \quad y = s
\]

Making use of long wave length approximation \( \epsilon \ll 1 \).

We write \( \psi = \psi_0 + \epsilon \psi_1 + \ldots \) \( (2.2.14) \)

Substituting (2.2.15) in (2.2.13) and separating the terms of different order of \( \epsilon \). The equation corresponding to zeroth order in \( \epsilon \) is

\[
\frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right) = 0
\]

The corresponding Boundary conditions are

\[
\psi_0 = 0 \quad \text{and} \quad \left( \psi_0 \right)_y = 0 \quad \text{on} \quad y = 0
\]
\[
\left( \psi_0 \right)_y = -1 \quad \text{on} \quad y = s \quad \text{and} \quad \psi_0(s) - \psi_0(0) = Q
\]

\( (2.2.17) \)
The equation corresponding to the order $\epsilon$ is
\[
\frac{\partial}{\partial x} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right) + \frac{\partial \psi_0}{\partial y} \frac{\partial}{\partial x} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right) - \frac{\partial \psi_0}{\partial x} \frac{\partial}{\partial y} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right) - 2R_p \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^{n-1} \frac{\partial^2 \psi_0}{\partial y^2}
\]
\[= R_p \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 \psi_1}{\partial y^2} \right)^n \quad (2.2.18)\]

The corresponding conditions are
\[
\psi_1 = 0 \text{ and } \left( \psi_1 \right)_{yy} = 0 \quad \text{on} \quad y = 0 \quad (2.2.19)
\]
\[
\left( \psi_1 \right)_y = 0 \quad \text{on} \quad y = s \quad \text{and} \quad \psi_1(s) - \psi_1(0) = 0
\]

On solving the equation (2.2.16) subjected to the conditions (2.2.17) we get

\[
\psi_0 = \left( \frac{2n+1}{n+1} \right) \left( Q + s \right) \left( \frac{y}{s} \right) - \left( \frac{n}{2n+1} \right) \left( \frac{y}{s} \right)^n - y \quad (2.2.20)
\]
\[
u_0 = \frac{\partial \psi_0}{\partial y} = \left( \frac{2n+1}{n+1} \right) \left( Q + s \right) \left[ 1 - \left( \frac{y}{s} \right)^n \right] \left( \frac{y}{s} \right)^n - 1 \quad (2.2.21)
\]
\[
u_0 = -\frac{\partial \psi_0}{\partial x} = -\left( \frac{2n+1}{n+1} \right) \left( Q + s \right) \left[ \left( \frac{y}{s} \right)^n - \left( \frac{y}{s} \right)^{2n+1} \right] s_x +
\]
\[
\left( \frac{y}{s} \right) - \left( \frac{n}{2n+1} \right) \left( \frac{y}{s} \right)^n \left( \frac{y}{s} \right)^{2n+1} s_x \quad (1.2.22)
\]

The solution of the equation corresponding to the first order in $\epsilon$ (2.2.18) subjected to the related conditions (2.2.19) involves a complicated particular integral which might be integrated either near the axis ($y \sim 0$) or near
the boundary \((y-s)\). On substituting (2.2.20) for \(\Psi_0\) and simplifying the left hand side of (2.2.18) reduces to

\[
R_p \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 \Psi}{\partial y^2} \right)^n = \frac{(2n+1)(1-n)}{(1+n)^2} \frac{(Q+s)S_y}{s \left( \frac{2n+1}{n} \right)} +
\]

\[
+ 2R_p \frac{n(1-n)}{(2n-1)} \frac{2n^2 + 3n + 1}{n^2} y^{-\left( \frac{2n-1}{n} \right)} = F_1(y,s) \text{ say} \]

(2.2.23)

The solution of (2.2.23) subjected to the conditions (2.2.19) may be represented as follows:

\[
\Psi_1 = F_3 - y \left( F_2 \right)_{y=s} \tag{2.2.24}
\]

Where \(F_2 = \int_0^{y} (F_1(y,s) + c_1) y^{-\frac{1}{n}} dy\) and \(F_3 = \int_0^{y} F_2(y,s) dy\)

Satisfying the conditions \(s(F_2)_{y=s} - (F_3)_{y=s} = 0\) which determines the arbitrary constant \(c_1\). In view of the complexity of the integrals in (2.2.24) in the following discussion of the flow phenomenon we confine to the solution to the zeroth order in \(\varepsilon\) under long wave length approximation \((\varepsilon \ll 1)\)

2.3. DISCUSSION OF THE FLOW PHENOMENON:

The volume flow rate \(Q\) with respect to the wave frame in terms of the time average volume flow rate with respect to the fixed frame \(q\) in dimensionless (after dropping the asterisks) is \(Q = q - 1 = \theta - 1\). Where \(\theta\) is dimensionless volume flow rate. Assuming that all the material between \(y=a-b\) and \(y=a+b\) were convected rightwards like a solid body with speed \(c\). The expression for the stream function now reduces to

\[
\Psi_0 = \left( \frac{2n+1}{n+1} \right) (s + \theta - 1) \left\{ \frac{y}{s} - \left( \frac{n}{2n+1} \right) \left( \frac{y}{s} \right)^{\frac{2n+1}{n}} \right\} - y \tag{2.3.1}
\]

On the boundary wall \(y = s, \ \Psi_{ow} = \theta - 1\)
When \( n=1 \) the expression for the stream function in case of Newtonian Fluids may be obtained (19).

We find from equation (2.2.21) that the axial velocity \( u_0 \) in the wave frame attains its extreme at position \( x = \pm \left( \frac{2n+1}{4} \right) \)

With respect to the fixed frame this axial velocity varies from zero at the boundary wall to a maximum (or minimum) at the axis of the channel. The actual maximum axial velocity is attained at the widest part of the channel and is minimum at the narrowest part of the channel and the respect values are

\[
\left( u_0 \right)_{\text{max}} = \left( \frac{2n+1}{n+1} \right) \left( \frac{\phi(1+\theta)}{1+\phi} \right)
\]

\[
\left( u_0 \right)_{\text{min}} = \left( \frac{2n+1}{n+1} \right) \left( \frac{\phi(\theta-1)}{1-\phi} \right)
\]

**STRESS ON THE WALL:**

The stress on the flexible wall in the non-dimensional form is

\[
\tau = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( 1 - \left( \frac{ds}{dx} \right)^2 \right) + \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \frac{ds}{dx} \left( 1 + \left( \frac{ds}{dx} \right)^2 \right)
\]

substituting for \( u \) and \( v \) from (2.2.21) and (2.2.22) in the above expression for stress we obtain

\[
\tau = \frac{1}{n} \left( \frac{2n+1}{n} \right) \left\{ \left( \frac{Q+s}{s^2} \right) \left( s_x^2 - 1 \right) - \frac{n}{2n+1} s_{xx} \right\} \left[ \frac{1-s_x^2}{2} \right] - 2 \left[ \frac{Q+s}{s^2} \right] s_x^2
\]

\[
\tau = \frac{1}{n} \left( \frac{2n+1}{n} \right) \left\{ \left( \frac{Q+s}{s^2} \right) \left( s_x^2 - 1 \right) \left[ \frac{1-s_x^2}{2} \right] - 2 \left[ \frac{Q+s}{s^2} \right] s_x^2 \right\}
\]

\[
\tau = \frac{1}{n} \left( \frac{2n+1}{n} \right) \left\{ \left( \frac{Q+s}{s^2} \right) \left( s_x^2 - 1 \right) \left[ \frac{1-s_x^2}{2} \right] - 2 \left[ \frac{Q+s}{s^2} \right] s_x^2 \right\}
\]

\[
\left( \frac{2n+1}{n} \right) \left\{ \left( \frac{Q+s}{s^2} \right) \left( s_x^2 - 1 \right) \left[ \frac{1-s_x^2}{2} \right] - 2 \left[ \frac{Q+s}{s^2} \right] s_x^2 \right\}
\]

(2.3.2)
PRESSURE RISE:

The magnitude of the pressure rise per wavelength in the non-dimensional form is

\[ \Delta p = \left| \int_0^1 \frac{dp}{dx} \, dx \right| \]

\[\Delta p = R_p \left( \frac{2n+1}{n} \right)^n \int_0^1 \left( \frac{(Q + s)^n}{s^{2n+1}} \right) dx\]

\[\Delta p = R_p \left( \frac{2n+1}{n} \right)^n \int_0^1 \left( \frac{(q + \phi \sin(2\pi x))^n}{(1 + \phi \sin(2\pi x))^{2n+1}} \right) dx\]

\[= R_p \left( \frac{2n+1}{n} \right)^n \left\{ \frac{n(2n+1)}{2} \phi^{n+1} \theta^{n-1} - \frac{n(n-1)}{4} \phi^n \theta^{n-2} - \phi^n \theta^n - \frac{(2n+1)(n+1)}{2} \phi^{n+2} \theta^n \right\} \]

(2.3.3)

The dimensionless time mean flow \( \theta_0 \) for zero pressure rise \( (\theta_0)_{\Delta p=0} \) is either zero or

\[ (\theta_0)_{\Delta p=0} = \frac{n(2n+1)\phi + \sqrt{n(1-n) + \frac{\phi^2 n(2n+1)(n+1)}{4}}}{2 \left( 1 + \frac{(2n+1)(n+1)\phi^2}{2} \right)} \]

for shear thinning fluid \( (n<1) \)

\[ \theta_0 = \frac{n(2n+1)\phi + \sqrt{4n(1-n) + n(n+1)(2n+1)\phi^2}}{2 \left[ 2 + (n+1)(2n+1)\phi^2 \right]} \]

The Newtonian fluid \( (n=1) \) thus time mean flow for zero pressure rise reduced to
\[ \theta_0 = \frac{3\phi}{2[1 + 3\phi^2]} \]

For shear thickening fluid \((n>1)\)

\[ \theta_0 = \frac{n(2n + 1)\phi + \sqrt{n(1+n)(2n+1)\phi^2 - 4n(n-1)}}{2 + (n + 1)(2n + 1)\phi^2} \]

If \(\phi < 2\sqrt{\frac{n-1}{(n+1)(2n+1)}}\) no such non-zero mean flow can exist at zero pressure rise. Also it follows that the dimensionless pressure rise for zero time mean flow is

\[ (\Delta p)_{\theta=0} = \frac{(-1)^n R_p}{2\pi} \left( \frac{2n+1}{n} \right)^n \phi^n \Pi \]

Where

\[ \Pi = \int_{0}^{2\pi} \left[ (\sin t)^n - (2n+1)\phi(\sin t)^{n+1} - n(2n+1)\phi^2 (\sin t)^{n+2} + (2n+1)^2 \phi^2 (\sin t)^{n+2} \right] dt \]

(2.3.4)

This dimensionless pressure rise for zero time mean flow in case of Newtonian fluids \((n=1)\) reduces to

\[ (\Delta p)_{\theta=0} = \frac{\theta\phi^2 R_p}{2} \]

The results in case of Newtonian fluids coincide with those of earlier investigations of Shaperio et al \((18)\) to the order of \(\phi^2\).

**Reflux Phenomenon:**

In the fixed frame the flow is unsteady and the path lines of the material particle are different from stream lines. However, in the wave frame the flow is steady and hence the path lines and the stream lines coincide. They are usually, similar to the wall shape but lesser amplitude near the axis. The stream function in the wave frame \(\psi_w\) may be used to identify the material.
particle in the fixed frame. To find the rate of reflux of the material we calculate the time average flow in the fixed frame between the axis and a particular material particle $\psi_w$. The instantaneous flow $\frac{q_{\psi_f}}{2}$ of all the material between the axis and the stream line $\psi_w$ of material particle in the fixed frame is given by:

$$\left(\frac{q_{\psi_f}}{2}\right) = \int_0^{y(x,y,t)} u(x,y,t) \, dy$$

In terms of wave stream function $\psi_w$:

$$\left(\frac{q_{\psi_f}}{2}\right) = \psi_w + cy$$

Averaging over one period of wave we get:

$$\left(\frac{q_{\psi_f}}{2}\right) = \psi_w + c\int_0^1 y \, dt$$  \hspace{1cm} (2.3.5)

Performing the integration in the wave frame we obtain:

$$\left(\frac{q_{\psi_f}}{2}\right) = \psi_w + c\int_0^1 y \, dx^*$$

This in terms of the non-dimensional average flux:

$$\left(\frac{q_{\psi_f}}{ac}\right) = \left(\frac{q_{\psi}}{\psi}\right)$$

And the wave stream function $\frac{\psi_w}{ac} = \psi^*$ we obtain $q_{\psi}^* = \psi^* + \int_0^1 y^* dx^*$.

After dropping the asterisks $q_{\psi} = \psi + \int_0^1 y \, dx$ \hspace{1cm} (2.3.6)

**REFLUX LIMIT:**

Consider the wave frame stream lines very close to the wall and introduce a small parameter.
\[ e = \psi_0 - \psi_{ow} = \psi_0 + 1 - \phi \theta \quad (e \ll 1) \]

The equation of stream line near the wall may be expressed as

\[ y = s(1 + ea_1 + e^2a_2 + \ldots) \quad (2.3.8) \]

Substituting (2.3.7) and (2.3.8) in (2.3.1) we obtain

\[ a_1 = -\frac{1}{s} \quad \text{and} \quad a_2 = -(s + \theta - 1) \left( \frac{2n+1}{2n} \right) \frac{1}{s^3} \]

and thus

\[ y = s - e - e^2 \left( \frac{2n+1}{2n} \right) \left( \frac{s + \phi - 1}{s^2} \right) + O(e^3) \quad (2.3.9) \]

Substituting (2.3.9) in (2.3.6) we get

\[
\begin{align*}
\bar{q}_\psi &= e + (\phi - 1) + \int_0^1 \left[ s - e - e^2 \left( \frac{2n+1}{2n} \right) \left( \frac{s + \phi - 1}{s^2} \right) \right] dx \\
&= \phi - e^2 \left( \frac{2n+1}{2n} \right) \left( \frac{\phi - 1}{s^2} \right) \quad \text{to the order of } e^2
\end{align*}
\]

\[ \bar{q}_w = \phi \theta \quad \text{on} \quad y = s 
\]

\[ \Rightarrow \bar{q}_w = 1 + e^2 \left( \frac{2n+1}{2n} \right) \left( \frac{\phi - \phi^2}{\phi \theta} \right) \frac{1}{\left( 1 - \phi^2 \right)^3} \quad (2.3.11) \]

Reflex occurs provided

\[ \left( \frac{\bar{q}_\psi}{\bar{q}_w} \right) > 1 \Rightarrow \phi \theta < \phi^2 \Rightarrow \theta < \phi \quad (2.3.12) \]

**REFLUX FRACTION FOR SMALL AMPLITUDE:**

This reflux fraction ratio may be determined for small values of \( \phi \) as follows. Examination of the stream function (2.3.1) suggests that \( y \) may be expanded in a power series in the small parameter \( \phi \).

\[ y = -\psi \left( 1 + b_1(x, \theta, \psi) \phi + b_2(x, \theta, \psi) \phi^2 + \ldots \right) \]
Substituting this in equation (2.3.1) for \( \psi_0 \) and determining the coefficients of \( b_1, b_2 \) etc. we obtain

\[
y = -\psi - \frac{2n+1}{n+1}(\theta + \sin 2\pi \xi) \left( \frac{n}{2n+1} \chi + 1 \right) \psi \phi - \left( \frac{2n+1}{n+1} \right)(\theta + \sin 2\pi \xi) x
\]

\[
x \left\{ \frac{2n+1}{n+1}(\theta + \sin 2\pi \xi) \left( \frac{n}{2n+1} \chi + 1 \right) - \chi \sin 2\pi \xi \right\} \psi \phi^2
\]

(2.3.13)

Where \( \chi = - (\psi)^{n+1} \)

Substituting (2.3.13) into integral (2.3.6) and evaluating we obtain

\[
\left( -q_{\psi} \right) = -\left( \frac{2n+1}{n+1} \right) \left\{ \begin{array}{c} 3n+1 \\ 2(n+1) \end{array} \chi + \frac{n}{2n+1} \chi^2 + \frac{n}{2n+1} \right\} \psi \phi^2 +
\]

\[
\left( 1 + \frac{n}{2n+1} \chi \right) \psi \phi \theta + \left\{ \begin{array}{c} 3n+1 \\ n+1 \end{array} \chi + \frac{n}{n+1} \chi^2 + \frac{2n+1}{n+1} \right\} \psi \phi^2 \theta^2
\]

(2.3.14)

The maximum of \( -q_{\psi} \) occurs at \( \psi = \left( -\psi_{\text{max}} \right)^{n+1} \) where

\[
\psi_{\text{max}} = \left\{ \begin{array}{c} \phi^2 + \frac{2n+1}{n+1}(Q+1) + \frac{n\phi^2}{2(Q+1)(n+1)} - \frac{2(n+1)(Q+1)\phi}{(3n+2)(2(Q+1)^2 + \phi^2)} \end{array} \right\}
\]

The reflux fraction ratio \( R \) may now be calculated using

\[
R = \frac{\left( -q_{\psi} \right)_{\text{max}}}{\phi \theta} - 1
\]

\[
= \frac{2n+1}{n+1} \left\{ \begin{array}{c} 3n+1 \\ 2(n+1) \end{array} \psi_{\text{max}} + \frac{n}{2(n+1)} \psi_{\text{max}}^2 + \frac{n}{2(n+1)} \right\} \frac{\phi^2}{Q+1} + \left( 1 + \frac{n}{2n+1} \psi_{\text{max}} \right) +
\]

\[
\left( \frac{3n+1}{n+1} \psi_{\text{max}} + \frac{n}{n+1} \psi_{\text{max}}^2 + \frac{2n+1}{n+1} \right)(Q+1) \left( \psi_{\text{max}} \right)^{n+1} - 1
\]

(2.3.15)
TRAPPING LIMIT:

From (3.1) it is evident that $\psi = 0$ not only on the axial plane of the channel but also on the surface whose cross sectional curve is defined by

$$y = \frac{s}{(n + 1)^{n+1}} \left\{ \frac{(2n + 1)\phi \theta + n\phi \sin 2\pi x - (n + 1)}{\phi(\theta + \sin 2\pi x)} \right\}^n$$

(2.3.16)

This equation shows that the surface is real cuts the central plane and lies within the channel if

$$\frac{n + 1}{n\phi} > \theta > \frac{n + 1}{(2n + 1)\phi}$$

(2.3.17)

This is the condition for which there will exist closed stream line forming a trapped bolus moving at the speed of the wave. Following Brasseur et al. (1), the trapping limit in terms of $q$ is obtained setting $\psi = 0$ in (2.2.20) with $y \neq 0$ which leads to the

$$y = s \left\{ \frac{(2n + 1)(s + q - 1) - (n + 1)s}{(n + 1)(s + q - 1)} \right\}^\frac{n}{n+1}$$

(2.3.18)

For trapping to occur both the numerator and denominator in (2.3.18) be of the same sign so that $y$ is real and positive. The condition for the existence of trapped fluid is $q_\text{--} < q < q_\text{+}$, where

$$q_\text{--} = \frac{n + 1 - 2n\phi - \phi}{2n + 1}; \quad q_\text{+} = \frac{n + 1 + 2n\phi + \phi}{n}$$

4. COMPUTATIONAL ANALYSIS OF THE FLOW PHENOMENON:

For computational purpose as already stated we consider the different flow variables under the long wave length approximation. The profiles of the stream function, the velocity, the pressure rise and the wall stresses for
variations in the governing parameters $n, Q, \phi, R_p$ are plotted in figures 1-18. The stream lines in the wave frame for different values of $Q, n$ are plotted in the figures 1 - 5. We observe that in contrast to the Newtonian fluids the trapping occurs even for small values of $Q$, as may be seen from figure 1 - 3. However, the trapped bolus of the fluid which occurs far above the axial region moves closer to the boundary for an increase either in $Q$ or $n$. Also the shape and growth of the trapping zone undergoes a significant distortion as it moves closer to the wall. We may also notice that in case of Newtonian fluid the formation of the trapping zone is visible only at higher values of $Q$, while at lower values separation seems to takes place in a given axial range (figs. 4 & 5). The behaviour of the axial and transfers velocities may be observe from figures 6 - 12. These axial velocity profiles in general exhibit a gradual reduction in their relative velocities with reference to the wave frame from its maximum attained on the boundary to the minimum on the axis. This however, implies that the actual axial velocity gradually rises from its minimum on the boundary to its maximum on the axis. An increase in the power law index $n$ reduces the magnitude of the velocity at all corresponding points (figs. 6 & 7). Also an increase in $Q$ gives rise to reversal flow in the axial region (fig. 7) shows a retardation in the velocity as we move along the axial direction of the channel. An increase in power law index in case of shear thinning fluids retards the axial velocity in the mid region while enhances it in the vicinity of the wall. The magnitude of the relative velocity is minimum in the mid region and maximum near the boundary. When $n>1$, a slight enhancement in the magnitude may be noticed near the boundary (fig. 8). The relative transverse velocity $v$ profiles are plotted in figures 9 -12. These profiles exhibit the variations of $v$ from its zero value attained on the axis to its maximum attained on the boundary depending on $\phi$ and $x$. The increase in $Q$ the relative flux reduces $v$ uniformly all over the upper half of the region (fig. 9). For fixed $Q$, $x$ and $n$ an increase in $\phi$ increases the magnitude of $v$ all over the region (fig. 10). The variation in $v$ with reference to the variation
in $n$ (power law index) may be observe from figure 11. The velocity $v$ enhances with increase in $n$ all over the flow field. It is interesting to note that this transverse velocity which is the sole consequence of the peristalsis over the flexible wall of the channel is greatly influenced by the power law index and thus its magnitude in the case of shear thickening fluids greater than its corresponding magnitude either in Newtonian or shear thinning fluids. The transverse velocity changes its direction at different axial distances due to variations in the amplitude of the sinusoidal wave at these distances. In a given wave length the $v$ is towards the axis and decreases in its magnitude for lengths $x \leq \pi/4$ and later changes its directions towards the boundary and increases in its magnitude in the range $\pi/4 < x < \pi$ (fig. 12). Figures 13 and 14 correspond to magnitude of pressure rise $v/s$ $Q$ or $R_p$. We find that the magnitude of the pressure rise monotonically increases with an increase in $Q$ for all fixed $n$, $\phi$ and $R_p$ (fig.13). An increase in $R_p$ for fixed $Q$ and $\phi$ increase the pressure rise ($\Delta p$) to some extent $R_p \leq 65$ but decreases for little higher values of $R_p = 70$ and later experience a steep rise for higher values of $R_p = 100$ (fig.14). The behaviour of the stress may be observed from figure (15-18). For given values of $n$, $\phi$ and $x$ stress gradually reduces from its maximum value at $Q = 0$ to lower value for higher $Q$ (figure 15). An increase in $n$ the behaviour of the stress $v/s$ $n$ for fixed $Q$, $\phi$ and $x$ shows that the stress changes its sign from negative to positive as $n$ increases through smaller values ($n < 1$). Thus, in case of shear thinning fluids the flexible wall within a wave length may experience a zero stress for some critical $n$ although this stress increases with an increase in $n$ higher than the critical value (figure 16). A rapid growth in the stress for an increase in $\phi$ for fixed $Q$, $n$ and $x$ may be observed from figure 17. Figure 18 consolidates the behaviour of the stress at different axial distances in a wave length for variations in $n$, $\phi$ and $Q$. We may note that in a given wave length along the axis the stress changes its sign at $z = \pi/2$ indicating a possible separation the flow region. Of For any given set of parameters $n, \phi$ and $Q$ the magnitude of the stress exhibits an oscillatory
behaviour with wide variations in a given wave length. This oscillatory variation is the direct consequence of the variation in amplitude of the flexible boundary. We may note that some of our observations related to pressure rise and stresses are in agreement with the conclusions drawn in earlier investigations related to the power law fluids by Srivastava and Srivastava(20).
Stream Lines $n=0.28$, $Q=5.0$, $\Phi = 0.5$

Fig-1

Stream Lines $n=0.28$, $Q=50$, $\Phi = 0.5$

Fig-2
Stream Lines  $n=0.33, \quad Q=50, \quad \Phi =0.5$

Fig-3
Stream Lines $n=1.0$, $Q=5.0$, $\Phi=0.5$

Fig-4
Stream Lines \( n=1.0, \ Q=50, \ \Phi=0.5 \)

Fig-5
Velocity Profiles - Fig. 7

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Velocity Profiles - Fig. 8

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dp Vs. Q - Fig. 13

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Q vs. d_p
Stress Vs Q - Fig. 15

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