Chapter 3

Phase Transitions and Bubble Nucleations for $\phi^6$ Model in (2+1) Dimensional Curved Spacetime

3.1 Introduction

(2+1) dimensional gravity [81]-[85] exhibits novel features of interest. There are several important differences between the three and four dimensional problems. First of all the divergences in the gravitation action induced by scalar loops in 4 dimensions can, by power counting, be proportional to $1$, $R$, $R^2$, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (or suitable combinations of them). In three dimensions the situation is simplified, as the only candidates are $1$ and $R$ [82]. General relativity is a geometric theory of spacetime, and quantizing gravity means quantizing spacetime itself. Ordinary quantum field theory is local, but the fundamental physical observables of quantum gravity are necessarily nonlocal. Ordinary quantum field theory takes causality as a fundamental postulate, but in quantum gravity the spacetime geometry and the causal structure, are themselves subject to quantum fluctuations. Again, perturbative quantum field theory depends on the existence
of a smooth, approximately flat background, but there is no reason to believe that
the short-distance limit of quantum gravity even resembles a smooth manifold.
Faced with these problems, it is natural to look for simpler models that share the
important conceptual features of general relativity while avoiding some of the con­
ceptual difficulties. General relativity in (2 + 1) dimensions is one such scheme of
formulation [81]. Another important feature of the conformally invariant scalar
theory in three dimensions is that its \( \phi^6 \) coupling can in principle, induce a di­
gerence in the four-point Green’s functions necessitating a \( \phi^4 \) coupling, which is
not conformally invariant [6].

Field theory of (2+1) dimensions may exhibit several features of interest in
condensed matter physics, which are not in (3+1) dimensional field theory, de­
scribing high energy physics. (2+1) dimensional \( \phi^6 \) theory finds applications in
the study of vortex solution of the abelian Chern-Simons theory [86], blackholes
in string theory [1], etc. In this chapter the first order phase transition in a
(2+1) dimensional curved spacetime for \( \phi^6 \) model is discussed. In the previous
chapter we have obtained a divergenceless expression for the one-loop effective
potential for the \( \phi^6 \) model in a (3+1) dimensional Bianchi type-I spacetime. This
chapter is organized in the following way. In section 3.2 we evaluate the one-loop
effective potential for \( \phi^6 \) theory in a (2 + 1) dimensional Bianchi type-I space­
time and obtain a divergenceless expression. A finite expression for the energy
momentum tensor for the \( \phi^6 \) theory in this spacetime is obtained in section 3.3.
The finite temperature effective potential for the same theory is evaluated and
the finite temperature effects on the phase transitions are discussed in the next
section. In section 3.5 the nature of phase transitions for the present model is
examined and is clarified to be of first order. The crucial dependence of phase
transitions on spacetime curvature and the gravitational-scalar coupling is made
clear in section 3.6. A first order phase transition proceeds by nucleation of bubbles of broken phase in the background of unbroken phase. In section 3.7 the interaction between the bubble field and the surrounding plasma is considered and the expansion of bubbles in such a damping environment is discussed. It is found that there exists an exact solution for the damped motion of the bubble in the thin wall regime. The discussions and conclusions are presented in the final section.

3.2 One-loop Effective Potential for $\phi^6$ Theory in (2+1) Dimensional Bianchi Type-I Spacetime

Let us consider a massive self interacting scalar field $\phi$ coupled arbitrarily to the gravitational back ground and described by the Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \sqrt{-g} \left\{ \frac{1}{2} [g^\mu\nu \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2] - \frac{1}{2} \lambda^2 \phi^2 (\phi^2 - m/\lambda)^2 \right\}$$ (3.2.1)

The equation of motion associated with the Lagrangian (3.2.1) is,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \left( m^2 + \xi R \right) \phi - 4 \kappa \phi^3 + 3 \lambda^2 \phi^5 = 0$$ (3.2.2)

in which we put $m\lambda = \kappa$. Writing $\phi = \phi_c + \phi_q$, where $\phi_c$ is the classical field and $\phi_q$ is a quantum field with vanishing vacuum expectation value, $\langle \phi_q \rangle = 0$, the field equation for the classical field $\phi_c$ is given by,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi_c + \left[ (m_r^2 + \delta m^2) + (\xi_r + \delta \xi) R \right] \phi_c - 4 (\kappa_r + \delta \kappa) \phi_c^3 - 12 (\kappa_r + \delta \kappa) \phi_c \langle \phi_q^2 \rangle + 3 (\lambda_r^2 + \delta \lambda^2) \phi_c^5 + 30 (\lambda_r^2 + \delta \lambda^2) \phi_c^3 \langle \phi_q^2 \rangle + 15 (\lambda_r^2 + \delta \lambda^2) \phi_c \langle \phi_q^4 \rangle = 0$$ (3.2.3)
where the bare parameters $m$, $\xi$, $\kappa$ and $\lambda$ are replaced by the renormalised terms. To the one loop quantum effect, the field equation for the quantum field $\phi_q$ is,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi_q + (m_r^2 + \xi R)\phi_q - 12\kappa_r\phi_c^2\phi_q + 15\lambda_c^2\phi_c^4\phi_q = 0 \quad (3.2.4)$$

The effective potential $V_{eff}$ is given by,

$$V_{eff} = \frac{1}{2}[(m_r^2 + \delta m^2) + (\xi_r + \delta \xi)R][\phi_c^2 + \langle \phi_c^2 \rangle] - (\kappa_r + \delta \kappa)\phi_c^4 - 6(\kappa_r + \delta \kappa)\phi_c^2\langle \phi_q^2 \rangle - (\kappa_r + \delta \kappa)\langle \phi_q^4 \rangle + \frac{1}{2}(\lambda_c^2 + \delta \lambda^2)\phi_c^6 + \frac{15}{2}(\lambda_c^2 + \delta \lambda^2)\phi_c^4\langle \phi_q^2 \rangle + \frac{1}{2}(\lambda_c^2 + \delta \lambda^2)\langle \phi_q^4 \rangle \quad (3.2.5)$$

To make $V_{eff}$ finite, the following renormalisation conditions are used,

$$m_r^2 = \left( \frac{\partial^2 V_{eff}}{\partial \phi_c^2} \right)_{\phi_c = R = 0}, \quad \xi_r = \left( \frac{\partial^3 V_{eff}}{\partial R \partial \phi_c^2} \right)_{\phi_c = R = 0},$$

$$\kappa_r = \left( \frac{\partial^4 V_{eff}}{\partial \phi_c^4} \right)_{\phi_c = R = 0}, \quad \lambda_r^2 = \left( \frac{\partial^6 V_{eff}}{\partial \phi_c^6} \right)_{\phi_c = R = 0} \quad (3.2.6)$$

Consider a $(2+1)$ dimensional Bianchi type-I spacetime with small anisotropy which has the line element

$$ds^2 = C(\eta)d\eta^2 - a_1^2(\eta)dx^2 - a_2^2(\eta)dy^2, \quad C = a_1a_2 \quad (3.2.7)$$

In this model the mode function of the quantum field $\phi_q$ can be written in the separated form as $u_k = C^{-1/4}(2\pi)^{-1}\exp(i\kappa x)\chi_k(\eta)$. The wave equation Eq. (3.2.4) will then lead to

$$\ddot{\chi}_k + \left\{ C \left[ m_r^2 + (\xi_r - \frac{1}{8})R - 12\kappa_r\phi_c^2 + 15\lambda_c^2\phi_c^4 + \sum_i \frac{k_i^2}{a_i^2} \right] + Q \right\} \chi_k = 0 \quad (3.2.8)$$
where the spacetime curvature function \( R \) and the anisotropic function \( Q \) are

\[
R = 8C^{-1}(\dot{H} + H^2 + Q), \quad H = \sum h_i, \quad h_i = \frac{\dot{a}_i}{a_i}, \quad Q = \frac{1}{16} \sum_{i<j} (h_i - h_j)^2 \tag{3.2.9}
\]

When the metric is slowly varying, Eq. (3.2.8) possesses WKB approximation solution:

\[
\chi_k = (2W_k)^{-\frac{1}{2}} \exp(-i \int d\eta W_k) \tag{3.2.10}
\]

where,

\[
W_k = \left\{ C \left[ m_r^2 + (\xi_r - \frac{1}{8})R - 12k_r \phi_c^2 + 15\lambda_r^2 \phi_c^4 + \sum_i \frac{k_i^2}{a_i^4} + Q \right] \right\}^{\frac{1}{2}}
\]

Using the above solution we get:

\[
\langle \phi_q^2 \rangle = \frac{1}{8\pi^2 C(\eta)} \int d^2 k \left[ m_r^2 + (\xi_r - \frac{1}{8})R - 12k_r \phi_c^2 + 15\lambda_r^2 \phi_c^4 + \sum_i \frac{k_i^2}{a_i^4} + Q \right]^{-1/2}
\]

\[
= \frac{1}{16\pi} \left[ \Lambda + \frac{A_2}{2\Lambda} - A_2^{1/2} \right] \tag{3.2.11}
\]

where \( A_2 = (m_r^2 + (\xi_r - \frac{1}{8})R - 12k_r \phi_c^2 + 15\lambda_r^2 \phi_c^4 + \frac{Q}{C}) \).

And we get,

\[
\langle \phi_q^4 \rangle = \frac{3}{16\pi^4 C} \int d^2 k \chi_k(\eta) \chi_k^*(\eta) \int d^2 k' \chi_{k'}(\eta) \chi_{k'}^*(\eta)
\]

\[
= \frac{3}{16\pi^2} \left[ 2A_2 + \Lambda^2 - 2\Lambda A_2^{1/2} \left( 1 + \frac{A_2}{2\Lambda^2} \right) \right] \tag{3.2.12}
\]

where a momentum cut-off \( \Lambda \) is introduced to regularize the \( k \)-integration. From the renormalisation conditions given by Eq. (3.2.8) the renormalisation counter terms are evaluated and substituting the renormalisation counter terms, we obtain \( \frac{\partial V_{\text{eff}}}{\partial \phi_c} \) from Eq. (3.2.5) as,
Thus it is clear that we can obtain a finite expression for the one loop effective potential for the $\phi^6$ model in (2+1) dimensional Bianchi type-I spacetime. In the previous chapter it is shown that $\phi^6$ potential can be regularized in (3+1) dimensional curved spacetime. In this section a divergenceless expression for the $\phi^6$ one-loop effective potential in a (2+1) dimensional Bianchi type-I background spacetime is obtained.

\[
\frac{\partial V_{\text{eff}}}{\partial \phi_c} = (m_r^2 + \xi_r R)\phi_c + \frac{\lambda_r^2[(m_r^2 + \frac{Q}{C})^{1/2} - A_2^{1/2}]}{2B_2} \phi_c
\]

\[- \frac{D_2[(m_r^2 + \frac{Q}{C}) + A_2]}{20\pi B_2 E_2} \phi_c + \frac{1}{4B_2} \left[ \frac{\lambda_r^2}{(m_r^2 + \frac{Q}{C})^{1/2}} - \frac{D_2}{5\pi E_2} \right] (\xi_r - \frac{1}{8}) R \phi_c
\]

\[+ \frac{\pi}{3B_2} \left[ 2\lambda_r^2 - \frac{D_2 A_2^{1/2}}{5\pi E_2} \right] \phi_c^3 + \frac{2D_2}{75B_2 E_2} \phi_c^5 \] (3.2.13)

where,

\[A_2 = (m_r^2 + (\xi_r - \frac{1}{8}) R - 12\kappa_r \phi_c^2 + 15\lambda_r^2 \phi_c^4 + \frac{Q}{C}),\]

\[B_2 = \left[ \frac{-45\lambda_r^2}{(m_r^2 + \frac{Q}{C})^{1/2}} + \frac{108\kappa_r^2}{(m_r^2 + \frac{Q}{C})^{3/2}} \right],\]

\[D_2 = \left[ -\lambda_r^2 \pi + \frac{117\kappa_r \lambda_r^2}{(m_r^2 + \frac{Q}{C})^{1/2}} + \frac{270\kappa_r^3}{(m_r^2 + \frac{Q}{C})^{3/2}} \right] \quad \text{and} \quad E_2 = \left[ \frac{9\kappa_r}{\pi} - (m_r^2 + \frac{Q}{C})^{1/2} \right]\] (3.2.14)
3.3 Energy-Momentum Tensor for $\phi^6$ Field in (2+1) Dimensional Bianchi Type-I Spacetime

Considering a (2+1) dimensional Bianchi type-I background spacetime, the finite expression for the expectation value of the quantum energy-momentum tensor is obtained by adopting momentum cut-off regularization technique:

$$
\langle T_{\mu}^{\nu}\rangle^{Q} = \frac{C^2[3A_2 - \frac{Q}{C}]}{768\pi C^4 A_2^{3/2}} - \frac{1}{48\pi} A_2^{3/2} - \frac{C^2 A_2^{1/2}}{256\pi C^3} - \frac{C^2 (A_2 + \frac{Q}{C})}{128\pi C^3 A_2^{1/2}}
$$

$$
+ \frac{C^2}{16\pi C^3} \left\{ \xi_r + \frac{(\xi_r - \frac{1}{8})\lambda_r^2}{4B_2} - \frac{D_2 (\xi_r - \frac{1}{8})}{20\pi B_2 E_2} \right\} \frac{(A_2 + \frac{Q}{C})}{A_2^{1/2}}
$$

$$
+ \frac{C'}{16\pi C^2} \left[ \frac{C'}{C} - 12 \right] \left\{ \xi_r + \frac{(\xi_r - \frac{1}{8})\lambda_r^2}{4B_2} - \frac{D_2 (\xi_r - \frac{1}{8})}{20\pi B_2 E_2} \right\} A_2^{1/2}
$$

$$
+ \frac{A_2^{1/2}}{8\pi} \left\{ m_r^2 + \frac{\lambda_r^2 (m_r^2 + \frac{Q}{C})^{1/2}}{2B_2} - \frac{D_2 (m_r^2 + \frac{Q}{C})}{20\pi B_2 E_2} \right\}
$$

$$
- \frac{A_2^{1/2}}{2B_2} \left\{ \lambda_r^2 - \frac{3A_2^{1/2} D_2}{20\pi E_2} \right\} \phi_c^2 - \frac{A_2^{1/2} D}{20B_2 E_2} \phi_c^4
$$

(3.3.1)

where $A_2$, $B_2$, $D_2$ and $E_2$ are defined by Eq. (3.2.14). For this case also it is clear that the energy-momentum tensor depends on the anisotropy of the spacetime.

A knowledge of $T_{\mu\nu}$ is important for two reasons. It can be used to assess the importance of quantum effects on the dynamics of the gravitational field itself,
that is the back-reaction problem. Also, it is frequently a more useful probe of the physical situation than a particle count. In regions of strong gravity, vacuum polarisation effects, akin to those in QED can lead to important phenomena even in the absence of actual particle creation.

3.4 Finite Temperature Behaviour

To evaluate the finite temperature effective potential, the vacuum expectation value is replaced by the thermal average \( \langle \phi \rangle_T = \sigma_T \). Considering the same Lagrangian density as above, the zero loop effective potential is temperature independent,

\[
V_0(\sigma) = \frac{1}{2} \xi R \sigma^2 + \frac{1}{2} \lambda^2 \sigma^2 (\sigma^2 - m/\lambda)^2
\]  

(3.4.1)

The one loop approximation to finite temperature effective potential \([65]-[68]\) is given by,

\[
V_1^\beta(\sigma) = \frac{1}{2\beta} \sum_n \int \frac{d^2 k}{(2\pi)^2} \ln(k^2 - M^2)
\]

\[
= \frac{1}{2\beta} \sum_n \int \frac{d^2 k}{(2\pi)^2} \ln \left( \frac{-4\pi^2 n^2}{\beta^2} - E_M^2 \right)
\]  

(3.4.2)

where,

\[
\beta = \frac{1}{T}, \quad E_M^2 = k^2 + M^2, \quad M^2 = m^2 + \xi R - 12\lambda m \sigma^2 + 15\lambda^2 \sigma^4
\]  

(3.4.3)

Proceeding as in the previous chapter, in the high temperature limit \([8]\) it is obtained that

\[
V_1^\beta(\sigma) = \frac{1}{4\pi \beta^3} \xi(3) - \frac{M^2}{8\pi \beta} \ln(M\beta)
\]  

(3.4.4)

where \( \xi(z) \) is the Riemannian Zeta function \([87]\), \( \xi(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \ |Re\ Z > 1| \).
In the (2+1) dimensional case also, the symmetry breaking present in this $\phi^6$ model can be removed if the temperature is raised above a certain value called the critical temperature. The expression for critical temperature in this case is obtained as

$$T_c = (m^2 + \xi R) \exp \left( \frac{2(m^2 + \xi R)^2}{3T_c \lambda m} \right)$$

(3.4.5)

The order parameter of the theory is temperature dependent. The temperature dependence of finite temperature effective potential leads to phase transitions.

### 3.5 First Order Phase Transitions

On shifting the field from $\phi$ to $\phi + \sigma$ in the equation (3.2.2) and taking the Gibbs average of the corresponding equation we get:

$$\Box \sigma_T + (m^2 + \xi R)\sigma_T - 4\kappa \sigma_T^3 - 12\kappa \sigma_T^2 \langle \phi^2 \rangle - 12\kappa \sigma_T^2 \phi > + 15\lambda^2 \sigma_T \langle \phi^4 \rangle$$

$$+ 30\lambda^2 \sigma_T^2 \langle \phi^3 \rangle + 30\lambda^2 \sigma_T^3 \langle \phi^2 \rangle + 15\lambda^2 \sigma_T^4 \langle \phi \rangle + 3\lambda^2 \sigma_T^5 = 0$$

(3.5.1)

Using the standard finite temperature Green’s-function methods we can find that in the high temperature limit,

$$\langle \phi^2 \rangle = \frac{1}{4\pi} \int \frac{dk}{(\text{Exp}(k/T) - 1)} \text{, } \langle \phi^4 \rangle = 3\langle \phi^2 \rangle^2 \text{ and } \langle \phi^3 \rangle = \langle \phi \rangle = 0. \text{ Thus equation (3.5.1) becomes}$$

$$\Box \sigma_T + (m^2 + \xi R)\sigma_T - 4\kappa \sigma_T^3 - 12\kappa \sigma_T^2 \langle \phi^2 \rangle + 15\lambda^2 \sigma_T \langle \phi^4 \rangle$$

$$+ 30\lambda^2 \sigma_T^2 \langle \phi^3 \rangle + 3\lambda^2 \sigma_T^5 = 0$$

(3.5.2)
Assuming that $\sigma_T$ is a constant we obtain,

$$
\sigma_T \left[ (m^2 + \xi R) - 4\kappa\sigma_T^2 - 12\kappa \langle \phi^2 \rangle + 15\lambda^2 \langle \phi^4 \rangle + 30\lambda^2\sigma_T^2 \langle \phi^2 \rangle + 3\lambda^2\sigma_T^4 \right] = 0
$$

(3.5.3)

This equation has degenerate solutions: $\sigma_T = 0$, and

$$
\sigma_T^2 = \frac{\left\{ (4m - 30\lambda \langle \phi^2 \rangle) \pm \sqrt{(4m^2 - 12\xi R - 96\lambda m \langle \phi^2 \rangle + 900\lambda^2(\langle \phi^2 \rangle)^2 + 180 \langle \phi^4 \rangle)\frac{1}{2}} \right\}}{6\lambda}
$$

(3.5.4)

Each of these solutions defines a possible phase of the field system with its characteristic excitations. On heating the field system from absolute zero, the two branches of $\sigma_T^2$ given by the above equation coincide at a temperature for which

$$
(4m^2 - 12\xi R - 96\lambda m \langle \phi^2 \rangle + 900\lambda^2(\langle \phi^2 \rangle)^2 + 180 \langle \phi^4 \rangle) = 0
$$

(3.5.5)

yielding a common value of $\sigma_T$. The existence of the separate branches of $\sigma_T^2$ implies that the phase transition is of first order [10, 88]. Numerical results obtained using equation (3.5.4) clearly shows that the order parameter does not vanish even for very high values of the temperature. Fig. 3.1 gives the variation of the two branches of $\sigma_T^2$ with respect to temperature. It is found that the two branches coincides at a particular value of $T$ given by equation (3.5.5). From the figure it is clear that there is a discontinuity for the variation of the order parameter with temperature, indicating a first order phase transition.

The temperature dependence of $V_{eff}$ obtained for the present $\phi^6$ model in the (2+1) dimensional background spacetime is shown in Fig. 3.2. It is found that for $T \gg T_c$ the effective potential attains a minimum at $\sigma = 0$, which corresponds to the completely symmetric case. When the temperature decreases, a global
Fig. 3.1: Variation of the two branches of $\sigma^2_T$ with respect to temperature. The two curves coincide after the temperature which satisfies equation (3.5.5), where $m = 3.9371$, $\lambda = 0.8$, $R = 0.9$ and $\xi = 0.2$

Fig. 3.2: The behaviour of finite temperature effective potential as a function of $\sigma$ for fixed $m = 0.9371$, $\lambda = 0.008$. Starting from the top the curves corresponds to the following values of the parameters: (i) $R = 3.3$, $\xi = 2.54$, $T = 25$ such that $T \gg T_c$, (ii) $R = 1.93$, $\xi = 0.198$ and $T = 18.5$ such that $T > T_c$, (iii) $R = 0.42$, $\xi = 0.02$ and $T = 9$ such that $T = T_c$, (iv) $R = 0.35$, $\xi = -0.3$ and $T = 5$ such that $T < T_c$
minimum appears at $\sigma = 0$ and two local minima at $\sigma \neq 0$, which shows the existence of a barrier between the global and local minima. At $T = T_c$, all the minima are degenerate, which implies that the symmetry is broken. For $T < T_c$ the minima at $\sigma \neq 0$ become global minima. If for $T \leq T_c$ the extremum at $\sigma = 0$ remains a local minimum, there must be a barrier between the minimum at $\sigma = 0$ and at $\sigma \neq 0$. Therefore the change in $\sigma$ in going from one phase to the other must be discontinuous, indicating a first order phase transition [10], [70, 88].

3.6 Dependence on Curvature $R$ and Scalar-Gravitational Coupling $\xi$

![Graph showing the behavior of finite temperature effective potential](image)

Fig. 3.3: The behaviour of finite temperature effective potential as a function of $\sigma$ for fixed $m = 0.9371$, $\lambda = 0.009$, $\xi = 0.1$ and $T = 4$. Starting from top the curves corresponds to the following values of the curvature: $R=20, 3, 0.99, 0.02, -0.72$.

Fig. 3.3 clearly shows the crucial role of scalar curvature $R$ in determining the fate of symmetry and the phase transitions for the present model. From the figure
it is clear that the first order phase transition takes place as $R$ changes. It is found that for $R = 0$ or $\xi = 0$ the system remains in the symmetry broken state for all values of $T \leq T_c$. As the temperature is increased above $T_c$, the symmetry is restored depending on the values of $R$ and $\xi$ also. It is also found that symmetry can be restored either by increasing the value of $R$ or by increasing the value of $\xi$ keeping the temperature constant, even below the critical temperature. It is clear from Fig. 3.4 that there is a barrier between the symmetric and broken phases. Thus the phase transition, induced by the coupling constant $\xi$ is also of first order. This shows that the scalar-gravitational coupling and the scalar curvature do play a crucial role in determining the nature of phase transitions.

![Graph](image)

Fig. 3.4: The behaviour of finite temperature effective potential as a function of $\sigma$ for fixed $m = 0.9371$, $\lambda = 0.009$, $R = 0.2$ and $T = 5$. Starting from top the curves corresponds to the following values of the curvature: $\xi = 9, 2, 0.85, 0.025, -0.35, -0.8$.

### 3.7 Bubble Nucleation and Expansion

A first order phase transition proceeds by nucleation of bubbles of broken phase in the background of unbroken phase [8]. Decay of metastable vacuum state with
\( \phi = 0 \) proceeds via quantum tunnelling [10] with the nucleation of bubbles of the asymmetric phase. The bubbles expand and eventually collide, while new bubbles are continuously formed, until the phase transition is completed.

Consider a massive self interacting complex field \( \Phi \) coupled arbitrarily to the gravitational back ground, with the \( \Phi^4 \) potential

\[
V(\Phi) = \frac{1}{2} \xi R |\Phi|^2 + \frac{1}{2} \lambda^2 |\Phi|^4 \left( |\Phi|^2 - m/\lambda \right)^2
\]

(3.7.1)

with a minimum at \( |\Phi| = 0 \) and a set of minima at \( |\Phi| = \left[ \frac{m \pm \sqrt{\xi R}}{\lambda} \right]^{1/2} \), connected by \( U(1) \) transformation. When the temperature is below \( T_c \), a false vacuum is found at \( \Phi = 0 \) and true vacuum at \( \Phi = \Phi_0 \neq 0 \). As the temperature increases, the false vacuum will decay to the true vacuum state via bubble nucleation.

During a first order phase transition as in Fig. 3.2, at a temperature \( T < T_c \), where \( |V(\Phi)| \) at a minimum with \( \Phi = \Phi_0 \neq 0 \) is much lower than the barrier height in \( V(\Phi) \) between \( \Phi = 0 \) and \( \Phi = \Phi_0 \), the thin wall approximation [8] is valid. The equation of motion for this system is

\[
\partial_\mu \partial^\mu \Phi = -\frac{\partial V}{\partial \Phi}
\]

(3.7.2)

Let us consider the minimally coupled case \( \xi = 0 \). For the thin wall regime, the approximate solution of Eq. (3.7.2) is obtained as

\[
|\Phi| = \left\{ \frac{m}{2\lambda} \left( \tanh \left[ m(\chi - R_0) \right] + 1 \right) \right\}^{1/2}
\]

(3.7.3)

where \( R_0 \) is the bubble radius at nucleation time and \( \chi^2 = |\vec{x}|^2 - t^2 \). The kink like shape of this solution, obtained numerically is shown in Fig. 3.5.

While discussing the bubble collisions [89]-[91], one has to consider the interaction between the bubble field and the surrounding plasma. As the bubble wall sweeps through a specific point, the Higgs field \( \phi \) acquires an expectation
value and the field coupled to it acquires mass. Thus particles without enough energy to acquire the corresponding mass inside the bubble will bounce-off the wall (thus imparting negative momentum to it), while the rest will get through. Obviously, the faster the wall propagates the stronger this effect will be, since the momentum transfer in each collision will be larger and thus a force proportional to the velocity with which the wall sweeps through the plasma appears. Thus considering the damping effect of the surrounding plasma on the motion of the walls we insert a frictional term in the equation of motion,

\[ \partial_\mu \partial^\mu \Phi + \gamma |\Phi| e^{i\theta} = -\frac{\partial V}{\partial \Phi} \]  

(3.7.4)

where \(|\Phi| = \frac{\partial |\Phi|}{\partial t}\), \(\theta\) is the phase of the field and \(\gamma\) stands for the friction coefficient.

To find the solution of Eq. (3.7.4) in the thin wall limit, first we suppose that solution for which the wall has the form of a travelling wave do exist [90]. Writing \(\Phi\) in polar form \(\Phi = \rho e^{i\theta}\), we can rewrite Eq. (3.7.4) and for a single
bubble configuration we take the phase of the bubble $\theta$ to be constant. Then the
equation for the modulus of the field is

\[ \partial_\mu \partial^\mu \rho + \gamma \dot{\rho} = -\frac{\partial V(\rho)}{\partial \rho} \]  

(3.7.5)

Because the wall thickness is much smaller than the radius of the bubble, we
can go to $(1+1)$ dimensions to get an approximate expression for the terminal
velocity of the bubble under this equation in the thin wall limit. Inserting the
ansatz $\rho = \rho(x - x_0(t))$ leads to

\[ (1 - \ddot{x}_0^2)\rho'' + (\ddot{x}_0 + \gamma \dot{x}_0)\rho' = \frac{\partial V(\rho)}{\partial \rho} \]  

(3.7.6)

where $\rho' = \frac{\partial \rho}{\partial x}$. Multiplying by $\rho'$ and integrating over $-\infty \leq x \leq +\infty$ we get,

\[ (\ddot{x}_0 + \gamma \dot{x}_0) \left( \int_{-\infty}^{+\infty} \rho'^2 dx \right) = \int_{-\infty}^{+\infty} V'(x) dx = \Delta V \]  

(3.7.7)

where $\Delta V$ is the potential energy difference between the false and the true vacuum
phases. For the initial conditions $x_0(t = 0) = R_0$, $\dot{x}_0(t = 0) = 0$, the solution of
Eq. (3.7.7) is

\[ x_0(t = 0) = \frac{1}{\gamma} \alpha t + \frac{\alpha}{\gamma^2} (e^{-\gamma t} - 1) + R_0 \]  

(3.7.8)

where $\alpha \equiv \Delta V / \left( \int \rho'^2 dx \right)$. Thus for values of $t \gg \gamma^{-1}$ the bubble walls will have
reached their terminal velocity

\[ v_{\text{ter}} = \frac{\Delta V}{\gamma \left( \int \rho'^2 dx \right)} \]  

(3.7.9)

To get an approximate expression for $\rho$ valid within this regime, it suffices to
rewrite Eq. (3.7.5) with an ansatz $\rho = \rho(r - r_0(t))$, where $r$ is the usual radial
coordinate. Using $\ddot{r}_0 = 0$, $\dot{r}_0 \approx v_{\text{ter}}$, we get,

\[ (1 - v_{\text{ter}}^2) \frac{\partial^2 \rho}{\partial r^2} + \left( \frac{2}{r} + \gamma v_{\text{ter}} \right) \frac{\partial \rho}{\partial r} = \frac{\partial V(\rho)}{\partial \rho} \]  

(3.7.10)
According to Eq. (3.7.9) the terminal velocity roughly goes like

$$v_{\text{ter}} = \frac{\Delta V}{\gamma(\rho_{\text{tv}}^2 \delta_m)} = \frac{\Delta V \delta_m}{\gamma \rho_{\text{tv}}^2}$$

(3.7.11)

where $\delta_m$ is the bubble wall thickness and $\rho_{\text{tv}}$ is the true vacuum value of the field. At the values of $r$ for which the first derivative of the field is important ($r \sim R$ for thin wall bubble), we have $R \gamma v_{\text{ter}} \sim \delta_m << 1$, and the second term in the second parenthesis of Eq. (3.7.10) is negligible when compared to the first. Again, since the radius of thin walled bubble is very large, we can also neglect the term $(\frac{2}{r}) \frac{\partial \rho}{\partial r}$ in the standard thin wall approximation. Thus we get,

$$(1 - v_{\text{ter}}^2) \frac{\partial^2 \rho}{\partial r^2} = \frac{\partial V(\rho)}{\partial \rho}$$

(3.7.12)

where $r$ is the radial coordinate and $v_{\text{ter}}$ is the terminal velocity of the bubble walls. For the present $\phi^6$ potential the solution for Eq. (3.7.12) is obtained as

$$\rho = \left\{ \frac{m}{2\lambda} \left( \tanh \left[ \frac{m(r - v_{\text{ter}}t - R_0)}{\sqrt{1 - v_{\text{ter}}^2}} \right] + 1 \right) \right\}^{1/2}$$

(3.7.13)

which is simply a Lorentz-contracted moving domain wall. Thus it is clear that there exists an exact solution for the damped motion of the bubble in the thin wall regime. Fig. 3.6 gives the kink like shape of this solution obtained numerically.

The bubbles of the new phase nucleated within the old one subsequently expand and collide with each other. This would take place via Kibble mechanism [92]. In order to justify Kibble mechanism, one must follow the evolution of amplitude and phase of $\Phi$ [93]-[95]. The regions of the old phase trapped within the new one give birth to topological defects [10].
3.8 Discussion and Conclusions

In this chapter a divergenceless expression for the $\phi^6$ one-loop effective potential and energy-momentum tensor in a (2+1) dimensional Bianchi type-I background spacetime is obtained. The temperature dependence of phase transitions for the $\phi^6$ model is closely examined and the nature of phase transitions for the present model is verified to be of first order.

In the present work, considering a $\phi^6$ potential we have proved that the gravitational effects are of particular interest in a (2+1) dimensional Bianchi type-I spacetime. The phase transition taking place in a (2+1) dimensional Bianchi type-I background spacetime, during such a SSB is of first order.

As the bubbles of the low temperature phase expand, they expel heat into their surroundings, heating the high temperature phase up to $T_c$. At this point the pressure of the high temperature phase prevents further expansion of the low temperature phase. After all, $T_c$ is the temperature at which the two phases have
equal pressures and can coexist.

To get the false vacuum and to study the bubble nucleation Ferrera et al. [90, 91] has introduced a $\phi^3$ term in the $\phi^4$ potential. But in the present $\phi^6$ model, false vacuum at $\phi = 0$ and true vacuum at $\phi = \phi_0 \neq 0$ occur naturally at temperatures $T < T_c$. Considering the interaction between the bubble field and plasma an exact solution for the damped motion of the bubble in the thin wall regime is obtained for the present model. Whether or not the universe recovers from a first order phase transition and any relics are left behind depends upon the nucleation, expansion and collision of bubble and on the process of eventual transition to the new phase.