Chapter 1

On Weakly Symmetric Riemannian Manifolds

1.1 Introduction

The study of Riemann symmetric manifolds began with the work of E. Cartan [24]. A Riemannian manifold \((M^n, g)\) is said to be locally symmetric due to Cartan if its curvature tensor \(R\) satisfies the relation \(\nabla R = 0\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the metric tensor \(g\). This condition of local symmetry is equivalent to the fact that at every point \(P \in M\), the local geodesic symmetry \(F(P)\) is an isometry [124]. The class of Riemann symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as recurrent manifold by A. G. Walker [169], semisymmetric manifold by Z. I. Szabó [157], pseudosymmetric manifold by M. C. Chaki [25], generalized pseudosymmetric manifold

\[^{1}\text{From now on the equation numbers are in the form (C.S.E), where C stands for chapter, S for section number and E stands for equation number. Thus (1.3.4) means equation number 4 of the third section of the first chapter and so on.}\]
by M. C. Chaki [28], weakly symmetric manifolds by A. Selberg [145] and weakly symmetric manifolds by L. Támassy and T. Q. Binh [161]. It may be noted that the notion of weakly symmetric Riemannian manifolds by Selberg [145] is different to that of Támassy and Binh [161] and throughout our study we confined ourselves with the weak symmetries of Riemannian manifolds [161].

This chapter deals with a study of weakly symmetric Riemannian manifolds. In 1989 L. Tamássy and T. Q. Binh [161] introduced the notions of weakly symmetric and weakly projective symmetric manifolds. A non-flat Riemannian manifold \((M^n, g)(n > 2)\) is called a weakly symmetric manifold if the curvature tensor \(R\) of type \((0, 4)\) satisfies the condition

\[
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\), where \(A, B, H, D\) and \(E\) are 1-forms (not simultaneously zero) and \(\nabla\) is the operator of covariant differentiation with respect to the Riemannian metric \(g\). The 1-forms are called the associated 1-forms of the manifold and an \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\). Then U. C. De and S. Bandyopadhyay [49] gave an example of a \((WS)_n\) by a suitable metric [140] and proved that in a \((WS)_n\), the associated 1-forms \(B = H\) and \(D = E\). Hence the defining condition of a \((WS)_n\) reduces to the following form:

\[
\]
In section 1.2 we derive some fundamental results of a \((WS)_n\). Section 1.3 deals with the conformally flat weakly symmetric Riemannian manifolds. The last section provides several non-trivial examples of a \((WS)_n\) \((n \geq 4)\).

### 1.2 Fundamental Results of a \((WS)_n\)

Let \(\{e_i : i = 1, ..., n\}\) be an orthonormal basis of the tangent space at any point of the manifold. Then setting \(Y = V = e_i\) in (1.1.2) and taking summation over \(i, 1 \leq i \leq n\) we get

\[
(\nabla_X S)(Z, U) = A(X)S(Z, U) + B(Z)S(X, U) + D(U)S(X, Z)
+ B(R(X, Z)U) + D(R(X, U)Z),
\]  

(1.2.1)

where \(S\) is the Ricci tensor of type \((0, 2)\). From (1.2.1), it follows that a \((WS)_n\) \((n > 2)\) is weakly Ricci symmetric (briefly \((WRS)_n\) \((n > 2)\)) [162] if

\[
B(R(X, Z)U) + D(R(X, U)Z) = 0
\]

(1.2.2)

for all \(X, U, Z\). This leads to the following:

**Theorem 1.2.1.** A \((WS)_n\) \((n > 2)\) satisfying the condition (1.2.2) is a \((WRS)_n\).

Again from (1.2.1) it follows that

\[
dr(X) = rA(X) + 2B(QX) + 2D(QX),
\]

(1.2.3)

where \(r\) is the scalar curvature of the manifold, \(Q\) is the Ricci-operator, i.e., \(g(QX, Y) = S(X, Y)\). From (1.2.3), we can state the following:
Theorem 1.2.2. If a (WS)\(_n\) \((n > 2)\) is of non-zero constant scalar curvature, then the 1-form \(A\) can be expressed as

\[
A(X) = -\frac{2}{r}[B(QX) + D(QX)]
\]

(1.2.4)

for all \(X\).

Further if \(r = 0\), then from (1.2.3) we obtain

\[
B(QX) + D(QX) = 0.
\]

Hence we have the following:

Corollary 1.2.3. If a (WS)\(_n\) \((n > 2)\) is of zero scalar curvature then the relation \(B(QX) + D(QX) = 0\) holds for all \(X\).

Interchanging \(Z\) and \(U\) in (1.2.1), and then subtracting the resultant from (1.2.1), we obtain by virtue of the Bianchi identity

\[
[B(Z) - D(Z)]S(X, U) - [B(U) - D(U)]S(X, Z)
\]

\[
- [B(R(Z,U)X) - D(R(Z,U)X)] = 0.
\]

(1.2.5)

Replacing \(X\) and \(U\) by \(e_i\), and taking summation over \(i, 1 \leq i \leq n\), we get

\[
r[B(Z) - D(Z)] = 2[B(QZ) - D(QZ)].
\]

(1.2.6)

We define the vector field \(\rho\) by \(T(X) = g(X, \rho) = B(X) - D(X)\) for all \(X\). Then (1.2.6) yields

\[
T(QX) = \frac{r}{2}T(X).
\]

(1.2.7)

Hence we can state the following:
Theorem 1.2.4. In a \((WS)_n(n > 2)\), \(\frac{\pi}{2}\) is an eigenvalue of the Ricci tensor \(S\) corresponding to the eigenvector \(\rho\) defined by \(T(X) = g(X, \rho) = B(X) - D(X) \neq 0\) for all \(X\).

Again from (1.2.5) we can state the following:

Theorem 1.2.5. In a \((WS)_n(n > 2)\) the relation

\[
T(Z)S(X, U) - T(U)S(X, Z) = 0.
\]

holds for all vector fields \(X, Z, U,\) and \(T\) is a 1-form defined by \(T(X) = B(X) - D(X) \neq 0\) for all \(X\).

1.3 Conformally Flat \((WS)_n\)

Let \((M^n, g) (n \geq 3)\) be a conformally flat \((WS)_n\). It is known that in a conformally flat Riemannian manifold \((M^n, g) (n \geq 3)\) the following relation holds:

\[
(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = \frac{1}{2(n-1)} [g(Y, Z)d\tau(X) - g(X, Y)d\tau(Z)].
\]

Interchanging \(X\) and \(U\) in (1.2.1), and then subtracting the resultant from (1.2.1), we obtain by virtue of (1.3.1) that

\[
[A(X) - D(X)]S(U, Z) - [A(U) - D(U)]S(X, Z) + B(R(X, U)Z) + 2D(R(X, U)Z) = \frac{1}{2(n-1)} [g(Z, U)d\tau(X) - g(X, Z)d\tau(U)].
\]

Let \(\rho_1, \rho_2, \rho_3\) be the associated vector fields corresponding to the 1-forms \(A, B, D\) respectively, i.e.,

\[
g(X, \rho_1) = A(X), \ g(X, \rho_2) = B(X) \text{ and } D(X) = g(X, \rho_3) \text{ for all } X.
\]
Substituting \( U \) by \( \rho_2 \) in (1.3.2), and then using (1.2.3), we get

\[
\begin{align*}
[A(X) - D(X)]B(QZ) &- [A(\rho_2) - D(\rho_2)]S(X, Z) \\
+R(X, \rho_2, Z, \rho_2) + 2R(X, \rho_2, Z, \rho_3) \\
= &B(Z)[rA(X) + 2B(QX) + 2D(QX)] \\
- &g(X, Z)[rA(\rho_2) + 2B(Q(\rho_2)) + 2D(Q(\rho_2))].
\end{align*}
\] (1.3.3)

If the manifold has non-zero constant scalar curvature, then (1.3.3) yields by virtue of (1.2.4) that

\[
\begin{align*}
[A(\rho_2) - D(\rho_2)]S(X, Z) &- [A(X) - D(X)]B(QZ) \\
+R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) = &0.
\end{align*}
\] (1.3.4)

Again, since the manifold under consideration is conformally flat, we have

\[
\begin{align*}
R(X, Y, Z, W) \\
= &\frac{1}{n-2}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\
+ &S(X, W)g(Y, Z) - S(Y, W)g(X, Z)] \\
+ &\frac{r}{(n-1)(n-2)}[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)].
\end{align*}
\] (1.3.5)

From (1.3.5), it follows that

\[
\begin{align*}
R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) \\
= &\frac{1}{n-2}[S(X, Z){B(\rho_2)} + 2B(\rho_3)] - B(QZ){B(X) + 2D(X)} \\
+ &g(X, Z){B(Q(\rho_2)) + 2B(Q(\rho_3))} - B(Z){B(QX) + 2D(QX)} \\
+ &\frac{r}{(n-1)(n-2)}[B(Z){B(X) + 2D(X)} \\
- g(X, Z){B(\rho_2) + 2B(\rho_3)}].
\end{align*}
\] (1.3.6)
Using (1.3.6) in (1.3.4) we obtain

\[ S(X, Z) = \alpha g(X, Z) + \alpha_1 B(X)B(Z) + \alpha_2 B(Z)D(X) - \alpha_3 B(X)\tilde{B}(Z) + \alpha_4 B(Z)\tilde{B}(X) + \alpha_5 B(Z)\tilde{D}(X) + \alpha_6 \tilde{B}(Z)D(X) + \alpha_7 A(X)\tilde{B}(Z), \] (1.3.7)

where \( \alpha, \alpha_1, \ldots, \alpha_7 \) are scalars in terms of \( r, B(\rho_2) \) and \( B(\rho_3) \), and \( \tilde{B}(X) = B(QX), \tilde{D}(X) = D(QX) \) for all \( X \). This leads to the following:

**Theorem 1.3.1.** In a conformally fiat \((WS)_n (n \geq 3)\) of non-zero constant scalar curvature, the Ricci tensor \( S \) has the form (1.3.7).

Again, using (1.2.4) in (1.3.7), we have

\[ S(X, Z) = \alpha g(X, Z) + \alpha_1 B(X)B(Z) + \alpha_2 B(Z)D(X) + \alpha_3 B(X)\tilde{B}(Z) + \alpha_4 B(Z)\tilde{B}(X) + \alpha_5 B(Z)\tilde{D}(X) + \alpha_6 \tilde{B}(Z)D(X) - \frac{2}{r} \alpha_7 |\tilde{B}(X) + \tilde{D}(X)|\tilde{B}(Z). \] (1.3.8)

According to Chen and Yano [42], a Riemannian manifold \((M^n, g) (n > 3)\) is said to be of quasi-constant curvature if it is conformally flat, and its curvature tensor \( R \) of type \((0, 4)\) has the form

\[ R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W) + g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)], \] (1.3.9)

where \( A \) is a 1-form and \( a, b \) are scalars of which \( b \neq 0 \).

Generalizing this notion we define the manifold of hyper quasi-constant curvature as follows:
**Definition 1.3.1.** A Riemannian manifold $(M^n, g)$ $(n > 3)$ is said to be of *hyper quasi-constant curvature* if it is conformally flat, and its curvature tensor $R$ of type $(0, 4)$ satisfies the condition

$$R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ g(X, W)P(Y, Z) - g(X, Z)P(Y, W)$$

$$+ g(Y, Z)P(X, W) - g(Y, W)P(X, Z), \quad \text{(1.3.10)}$$

where

$$P(Y, Z) = (\beta BD)(Y, Z) = \beta_1 B(Z)D(Y) + \beta_2 B(Z)B(Y) + \beta_3 \bar{B}(Z)B(Y)$$

$$+ \beta_4 B(Z)\bar{B}(Y) + \beta_5 B(Z)\bar{D}(Y) + \beta_6 \bar{B}(Z)D(Y)$$

$$+ \beta_7 \bar{B}(Z)\bar{B}(Y) + \beta_8 \bar{B}(Z)\bar{D}(Y),$$

and $\beta_1, \beta_2, ..., \beta_8$ are non-zero scalars.

Now in view of (1.3.8) we obtain from (1.3.5) that

$$R(X, Y, Z, W) = a_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ g(X, W)P'(Y, Z) - g(X, Z)P'(Y, W)$$

$$+ g(Y, Z)P'(X, W) - g(Y, W)P'(X, Z), \quad \text{(1.3.11)}$$

where

$$P'(Y, Z) = (\beta' BD)(Y, Z) = \beta'_1 B(Z)D(Y) + \beta'_2 B(Z)B(Y) + \beta'_3 \bar{B}(Z)B(Y)$$

$$+ \beta'_4 B(Z)\bar{B}(Y) + \beta'_5 B(Z)\bar{D}(Y) + \beta'_6 \bar{B}(Z)D(Y)$$

$$+ \beta'_7 \bar{B}(Z)\bar{B}(Y) + \beta'_8 \bar{B}(Z)\bar{D}(Y),$$

and $\beta'_1, \beta'_2, ..., \beta'_8$ are non-zero scalars.
and $a_1, \beta_1, \beta_2, \ldots, \beta_8$ are non-zero scalars.

Comparing (1.3.10) and (1.3.11), it follows that the manifold is of hyper quasi-constant curvature. This leads to the following:

**Theorem 1.3.2.** A conformally flat $(WS)_n$ ($n > 3$) of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

Now putting $U = \rho$ in (1.2.8), and then using (1.2.7), we get

$$\frac{r}{2} T(X)T(Z) - T(\rho)S(X, Z) + R(\rho, Z, X, \rho) = 0. \quad (1.3.12)$$

Let us now suppose that a $(WS)_n$ ($n > 3$) is conformally flat, and of non-zero scalar curvature. Then (1.3.5) yields

$$R(\rho, Z, X, \rho) = \frac{1}{n-2} \left[ T(\rho)S(X, Z) - rT(X)T(Z) + \frac{r}{2} T(\rho)g(X, Z) \right]$$

$$+ \frac{r}{(n-1)(n-2)} [T(Z)T(X) - T(\rho)g(X, Z)]. \quad (1.3.13)$$

Using (1.3.13) in (1.3.12), it follows that

$$2(n-1)T(\rho)S(X, Z) = rT(\rho)g(X, Z) + r(n-2)T(X)T(Z). \quad (1.3.14)$$

We shall now show that $T(\rho) \neq 0$. For if $T(\rho) = 0$ then (1.3.14) implies that

$$r(n-2)T(X)T(Z) = 0.$$

Since $T(X) \neq 0$ for all $X$, and $n > 3$, the above relation yields $r = 0$, a contradiction to the assumption that the manifold is of non-zero scalar curvature. Thus we have $T(\rho) \neq 0$. Consequently (1.3.14) yields

$$S(X, Z) = \alpha g(X, Z) + \beta T(X)T(Z), \quad (1.3.15)$$
where $\alpha, \beta$ are non-zero scalars.

Again, according to Chaki and Maity [38], a Riemannian manifold is said to be quasi-Einstein if its Ricci tensor is of the form

$$S = pg + q\omega \otimes \omega,$$

where $p, q$ are scalars of which $q \neq 0$ and $\omega$ is a 1-form. This leads to the following:

**Theorem 1.3.3.** A conformally flat $(WS)_n$ $(n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0$ for all $X$.

Again, using (1.3.15) in (1.3.5), it follows that

$$R(X, Y, Z, W) = \gamma[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ \delta[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)]$$

$$+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z),$$

(1.3.16)

where $\gamma$ and $\delta$ are non-zero scalars. Comparing (1.3.16) and (1.3.9), we can state the following:

**Theorem 1.3.4.** A conformally flat $(WS)_n$ $(n > 3)$ of non-vanishing scalar curvature is a manifold of quasi-constant curvature with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0$ for all $X$.

Using the expression of $T$ in (1.3.16), it can be easily seen that

$$R(X, Y, Z, W) = \gamma[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ g(X, W)\{\delta BD\} (Y, Z) - g(X, Z)\{\delta BD\} (Y, W)$$

$$+ g(Y, Z)\{\delta BD\} (X, W) - g(Y, W)\{\delta BD\} (X, Z),$$
where \( \delta BD = \delta(BB - BD - DB + DD) \).

Comparing the above relation with (1.3.10), we can state the following:

**Theorem 1.3.5.** A conformally flat \((WS)_n\) \((n > 3)\) of non-zero scalar curvature is a manifold of hyper quasi-constant curvature.

### 1.4 Some Examples of \((WS)_n\)

This section deals with several examples of \((WS)_n\). We calculate the components of the curvature tensor and its covariant derivative and then we verify the defining relation (1.1.2). In the first two examples we have considered the metrics which are the special forms of the metrics due to W. Roter [140].

**Example 1.4.1.** Let \( M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4\} \) be an open subset of \( \mathbb{R}^4 \) endowed with the metric

\[
ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2,
\]

(\(i, j = 1, 2, 3, 4\)), where \( f = a_0 + a_1x^3 + a_2(x^3)^2\), \(a_0, a_1, a_2\) are non-constant functions of \(x^1\) only. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

\[
\Gamma^2_{11} = \frac{1}{2}f_1, \quad \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_3, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^2_{44} = -x^1,
\]

\[
R_{1331} = \frac{1}{2}f_{33} = a_2 \neq 0,
\]

and the components which can be obtained from these by the symmetry properties. Here ' denotes the partial differentiation with respect to the coordinates. Using the above relations, it can be easily shown that the scalar curvature of the manifold is
zero. Therefore \( \mathbb{R}^4 \) with the considered metric is a Riemannian manifold \( M^4 \) whose scalar curvature is zero. The only non-zero covariant derivatives of \( R \) are

\[
R_{1331,1} = \frac{1}{2} f_{331} = (a_2)_{1} \neq 0
\]

and the components which can be obtained from (1.4.3) by the symmetry properties, where ‘,’ denotes the covariant derivative with respect to the metric tensor. Hence our \((M^4, g)\) is neither flat nor locally symmetric. We shall now show that this \( M^4 \) is a \((WS)_4\), i.e., it satisfies (1.1.2). Let us now consider the 1-forms

\[
A_i(\partial_i) = A_i = \begin{cases} 
-d(x^2 x^3) & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_i(\partial_i) = B_i = \begin{cases} 
d(x^2 x^3) & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

\[
D_i(\partial_i) = D_i = \begin{cases} 
d(\log a_2) & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

where \( \partial_i = \frac{\partial}{\partial x_i} \) at any point \( x \in M \). In our \( M^4 \), (1.1.2) reduces with these 1-forms to the following equations

\[
(i) \quad R_{1331,1} = A_1 R_{1331} + B_1 R_{1331} + B_3 R_{1131} + D_3 R_{1311} + D_1 R_{1331}
\]

\[
(ii) \quad R_{1131,3} = A_3 R_{1131} + B_1 R_{3131} + B_1 R_{1331} + D_3 R_{1131} + D_1 R_{1133}
\]

\[
(iii) \quad R_{1311,3} = A_3 R_{1311} + B_1 R_{3311} + B_3 R_{1311} + D_1 R_{1331} + D_1 R_{1313},
\]

since for the cases other than (i), (ii) and (iii) the components of each term of (1.1.2) vanishes identically, and the relation (1.1.2) holds trivially. Now, from (1.4.2), (1.4.3) and (1.4.4) we get the following relations for the right hand side (R.H.S.) and left
hand side (L.H.S.) of (i):

\[ \text{R.H.S. of (i)} = (A_1 + B_1 + D_1)R_{1331} = d(\log a_2)R_{1331} = (a_2)_{,1} = \text{L.H.S. of (i)}. \]

Also R.H.S. of (ii) = \(-d(x^2x^3)(R_{3131} + R_{1331})\)

= 0 (by the skew symmetric property of \(R\))

= L.H.S. of (ii).

By a similar argument as in (ii) it can be shown that the relation (iii) is true. Hence we can state the following:

**Theorem 1.4.1.** Let \((M^4, g)\) be a Riemannian manifold endowed with the metric

\[ ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2 \quad (i, j = 1, 2, 3, 4), \]

where \(f = a_0 + a_1x^3 + a_2(x^3)^2\). \(a_0, a_1, a_2\) are non-constant functions of \(x^1\) only. Then \((M^4, g)\) is a weakly symmetric manifold of vanishing scalar curvature which is not locally symmetric.

In particular, if we take \(a_2 = e^{x^1}\), then (1.4.2) and (1.4.3) respectively reduce to the following:

\[ R_{1331} = e^{x^1} \neq 0, \]

\[ R_{1331,1} = e^{x^1} \neq 0, \]

and hence the manifold under consideration is not locally symmetric. We consider
the 1-forms as follows

\[ A_i(\partial_i) = A_i = \begin{cases} \frac{1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \]

\[ B_i(\partial_i) = B_i = \begin{cases} -\frac{1}{4} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (1.4.7) \]

\[ D_i(\partial_i) = D_i = \begin{cases} \frac{3}{4} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \]

where \( \partial_i = \frac{\partial}{\partial x_i} \) at any point \( x \in M \). Then proceeding similarly as in the previous case, it can be easily shown that the manifold under consideration satisfies (i)-(iii), and hence is a \((W_4)\). Thus we have the following:

**Theorem 1.4.2.** Let \((M^4, g)\) be a Riemannian manifold endowed with the metric

\[ ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^4)^2 \quad (i, j = 1, 2, 3, 4), \]

where \( f = a_0 + a_1x^3 + e^{x^1}(x^3)^2 \), \( a_0, a_1 \) are non-constant functions of \( x^1 \) only. Then \((M^4, g)\) is a weakly symmetric manifold with vanishing scalar curvature, and is not locally symmetric.

**Example 1.4.2.** Let \( M^n \) be an open subset of \( \mathbb{R}^n (n \geq 4) \) endowed with the metric

\[ ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + \sum_{k=3}^{n}(dx^k)^2, \quad (1.4.8) \]

\( (i, j = 1, 2, ..., n) \), where

\[ f = a_0 + a_1x^3 + e^{x^1}\left\{ \frac{1}{2}(x^3)^2 + \frac{1}{6}(x^3)^3 + ... + \frac{1}{(n-2)(n-3)}(x^3)^{n-2} \right\}, \]
\(a_0, a_1\) are non-constant functions of \(x^1\) only and \(0 < x^3 < 1\). Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and their covariant derivatives are

\[
\Gamma^2_{11} = \frac{1}{2} f_1, \quad \Gamma^2_{13} = -\Gamma^2_{11} = \frac{1}{2} f_3, \\
R_{1331} = \frac{1}{2} f_{33} = \frac{1}{2} e^{x^1} \left[ \frac{1 - (x^3)^{n-3}}{1 - x^3} \right] \neq 0, \\
R_{1331,1} = \frac{1}{2} f_{331} = \frac{1}{2} e^{x^1} \left[ \frac{1 - (x^3)^{n-3}}{1 - x^3} \right] \neq 0, \\
R_{1331,3} = \frac{1}{2} f_{333} = \frac{1}{2} e^{x^1} \left[ \frac{1 - (n - 3)(x^3)^{n-4} + (n - 4)(x^3)^{n-3}}{(1 - x^3)^2} \right] \neq 0
\]

and the components which can be obtained from these by the symmetry properties.

Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore our \(M^n\) with the considered metric is a Riemannian manifold, which is neither locally symmetric nor recurrent. We shall now show that this \(M^n\) is a \((WS)_n\), i.e., it satisfies (1.1.2). If we consider the 1-forms

\[
A_i(\partial_i) = A_i = \begin{cases} 
\frac{n-3}{n} & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_i(\partial_i) = B_i = \begin{cases} 
\frac{2}{n} & \text{for } i = 1 \\
\frac{1}{1 - x^3} & \text{for } i = 3 \\
0 & \text{otherwise},
\end{cases}
\]

\[
D_i(\partial_i) = D_i = \begin{cases} 
\frac{1}{n} & \text{for } i = 1 \\
-\frac{(n-3)(x^3)^{n-4}}{1-(x^3)^{n-3}} & \text{for } i = 3 \\
0 & \text{otherwise},
\end{cases}
\]

where \(\partial_i = \frac{\partial}{\partial x^i}\) at any point \(x \in M\), then proceeding similarly as in Example 1.4.1, it can be shown that the manifold under consideration is a \((WS)_n\). Thus we can state the following:
Theorem 1.4.3. Let \((M^n, g)\) \((n \geq 4)\) be a Riemannian manifold endowed with the metric given in (1.4.8). Then \((M^n, g)\) is a weakly symmetric manifold with vanishing scalar curvature which is neither locally symmetric nor recurrent.

Example 1.4.3. Let \(M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4\}\) be an open subset of \(\mathbb{R}^4\) endowed with the metric

\[
d s^2 = g_{ij} dx^i dx^j = e^{x^1} (dx^1)^2 + e^{x^1} (dx^2)^2 + e^{x^1} (\sin x^2)^2 (dx^3)^2 + e^{x^4} (dx^4)^2 \tag{1.4.9}
\]

\((i, j = 1, 2, 3, 4)\), where \(0 < x^2 < \pi/2\). Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and their covariant derivatives are

\[
\Gamma^1_{22} = -\frac{1}{2}, \quad \Gamma^1_{33} = -\frac{1}{2} (\sin x^2)^2, \quad \Gamma^2_{33} = -\sin x^2 \cos x^2,
\]

\[
\Gamma^3_{23} = \cot x^2, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{44} = \frac{1}{2},
\]

\[
R_{2332} = -\frac{3}{4} e^{x^1} (\sin x^2)^2, \tag{1.4.10}
\]

\[
R_{2332,1} = \frac{3}{4} e^{x^1} (\sin x^2)^2 \neq 0, \tag{1.4.11}
\]

\[
R_{2331,2} = \frac{3}{8} e^{x^1} (\sin x^2)^2 \neq 0, \tag{1.4.12}
\]

\[
R_{3221,3} = -\frac{3}{8} e^{x^1} (\sin x^2)^2 \neq 0, \tag{1.4.13}
\]

and the components which can be obtained from these by the symmetry properties. Here ‘, ’ denotes the covariant differentiation. Using the above relations, it can be easily shown that the scalar curvature \(r\) of the manifold is given by

\[
r = -\frac{3}{2} e^{-x^1} \neq 0 \text{ for } x^1 \text{ is finite.}
\]

Thus the scalar curvature of the manifold is negative, non-vanishing and non-constant. Therefore our \(M^4\) with the considered metric is a Riemannian manifold, which is
neither locally symmetric nor of vanishing scalar curvature. We shall now show that this $M^4$ is \((WS)_4\), i.e., it satisfies (1.1.2). We consider the 1-forms

\[
A_i(\partial_i) = A_i = \begin{cases} 
-1 & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_i(\partial_i) = B_i = \begin{cases} 
-\frac{1}{2} & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

\[
D_i(\partial_i) = D_i = \begin{cases} 
-\frac{1}{2} & \text{for } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]

where $\partial_i = \frac{\partial}{\partial x_i}$ at any point $x \in M$. In our $M^4$, (1.1.2) reduces with these 1-forms to the following equations

\[
(i) \quad R_{2332,1} = A_1 R_{2332} + B_2 R_{1332} + B_3 R_{2132} + D_3 R_{2312} + D_2 R_{2331},
\]

\[
(ii) \quad R_{2331,2} = A_2 R_{2331} + B_2 R_{2331} + B_3 R_{2231} + D_3 R_{2321} + D_1 R_{2332},
\]

\[
(iii) \quad R_{1332,2} = A_2 R_{1332} + B_1 R_{2332} + B_3 R_{1232} + D_3 R_{1322} + D_2 R_{1332},
\]

\[
(iv) \quad R_{3221,3} = A_3 R_{3221} + B_3 R_{3221} + B_2 R_{3321} + D_2 R_{3231} + D_1 R_{3223},
\]

\[
(v) \quad R_{1223,3} = A_3 R_{1223} + B_1 R_{3223} + B_3 R_{1333} + D_3 R_{1233} + D_2 R_{1323},
\]

\[
(vi) \quad R_{2232,3} = A_3 R_{2232} + B_2 R_{2332} + B_3 R_{2232} + D_3 R_{2232} + D_2 R_{2233},
\]

\[
(vii) \quad R_{2323,3} = A_3 R_{2332} + B_2 R_{3322} + B_3 R_{2322} + D_3 R_{2332} + D_2 R_{2323},
\]

since for the cases other than (i) - (vii) the components of each term of (1.1.2) vanishes identically and the relation (1.1.2) holds trivially. Now from (1.4.10) - (1.4.14) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

\[
\text{R.H.S. of (i)} = A_1 R_{2332} = -R_{2332} = \frac{3}{4}e^{x^1}(\sin x^2)^2 = \text{L.H.S. of (i)}.
\]
Also
\[ \text{R.H.S. of (ii)} = D_1 R_{2332} = \frac{3}{8} e^{x_1} (\sin x^2)^2 = \text{L.H.S. of (ii)}. \]

Similarly it can be shown that the relations (iii)-(vii) are true. Hence we can state the following:

**Theorem 1.4.4.** Let \((M^4, g)\) be a Riemannian manifold endowed with the metric
\[ ds^2 = g_{ij} dx^i dx^j = e^{x_1} (dx^1)^2 + e^{x_2} (dx^2)^2 + e^{x_3} (\sin x^4)^2 (dx^3)^2 + e^{x_4} (dx^4)^2 \]
\((i, j = 1, 2, 3, 4)\), where \(0 < x^2 < \pi/2\) and \(x^1\) is finite. Then \((M^4, g)\) is a weakly symmetric manifold with non-vanishing and non-constant scalar curvature, which is neither locally symmetric nor recurrent.

**Example 1.4.4.** Let \(M^n\) be an open subset of \(\mathbb{R}^n\) \((n \geq 4)\) equipped with the metric
\[ ds^2 = g_{ij} dx^i dx^j = (e^{x_1} - 1) [(dx^1)^2 + (dx^2)^2] + (e^{x_4} + x^2 - 1)(dx^3)^2 + (e^{x_4} - 1)(dx^4)^2 + \delta_{ij} dx^i dx^j \]
\((i, j = 1, 2, ..., n)\), where \(x^1\) is finite and \(\delta_{ij}\) denotes the Kronecker delta.

Then the only non-vanishing components of the Christoffel symbols, curvature tensor and its covariant derivatives are given by

\[ \Gamma^1_{11} = -\Gamma^2_{22} = \Gamma^3_{13} = \Gamma^3_{23} = \Gamma^4_{44} = \Gamma^2_{12} = \frac{1}{2}, \quad \Gamma^1_{33} = \Gamma^2_{33} = -\frac{1}{2} e^{x^2}, \]
\[ R_{2332} = \frac{1}{2} e^{x_1 + x^2}, \]
\[ R_{2332,1} = -\frac{1}{2} e^{x_1 + x^2} \neq 0, \]
\[ R_{2331,2} = -\frac{1}{4} e^{x_1 + x^2} \neq 0, \]
\[ R_{3221,3} = -\frac{1}{4} e^{x_1 + x^2} \neq 0. \]
and the components which can be obtained from these by the symmetry properties, where \( \cdot \cdot \cdot \) denotes the covariant differentiation. Using the above relations, it can be easily shown that the scalar curvature \( r \) of the manifold is given by

\[
r = e^{-x^1} \neq 0 \quad \text{for } x^1 \text{ is finite.}
\]

Hence the manifold under the considered metric is of non-zero scalar curvature, and it is a Riemannian manifold. If we consider the 1-forms

\[
A_i(\partial_i) = A_i = \begin{cases} -1 & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}
\]

\[
B_i(\partial_i) = B_i = \begin{cases} -1/2 & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}
\]

\[
D_i(\partial_i) = D_i = \begin{cases} -1/2 & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}
\]

where \( \partial_i = \frac{\partial}{\partial x^i} \) at any point \( x \in M \), then proceeding similarly as in Example 1.4.3, it can be easily shown that the manifold under consideration is a \((W.S)_n\). Thus we can state the following:

**Theorem 1.4.5.** Let \((M^n, g) \ (n \geq 4)\) be a Riemannian manifold equipped with the metric

\[
ds^2 = g_{ij}dx^idx^j = (e^{x^1} - 1)\left[(dx^1)^2 + (dx^2)^2\right] + (e^{x^1 + x^2} - 1)(dx^3)^2 + (e^{x^4} - 1)(dx^4)^2 + \delta_{ij}dx^idx^j
\]

\((i, j = 1, 2, \ldots, n)\). Then \((M^n, g)\) is a weakly symmetric manifold of non-zero and non-constant scalar curvature, which is neither locally symmetric nor recurrent.
Example 1.4.5. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4\}$ be an open subset of $\mathbb{R}^4$ equipped with the metric

$$ds^2 = g_{ij}dx^i dx^j = x^3 e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4(dx^4)^2, \ (i,j = 1,2,3,4),$$

where $x^3 > 0$ and $x^4 > 0$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{11} = \frac{1}{2}, \ \Gamma^3_{11} = -\frac{1}{2}e^{x^1}, \ \Gamma^3_{13} = \frac{1}{2x^3}, \ \Gamma^4_{44} = \frac{1}{2x^4},$$

$$R_{1331} = -\frac{1}{4x^3} e^{x^1} \neq 0,$$

$$R_{1331,3} = \frac{1}{2(x^3)^2} e^{x^1} \neq 0$$

and the components which can be obtained from these by the symmetry properties.

Using the above it can be easily shown that the scalar curvature $r$ of the manifold is given by

$$r = -\frac{1}{2(x^3)^2} \neq 0.$$

Hence the manifold with the considered metric is a Riemannian manifold of non-constant negative scalar curvature, which is neither locally symmetric nor recurrent.

If we consider the 1-forms

$$A_i(\partial_i) = A_i = \begin{cases} \frac{1}{x^3} & \text{for } i = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$B_i(\partial_i) = B_i = \begin{cases} \frac{2}{x^3} & \text{for } i = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$D_i(\partial_i) = D_i = \begin{cases} -\frac{3}{x^3} & \text{for } i = 3 \\ 0 & \text{otherwise,} \end{cases}$$
where $\partial_i = \frac{\partial}{\partial x^i}$ at any point $x \in M$, then proceeding similarly as in the previous examples it can be easily shown that the manifold under consideration is weakly symmetric. Hence we can state the following:

**Theorem 1.4.6.** Let $(M^4, g)$ be a Riemannian manifold equipped with the metric

$$ds^2 = g_{ij}dx^idx^j = x^3e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4(dx^4)^2$$

$(i, j = 1, 2, 3, 4)$, where $x^3 > 0$ and $x^4 > 0$. Then $(M^4, g)$ is a weakly symmetric manifold of non-constant and negative scalar curvature, which is neither locally symmetric nor recurrent.

* * * * *
ON WEAKLY QUASI-CONFORMALLY SYMMETRIC MANIFOLDS