CHAPTER IV

HELI CON-PHONON INTERACTION
IN METALS PARALLEL PROPAGATION
1. Introduction

As stated in Chapter I, when a strong external magnetic field is imposed over a metal or a doped semiconductor, the plasma can support a variety of waves, which are otherwise not possible. Aigrain (1960) discovered that a class of low-frequency, circularly polarised waves propagate with very little damping, in metals and these are now well known as the helicons. The phase velocity of the helicon wave is a function of the applied magnetic field and is far smaller than the Fermi velocity \(v_F\) of the carriers. Also, helicon wave can propagate either parallel or at an angle to the magnetic field, with a frequency less than the electron cyclotron frequency.

When an electromagnetic wave propagates through a metal, it gives rise to a volume force of the Lorentz type equal to \(\frac{j \times H_e}{c}\), where \(j\) and \(H_e\) denote respectively, the current density and the external magnetic field. As can be seen in subsequent sections, since \(j\) depends on the elastic displacement, this force produces a coupling between the elastic and electromagnetic wave fields. Since helicons are slow waves, their phase velocity can be made to match the velocities of the acoustic waves and in the region where their velocities coincide, strong resonant interaction can take place between these two excitations. Experiments by Grimes and Buchsbaum (1964) and others have shown that this interaction splits the degeneracy at the resonant point. This effect has been studied
theoretically by several authors (Akramov 1963, Langenberg and Bok 1963, Skobov and Kaner 1964), for metals. Generally, a magnetic field of the order of $10^5 \text{G}$, is required to observe the splitting of the degeneracy arising from the resonant interaction, for the case of metals. Recently, Viswanathan (1975) has derived the dispersion relation for propagation along any general direction, inside the crystal, for the case wherein the applied field is parallel to the propagation direction.

Of the two types of circularly polarised modes that can propagate in a solid, the helicon wave whose polarisation is the same as the sense of rotation of the electrons in the magnetic field, propagates freely, whereas the other mode is damped within a distance of few wavelengths. We have investigated in Section 3. of this chapter, the effect of the surface helicon mode on helicon-phonon interaction and it is shown that this is of such a small order of magnitude, as to be negligible.

In section 4 an expression for the Lagrangian density of the system is derived, which includes both elastic as well as electromagnetic wave fields and this throws light on the energy density of the interacting fields. From this, we have derived an expression for energy-flux density vector and the ray velocity vector. As the ray velocity vector is identical with the group velocity vector (Auld, 1973), this expression could be a starting point for future investigations on the characteristics of energy surfaces, for the hybrid modes of helicons and phonons.
However, the major aim of this chapter is a detailed numerical investigation of the helicon-phonon interaction for different magnetic fields, in the region of $0.5 \times 10^5 \text{G}$ to $2.3 \times 10^5 \text{G}$. This study reveals the dependence of the hybrid modes on the magnetic field, the nature of the inverse velocity surfaces and their variation with magnetic field. This is probably the first instance, when detailed numerical studies of helicon-phonon interaction for directions other than the symmetry axes, has been reported. The results of numerical investigations has been given in Section 5 for four metals, lithium, potassium, sodium and Nb$_3$Sn. From these it is seen that, with the exception of longitudinal elastic mode, the remaining three hybrid modes show interesting variation with magnetic field. It is observed that the helicons and the transverse acoustic waves merge with each other at the resonance point in such a manner that it is rather impossible to point out the basic nature of these after interaction. They can be classified only as hybrid modes consisting of electromagnetic and elastic vibrations, rather than purely elastic and electromagnetic waves.

2. **Dispersion Relation**

Here we give a brief review of the dispersion equation for helicon-phonon interaction for parallel propagation, as discussed by Viswanathan (1975).

The propagation of an elastic wave gives rise to a volume force $f$ of magnitude $\frac{1}{c}(j \times \mathbf{E})$ which acts on the lattice. The current density $j$ has the form:
\[ j = \sigma \frac{\varepsilon}{\omega} E + \frac{1}{\omega} (u \times H_0) \]  

\( E \) being the electric field, \( u \) the elastic displacement and \( \sigma \) the conductivity tensor of rank 2. \( \dot{u} \) represents the time derivative of \( u \). The equations for lattice vibrations are,

\[ \frac{\rho \partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} + f_i \]

with \( i, j = 1, 2, 3 \).

where \( T_{ij} \) are components of the stress tensor. Choosing an orthogonal system of axes such that \( e_1 \) coincides with the direction of propagation, \( k \) and assuming that \( u \) and \( E \) vary as \( \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{x})] \), equation (2) becomes,

\[ (\tilde{G}_{iik} k^2 - \rho \omega^2 \delta_{ik}) u_k = f_i \]

\( \tilde{G}_{iik} \) are the elastic constants referred to the axes, \( e_1, e_2, e_3 \). In this system of co-ordinates, \( f \) takes the form:

\[ f = \frac{H_0}{C} \left[ e_2 j_3 - e_3 j_2 \right] \]

Let us denote by \( (A) \), the matrix whose components are given by,

\[ a_{ik} = (\tilde{G}_{iik} k^2 - \rho \omega^2 \delta_{ik}) \]

The inverse matrix \( (B) \) of this can be written as,

\[ b_{ik} = \frac{A_{ik}}{\Delta} \]
\( A_{ik} \) being the cofactor of \( a_{ik} \) and \( \Delta \) the determinant of the matrix \( (A) \).

We shall next pass on to the electromagnetic wave equations,

\[
-k \times (k \times E) = \left( 4\pi i \omega / c^2 \right) j
\]  

(7)

or,

\[
k^2 \mathbf{E}_\perp = \left( 4\pi i \omega / c^2 \right) j
\]  

(8)

\( \mathbf{E}_\perp \) are the perpendicular components of the electric field. Now, \( j_\perp \), the perpendicular components of \( j \) can be written as:

\[
\mathbf{j}_\perp = \sigma \left[ \mathbf{E}_\perp - \frac{i\omega}{c} (\mathbf{u} \times \mathbf{H}_\perp) \right]
\]  

(9)

where \( \sigma \) is the 2 x 2 conductivity tensor, which is anti-symmetric. That is,

\[
\sigma = \begin{pmatrix} \sigma_{22} & \sigma_{23} \\ -\sigma_{23} & \sigma_{22} \end{pmatrix}
\]  

(10)

Let us now write,

\[
\mathbf{j}_\perp = j_2 \mathbf{i}_1 + j_3 \mathbf{i}_3 \quad \text{and} \quad \mathbf{u}_\perp = u_2 \mathbf{i}_1 + u_3 \mathbf{i}_3. \quad \text{Eliminating } \mathbf{E}_\perp
\]

from equations (8) and (9), we get,

\[
\left(1 - \frac{k^2 c^2}{4 \pi \omega \sigma_+} \right) j_+ = \left( -\frac{i k^2 c H_\perp}{4 \pi} \right) u_+ \quad \text{(11)}
\]

and

\[
\left(1 - \frac{k^2 c^2}{4 \pi \omega \sigma_-} \right) j_- = \left( \frac{i k^2 c H_\perp}{4 \pi} \right) u_-
\]  

(12)
where we write,

$$\sigma_{\pm} = \pm \sigma_{23}^* + i \sigma_{22}$$

(13)

The mode corresponding to $j_+$ (or $E_+$) represents the surface helicon mode, which decays within a distance of the order of a few wavelengths. We shall, therefore, pass on to the limit $j_+ = 0$ in equation (3), (11) and (12), which give four equations in the variables, $u_1$, $u_2$, $u_3$ and $j_-$. By eliminating them, we obtain the following determinantal equation,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & -\frac{He}{2c} \\ a_{31} & a_{32} & a_{33} & \frac{He}{2c} \\ 0 & -\frac{k_c H_e}{4\pi} & -\frac{k_c H_e}{4\pi} & 1 - \frac{k_c^2}{4\pi \omega_0^2 c^2} \end{vmatrix} = 0$$

(14)

On expanding this determinant along the last row or column, this can be written as:

$$\left(1 - \frac{k_c^2}{4\pi \omega_0^2 c^2}\right) + \frac{k_c^2 H_e^2}{8\pi} \left( b_{22} + b_{33} \right) = 0$$

(15)

which is the equation (20) derived by Viswanathan. This is the equation that is solved numerically in section 5 for four metals of cubic class, namely, lithium, potassium, sodium and Nb3Sn.
3. **Effect of Surface Helicon Mode on the Dispersion relation**

In the previous section, we have taken the limit when \( j_+ \) or \( E_+ \) tends to zero, since it represents a surface mode. In this section, an attempt is made to justify this approximation and to bring out a correction factor due to this term on the dispersion relation and thereby establish numerically for a few chosen values, the exact nature of dependence of the dispersion relation on \( j_+ \) or \( E_+ \).

Equations (3), (11) and (12) give a set of five equations in variables, \( u_1 \), \( u_2 \), \( u_3 \), \( j_- \) and \( j_+ \). The determinant formed by the coefficients of these variables, gives the dispersion relation including the surface mode. That is,

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} & 0 & 0 \\
  a_{21} & a_{22} & a_{23} & -iH_e/2c & iH_e/2c \\
  a_{31} & a_{32} & a_{33} & H_e/2c & H_e/2c \\
  0 & -ikcHe/4\pi & -k^2cH_e/(4\pi) & (1-k_c^2/(4\pi\omega_0)) & 0 \\
  0 & i(kcHe/4\pi) & -k^2cH_e/(4\pi) & 0 & (1-k_c^2/(4\pi\omega_0))
\end{vmatrix} = 0 \quad (16)
\]

Expanding (16) along the last row, we get

\[
\begin{bmatrix}
  B_3 \\
  (ikcHe/4\pi)
\end{bmatrix} - \left( \frac{k^2c^2H_e}{4\pi \omega_0} \right) \begin{bmatrix}
  B_1 - iB_2
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
  B_3 \\
  (ikcHe/4\pi)
\end{bmatrix} - \left( \frac{k^2c^2H_e}{4\pi \omega_0} \right) \begin{bmatrix}
  B_1 - iB_2
\end{bmatrix} = 0 \quad (17)
\]
provided \( 1 - \frac{k^2c^2}{\omega \sigma^+} \neq 0 \).

Here \((B_1)\) represents the cofactor of the second term in the last row, \([B_2]\) that of the third term and \([B_3]\), the cofactor of \(1 - \frac{k^2c^2}{\omega \sigma^+}\).

It is seen that \([B_3]\) is none other than the dispersion relation for helicon-phonon interaction in the limit \(j_+ \to 0\). Hence the second and third terms of (17) give the correction factor \(\delta\) to the dispersion equation.

\[
\delta \sim -\left( \frac{i k_c^2 H_e}{4\pi} \right) \left[ \frac{k_c^2 H_e}{4\pi} C_1 - \left( 1 - \frac{k^2c^2}{4\pi \omega \sigma^+} \right) C_2 \right] \tag{18}
\]

where,

\[
C_1 = -\frac{i H_e^2 a_{11}}{2c^2} \tag{19}
\]

and,

\[
C_2 = \frac{H_e}{2c} \left[ (a_{11} a_{23} - a_{21} a_{13}) - i(a_{11} a_{33} - a_{31} a_{13}) \right] \tag{20}
\]

Giving typical values to the parameters \(k\) and \(\omega\), we find that \(\delta\) is smaller than \((B_3)\) by a factor of \(10^{-5}\), which justifies the assumption of the last section.

The effect of \(j_+\) is calculated again for the simple case, when the propagation direction is along the \([100]\) direction of a cubic crystal. The expression for the dispersion relation along this direction, splits up into three factors:

\[
\left( c_{44} k^2 - \rho \omega^2 \right) \left( 1 - \frac{k^2c^2}{4\pi \omega \sigma^+} \right) + \frac{k^2H_e^2}{4\pi} = 0 \tag{21}
\]
\[
\left( C_{44} k^2 - \rho \omega^2 \right) \left( 1 - k^2 c^2 \frac{\omega}{4 \pi \omega c} \right) + \frac{k^2 H_e^2}{4 \pi} = 0
\]  \hspace{1cm} (22)

and

\[
\left( C_{11} k^2 - \rho \omega^2 \right) = 0
\]  \hspace{1cm} (23)

It is seen that the surface helicon mode interacts with only one of the transverse elastic modes (21), while it does not affect the longitudinal mode and the other transverse mode. Of course, for directions other than the symmetry axes, the surface mode will be having interaction with other modes too. Any way, this gives an idea of the nature of interaction between the surface helicon and elastic wave.

Solving (21), we get,

\[
k^2 = \left( C_{44} + \frac{\rho \omega c^2}{4 \pi T + \omega} + H_e^2 \frac{\omega_c^2}{4 \pi} \right) + \left[ \left( C_{44} + \frac{\rho \omega c^2}{4 \pi T + \omega} + H_e^2 \frac{\omega_c^2}{4 \pi} \right)^2 - \frac{C_{44} \frac{\rho \omega c^2}{4 \pi T + \omega}}{C_{11} \frac{\omega_c^2}{4 \pi T + \omega}} \right]^{1/2}
\]  \hspace{1cm} (24)

It can be seen that for small values of \(H_e\), the \(k\)-mode behaves as an elastic mode, while \(k_+\) mode exhibits surface helicon properties.

Also, it is seen that even as the magnetic field increases, the damping rate of \(k_+\) is much high even though it decreases with the magnetic field, which points out that the wave gets damped, even within a small distance and this justifies the assumption used. The acoustic wave is also affected, by a small amount.
4. **Lagrangian Density and Ray Velocity**

For acoustic waves, it is well-known that the group velocity, which also signifies the velocity of transport of energy in the system, is different from the wave velocity both in magnitude as well as in direction. In order to obtain an insight into the nature of energy density and energy flux for a field consisting of interacting phonons and helicons, we derive in this section, the Lagrangian density $L$ and the power flux vector $P$ for the hybrid modes.

When the electromagnetic wave (Helicon) propagates through a metal, with a low phase velocity, it produces lattice vibrations which in turn brings about a term in the electric vector potential $A$, which is proportional to the displacement vector $u$. This term causes induction interaction of electrons with the lattice vibrations. The scalar electric potential is chosen to be zero, for convenience.

The energy of the system is manifested in three types of terms. The first corresponds to the elastic energy associated with the lattice vibrations; the second is the electromagnetic energy of the helicons and the third is the interaction energy of lattice vibrations and helicons. The Lagrangian density $L$ of the system, can be expressed in the following form (Morse and Feshbach, 1953):

$$
L = \frac{1}{2} \rho u^2 - \frac{1}{2} C_{ijkl} S_{ij} S_{kl} + \frac{1}{8\pi} \frac{\varepsilon E^2}{\varepsilon n} - \frac{H^2}{8\pi} + \frac{1}{c} \epsilon \cdot A
$$

(25)

Here, all the quantities are referred to the laboratory frame.
of reference. $S_{ij}$ represents the components of strain tensor and is given by

$$S_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$ (26)

Also,

$$A = A_e + \mathcal{A}' + (\mathbf{u} \times H_e)$$ (27)

$A_e$ is the vector potential relating to $H_e$, the external magnetic field. $\mathcal{A}'$ is that due to the electromagnetic field of helicons and $(\mathbf{u} \times H_e)$ is the potential arising from the interaction of $H_e$ with the lattice. From this, the electric and magnetic fields are written as follows:

$$\mathcal{E} = \mathcal{E}' - \left( \frac{\mathbf{u} \times H_e}{c} \right)$$ (28)

where $\mathbf{\dot{u}} = \frac{\partial \mathbf{u}}{\partial t}$,

and

$$\mathcal{H} = H_e + H_e' + \mathbf{\nabla} \times (\mathbf{u} \times H_e)$$ (29)

The Lagrangian density $L$, can give rise to equations of lattice vibrations and electromagnetic equations when substituted into Euler's equations, (given in equations (18) of chapter II).

The first set of Euler equations gives,

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_j}{\partial x_k \partial x_l} + \frac{1}{c} \left[ \mathcal{E} \times \mathcal{H} \right]_i$$ (30)

for $i, j, k, l = 1, 2, 3$,

which are same as eqn.(2) in section 2 of this chapter.
In the same way, the second Euler equation yields,

\[ \nabla \times H = \frac{1}{c} \varepsilon \frac{\partial E}{\partial t} + \frac{4\pi}{c} j \]  

(31)

which is Maxwell's electromagnetic equation, since \(-\frac{1}{c} \frac{\partial A}{\partial t} = \frac{E}{c}\)
and \(\text{Curl } \vec{A} = \vec{H} \quad (\mu = 1)\). Next, the total energy density per unit volume of the system \(E_0\), can be written as

\[ E_0 = E_S + U_e - \frac{1}{c} \vec{j} \cdot \vec{A} \]  

(32)

in which,

\[ E_S = \frac{1}{2} \rho \dot{u}^2 + \frac{1}{2} C_{ijkl} S_{ij} S_{kl} \]  

(33)

is the elastic energy, and

\[ U_e = \frac{1}{8\pi} (\varepsilon E^2 + H^2) \]  

(34)

represents the electromagnetic energy stored in the system and \(-\frac{1}{c} \vec{j} \cdot \vec{A}\) is the work done by \(\vec{A}\) on the current density \(\vec{j}\).

The rate of change of energy of the entire system with time, is given by,

\[ \frac{dE}{dt} = \int \left[ \frac{\partial E_S}{\partial t} + \frac{\partial U_e}{\partial t} - \frac{1}{c} \left( \vec{j} \cdot \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{J}}{\partial t} \cdot \vec{A} \right) \right] dV \]

\[ = \int \left[ \frac{\partial E_S}{\partial t} + \frac{\partial U_e}{\partial t} + \vec{j} \cdot \dot{E} - \frac{1}{c} \frac{\partial \vec{J}}{\partial t} \cdot \vec{A} \right] dV \]  

(35)

since \(E = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}\).

Simplifying the above, we get

\[ \frac{dE}{dt} = \int \left\{ \rho \ddot{u} \cdot \ddot{u} - \dot{u} \cdot \frac{\partial \vec{U}}{\partial t} - \frac{1}{c} \dot{u} \cdot \left( \vec{j} \times \vec{H} \right) \right\} dV + \]

\[ \int \left( \vec{T} \cdot \ddot{u} \right) \cdot dS + \int \left[ \frac{\partial U_e}{\partial t} + \vec{j} \cdot \dot{E} - \frac{1}{c} \frac{\partial \vec{J}}{\partial t} \cdot \vec{A} \right] dV + \int \vec{J} \cdot \left( \vec{J} \times \vec{H} \right) dV \]  

(36)
The first volume integral vanishes in view of the equation of lattice vibration (2). \( P_A = - (T \cdot \dot{u}) \) in which \( T = (T_{ij}) \) is the stress tensor and \( P_E = \frac{c}{4\pi} (E \times H) \) is the Poynting vector, representing the electromagnetic flux. \( \ddot{u} \) represents the second derivative of \( u \) with time can be further simplified as,

\[
\frac{dE}{dt} = \int [P_A - \frac{1}{4\pi} (A \times \frac{\partial H}{\partial t})] \cdot dS + \int \left[ \frac{\varepsilon}{4\pi} E \cdot \frac{\partial E}{\partial t} + J \cdot E \right] dV \\
- \frac{1}{c} J \cdot (\dot{\varepsilon} x H) + \frac{1}{4\pi c} \frac{\partial^2 A}{\partial t^2} \cdot A \] dV \tag{37}
\]

Since, we are considering the case of metals, in which the displacement current, \( J_d = \frac{1}{\varepsilon} \frac{\partial D}{\partial t} \) is \( \omega/4\pi\sigma \) times smaller than the conductivity current \( J \), we can safely neglect \( \frac{\partial D}{\partial t} \) and \( \frac{\partial^2 D}{\partial t^2} \) terms in comparison with the other terms. Hence, we get:

\[
\frac{dE}{dt} + \int P \cdot dS - \int \left[ J \cdot \varepsilon - J \cdot (\dot{\varepsilon} x H) \right] dV = 0 \tag{38}
\]

where

\[
P = P_A - \frac{1}{4\pi} (A \times \frac{\partial H}{\partial t}) \tag{39}
\]

The above equation is similar to the equation of conservation of energy (Jackson, 1962), with \( P \) representing the energy flux density vector, the magnitude of which gives the amount of energy flowing in unit time through unit area of the surface perpendicular to \( \dot{\varepsilon} \).
The direction of energy flow can also be expressed in terms of the group velocity $v_G$, (Federov, 1968) which is the ratio between the time averages of energy-flux density vector $\mathbf{P}$ and energy density $\varepsilon_o$.

Assuming the spatial and temporal variation of $u$ and $A$ as $\cos(\omega t - k \cdot \mathbf{z})$, we get,

$$
\langle \varepsilon_o \rangle = \frac{\int_0^T \varepsilon_0 dt}{T} = \frac{1}{2} \rho u_o^2 \omega^2 + \frac{1}{16 \pi} \left( \varepsilon \varepsilon_o^2 + H_o^2 \right)
$$

(40)

with $T = 2\pi/\omega$ and $u_o, A_o', E_o, H_o$ are the amplitudes of $u, A', E$ and $H$, respectively.

Similarly,

$$
\langle P_i \rangle = \frac{\int_0^T P_i dt}{T} = \left( \frac{1}{2} C_{ijkl} k \varepsilon u_{ok} u_{oj} \right) - 
\left( \frac{\omega}{8\pi} \right) \left\{ \frac{1}{A_{oj}' + (u_o \times H_o) \cdot j} \left[ (k \times A_o) + k \times (u_o \times H_e) \right] k \Lambda i j k \right\}
$$

(41)

where $\Lambda i j k$ is the anti-symmetric tensor of rank three.

Hence,

$$
\frac{(v_G)_i}{\langle \varepsilon_o \rangle} = \frac{1}{\langle \varepsilon_o \rangle} \frac{\int_0^T P_i dt}{\int_0^T \varepsilon_0 dt} = \left( \frac{1}{2} C_{ijkl} k \varepsilon u_{ok} u_{oj} \right) - 
\left( \frac{\omega}{8\pi} \right) \left\{ \frac{1}{A_{oj}' + (u_o \times H_o) \cdot j} \left[ (k \times A_o) + k \times (u_o \times H_e) \right] k \Lambda i j k \right\}
$$

(42)
Here, we get the first term representing the elastic energy flux and the second one, the electromagnetic flux density, while the remaining terms are interaction terms. The above expression for the group velocity of the waves will be very useful for studying the energy surfaces or ray surfaces of the hybrid modes of the crystal.

5. Numerical Computation

The dispersion equation which has been numerically solved, is given in the equation (15), which can be rewritten in terms of $v^2$ as follows:

$$\left[ v^2 - \mu_0^2(\omega) \left( 1 - \frac{\nu^2}{\omega} \right)^{-1} \right] \Delta + \frac{\mu^2 \mu_0^2}{8 \pi} \left[ A_{22} + A_{33} \right] = 0$$

(43)

Here,

$$\Delta = \nu^6 \left[ -\rho^3 \right] + \nu^4 \rho^2 \left[ \tilde{C}_{11} + \tilde{C}_{66} + \tilde{C}_{55} \right] + \nu^2 \rho \left[ -\tilde{C}_{11} (\tilde{C}_{66} + \tilde{C}_{55}) - \tilde{C}_{66} \tilde{C}_{55} + \tilde{C}_{56} \tilde{C}_{65} + \tilde{C}_{61} \tilde{C}_{16} + \tilde{C}_{54} \tilde{C}_{15} \right] +$$

$$\left[ \tilde{C}_{11} \tilde{C}_{66} \tilde{C}_{55} - \tilde{C}_{11} \tilde{C}_{56} \tilde{C}_{65} - \tilde{C}_{61} \tilde{C}_{16} \tilde{C}_{55} + \right.$$

$$\left. \tilde{C}_{61} \tilde{C}_{56} \tilde{C}_{15} + \tilde{C}_{51} \tilde{C}_{16} \tilde{C}_{55} - \tilde{C}_{51} \tilde{C}_{66} \tilde{C}_{15} \right]$$

(44)

Also,

$$\left[ A_{22} + A_{33} \right] = \nu^4 \left[ 2 \rho^2 \right] + \nu^2 \rho \left[ -2 \tilde{C}_{11} - \tilde{C}_{55} - \tilde{C}_{66} \right] +$$

$$\left[ \tilde{C}_{11} \tilde{C}_{55} + \tilde{C}_{11} \tilde{C}_{66} - \tilde{C}_{51} \tilde{C}_{15} - \tilde{C}_{61} \tilde{C}_{16} \right]$$

(45)
and \( \nu \) is the collision frequency,

The above elastic constants, expressed using the two index notation, refer to the co-ordinate system \((S')\), associated with the propagation direction. They are determined, using proper co-ordinate transformations, from elastic constants with respect to the crystallographic axes \((S)\).

That is,

\[
\tilde{C}_{ijkl} = \sum_{p,q,r,s} a_{ip} a_{jq} a_{kr} a_{ls} C_{pqrs}
\]

where \( i, j, k, l, p, q, r, s = 1, 2, 3 \).

Here \( a_{ip}, a_{jq} \) etc., are the direction cosines which the system \((S')\) makes with the system of axes \((S)\). \( u_0(\omega) \) is the helicon wave velocity and is equal to \( \left( \frac{\omega c H_e}{4\pi N e} \right)^2 \) and \( \Omega \) is the cyclotron frequency of the charge carrier.

\[
\Omega = \frac{e H_e}{mc}
\]

In the subsequent parts, the effect due to collision is neglected. We may mention that we made also a detailed numerical investigation starting from eqn. (43) involving the collision frequency; but the numerical results displayed consistently that the damping effect arising from collision is too small to be negligible. This is because \( \nu/\Omega \ll 1 \) for all the cases, we have studied. Hence we have investigated the real part of the phase velocities only and we report in the following section, the roots of the equation (43), neglecting \( \nu/\Omega \).
This is an eighth degree equation in $v$ and yields 4 pairs of modes of which three are elastic in nature and the fourth relates to the helicon.

A computer program is written for this equation for the (001) plane ($xy$ plane) of a cubic crystal, for angles in the range $0°$ - $90°$ in steps of $10°$. Since the equation is a biquartic equation in $v^2$, we have used the solution of quartic equation, to obtain $v^2$. The roots were obtained for different magnetic fields, ranging from $0.5 \times 10^5 \text{G}$ to $2.3 \times 10^5 \text{G}$, at intervals of $0.2 \times 10^5 \text{G}$, using an IBM 360 computer. Also, the values of the inverse of phase velocities are obtained, in each case. The computer program, used for the calculation, is given in Appendix I. The results of the numerical calculations for the four metals, lithium, potassium, sodium and $\text{Nb}_3\text{Sn}$ are described below, for a frequency $\frac{9}{8} \times 10^5 \text{cps}$.

(a) Lithium

In figures (4.1), (4.2) and (4.3), we have plotted the sections of the inverse velocity surface of lithium in the (001) plane, for three different magnetic fields, namely $B = 0.5 \times 10^5 \text{G}$, $B = 1.5 \times 10^5 \text{G}$ and $B = 2.3 \times 10^5 \text{G}$. We have also given in figures (4.4), (4.5) and (4.6), the variation of the phase velocity $v$ with respect to the magnetic field, for the three different acoustic modes as well as the helicon, for propagation along the $[100]$ axis, at $20°$ and $45°$ with respect to the $[100]$ axis.

It can be seen from the graphs of the inverse phase velocities that there is no significant interaction between
FIG. 4.1 INVERSE VELOCITY SURFACE
LITHIUM (001) PLANE  \( B = 0.5 \times 10^5 \) G

FIG. 4.2 INVERSE VELOCITY SURFACE
LITHIUM (001) PLANE  \( B = 1.5 \times 10^5 \) G
FIG. 4.3 INVERSE VELOCITY SURFACE
LITHIUM (001) PLANE $B = 2.3 \times 10^5$ G

FIG. 4.5 VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
LITHIUM (001) PLANE AT 20° WITH [100] AXIS
FIG. 4.6: VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD LITHIUM (001) PLANE AT 45° WITH [100] AXIS

FIG. 4.4: VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD LITHIUM (001) PLANE ALONG [100] AXIS
the helicon and the phonons, in the region of the magnetic field chosen for lithium. Figures (4.4), (4.5) and (4.6), show that the curves marked as (1) and (3) are completely unaffected by the magnetic field, suggesting that these modes represent the pure shear and quasi-longitudinal modes. For propagation at 20° with respect to the [100] axis, the curves corresponding to the modes (1), (3) and (4) are unaffected by the magnetic field, whereas (2) shows a parabolic variation, suggesting that this could represent the helicon, while the others are acoustic modes. For propagation at an angle 45° with the x-axis ( [100] axis), the curves (1) and (3) show no variation with the magnetic field and these represent the longitudinal and pure shear modes, as stated earlier. Curve (4) does not show any variation with respect to the magnetic field, whereas curve (2) has a parabolic change, confirming that this mode represents the helicon for lithium. We have found that the same type of behaviour is also exhibited for propagation at an angle 60°, though we have not reproduced the figure for this direction. For propagation along the [100] axis, two of the acoustic modes, namely (3) and (4) become equal and the graphs show that except for helicon, the other modes show little variation with regard to the magnetic field.

Figures (4.1), (4.2) and (4.3), give the inverse velocity surfaces for the four different modes. Curve (1) is the innermost, whereas curve (2) is the outermost suggesting that the helicon velocity is the smallest among the four modes while the longitudinal velocity is the highest. It can be
seen by a comparison of the figures that as the intensity of the magnetic field is increased, the inverse velocity surface for the helicon (mode (2)) gradually shrinks and approaches the mode (4). Again the shape of this mode, which is isotropic and circular at low magnetic fields, gradually becomes anisotropic for higher magnetic fields and becomes square like at $B = 2.3 \times 10^5$ G.

It can be seen that the inverse velocity surface for the quasi-shear mode (4) shows a long spike at an angle 45° with respect to the x-axis. This is because the anisotropy factor $A = \frac{2 \lambda_{4k}}{c_{11} - c_{22}}$ is much greater than unity for lithium. It has been shown earlier by Jacob Philip and Viswanathan (1977), that if the section of the inverse velocity surface shows a spike along the diagonal direction, then the section of the group velocity surface should show a cusp along the x-axis. Our figures suggest that lithium must show a large cusp along the x-axis for the (001) section of the energy surface for the quasi-shear mode.

(b) **Potassium**

In figures (4.7), (4.8) and (4.9), we give the section of the inverse velocity surfaces by the (001) plane for potassium, for magnetic field strengths, $B = 1.5 \times 10^5$ G, $B = 1.7 \times 10^5$ G and $B = 1.9 \times 10^5$ G respectively. Figures (4.10), (4.11) and (4.12), depict the variation of the phase velocities of the four modes with respect to the magnetic field, for propagation at angles of 20°, 45° with the x-axis and along the [100] axis. It can be seen from these three figures
FIG. 4.7. INVERSE VELOCITY SURFACE POTASSIUM (001) PLANE $B = 1.5 \times 10^5 \text{G}$

FIG. 4.8. INVERSE VELOCITY SURFACE POTASSIUM (001) PLANE $B = 1.7 \times 10^5 \text{G}$
FIG. 4.9. INVERSE VELOCITY SURFACE
POTASSIUM (001) PLANE  $B = 1.9 \times 10^5$ G

FIG. 4.10. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
POTASSIUM (001) PLANE AT 20° WITH [100] AXIS
FIG. 4. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
POTASSIUM (001) PLANE AT 45° WITH [100] AXIS
that the mode (4) shows a variation with respect to the magnetic field even at low magnetic fields and varies, besides, in a parabolic fashion. This suggests that this mode could be identified with the helicon. For propagation at an angle 20° as well as at 45° with the axis, the mode (3) shows a variation with the magnetic field, suggesting interaction between this mode and the helicon, though in the region of the magnetic fields that we have computed, the helicon never intersects an acoustic mode for potassium. At 60° also, the behaviour has been found to be similar to the case of propagation along the diagonal direction.

From the inverse velocity curves, we find that modes (3) and (4) are nearly circular and both cross the quasi-shear mode at an angle of about 10°. The longitudinal mode, which is the innermost curve of the four, has the shape of a square. As in the case of lithium, potassium also has a long spike along the diagonal direction for the quasi-shear surface and this suggests that the section of the energy surface by the (001) plane should contain a large cusp for potassium too. There is not any significant change in the shape of the graphs as the magnetic field is increased from 1.5 x 10⁵ G to 1.9 x 10⁵ G, except for a slight change in the velocities of the modes (2) and (4). In all the graphs, the curves (3) and (4) show an upward spike along the axes, suggesting a diminution in the velocities of these branches along the principal axes.
(c) Sodium

In figures (4.13) and (4.14), we give the sections of the inverse velocity surfaces for sodium, which is another metal for which extensive experimental work has been carried on, for helicon wave propagation (Grimes and Buchsbaum, 1964). Figures (4.15) to (4.18) show the variation of the phase velocities of the four modes for different angles of propagation.

The inverse phase velocity surfaces present an interesting pattern in this case. The longitudinal and pure shear modes have their shapes unchanged as a square within a circle, as in the case when helicon-phonon interaction is absent. Both the curves (2) and (4) show a spike along the diagonal direction, but along the axis, the curve (2) shows an upward maximum whereas, curve (4) has a depression. As the magnetic field increases from $1.5 \times 10^5 \text{G}$ to $2.3 \times 10^5 \text{G}$, the velocities of the modes (2) and (4) tend to increase.

It is also interesting to note that the velocities of the modes (1), (3) and (4) merge along the axis, which is unique for sodium. This is because the longitudinal and pure shear modes have velocities nearly equal to each other for sodium in the absence of interaction, since $c_{11} = 0.655 \times 10^{11} \text{dynes/cm}^2$ and $c_{44} = 0.592 \times 10^{11} \text{dynes/cm}^2$, are very nearly equal.

In figures (4.15) to (4.18), we plot the variation of the phase velocities of the different modes for different angles of propagation. The figures for the variation with regard to magnetic field, for propagation at $20^\circ$ and $45^\circ$ show that...
FIG. 4.13 INVERSE VELOCITY SURFACE
SODIUM (001) PLANE  B = 1.5 \times 10^5 G

FIG. 4.14 INVERSE VELOCITY SURFACE
SODIUM (001) PLANE  B = 2.3 \times 10^5 G
FIG. 4.15. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD SODIUM (001) PLANE AT 20° WITH [100] AXIS

FIG. 4.16. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD SODIUM (001) PLANE AT 45° WITH [100] AXIS
FIG. 4.17. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD. SODIUM (001) PLANE AT 60° WITH [100] AXIS.

FIG. 4.18. VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD. SODIUM (001) PLANE ALONG [100] AXIS.
the branches (1) and (3) are nearly unaffected by the magnetic field, whereas there is noticeable interaction between the branches (2) and (4). Though it is hard to specify which of these modes represent a helicon, it can be inferred from fig. (4.15) for propagation along the axis and fig. (4.15) that for low magnetic fields, that the branch (2) shows a parabolic variation whereas branch (4) remains unaffected by the magnetic field. We could therefore call the mode (2) as the helicon. It is interesting to note that the quasi-shear mode shows strong magnetic field dependence. The figure for variation with respect to the magnetic field at 60° is interesting. It shows that at least three branches namely (2), (3) and (4) show noticeable variation with respect to the magnetic field. Besides, the quasi-shear mode shows a maximum near \( B = 0.9 \times 10^5 \text{G} \), diminishes and then again increases from \( B = 1.3 \times 10^5 \text{G} \) onwards. Though in the region of the magnetic field from \( 0.5 \times 10^5 \text{G} \) to \( 2.3 \times 10^5 \text{G} \) that we have investigated the acoustic modes also show strong field dependence, nowhere do we obtain a resonance region, wherein a helicon intersects one of the shear modes.

(d) \( \text{Nb}_3\text{Sn} \) (Non-transforming)

We finally consider the case of \( \text{Nb}_3\text{Sn} \) which is very interesting. This metal belongs to the class of A15 compounds, which are superconductors and show interesting anomalous elastic behaviour below a particular low temperature. For all these substances, the shear mode propagating along the [110] direction with (110) polarisation, becomes soft at
low temperatures. It has been pointed out by Testardi (1973), that for Nb$_3$Sn this shear mode could not be propagated below 32°K and the relation $c_{11} = c_{12}$ holds good below this temperature. In the following paragraphs, we give the results of the helicon-phonon interaction for Nb$_3$Sn at low temperatures and we have made use of the relation $c_{11} = c_{12}$ in our calculations. Since this substance also happens to be a superconductor, with a superconducting critical temperature of 18°K, our results are valid for the low temperature range 18°K to 32°K.

In figures (4.19) and (4.20), we show the sections of inverse velocity surfaces for the four branches corresponding to the magnetic fields $B = 1.5 \times 10^5$G and $B = 2.3 \times 10^5$G. In fig. (4.19) and (4.20), the mode (2) has a long spike tending to infinity along the direction of the diagonal and this suggests that at low magnetic fields, this mode behaves like the shear mode that becomes soft below 32°K. The curve (4) has an inward dip along the principal axes. The curve (1) which is the innermost of all the four modes, has the shape of a square and without difficulty, one can assign this mode as the longitudinal wave. The curve (3) which represents the pure shear mode has a spike along the axes as well as along the face diagonal and it touches the modes (2) and (4) along the axes. Thus, along the principal axes, the velocities of three modes become equal for a magnetic field of $B = 1.5 \times 10^5$G. When the magnetic field is increased to $2.3 \times 10^5$G, very strong
FIG. 4-19 INVERSE VELOCITY SURFACE

\( \text{Nb}_3\text{Sn} \) (001) PLANE     \( B = 1.5 \times 10^5 \text{G} \)
FIG. 4: INVERSE VELOCITY SURFACE
Nb₃Sn (001) PLANE  B = 2.3 x 10⁵ G

The dotted lines represent points outside the scale of the graph.
resonant interaction takes place between the modes (2) and (4) and it is no longer possible to label them either as helicons or shear modes. In this region, these waves behave as hybrid modes, having the characteristics of both electromagnetic waves as well as acoustic waves. The figure (4.20) shows that the mode (4) assumes a very low value for higher magnetic fields at a direction of about 60° with the axis, and these curves intersect at angles of about 20°, 40°, 50° and 70°, where very strong resonant interaction could be expected.

In figures (4.21), (4.22) and (4.23), we reproduce the graphs for the variation of the four different modes with magnetic field. Along the axis, the modes (2), (3) and (4) are very close to each other in the region of magnetic fields between $1 \times 10^5$ G to $1.7 \times 10^5$ G and again, the modes (2) and (3) are very close to each other beyond this magnetic field. These suggest that helicon-phonon interaction can be experimentally observed for Nb$_3$Sn along the direction of the principal axes in this range of magnetic field. Fig. (4.22), which represents the velocity variation at an angle 20°, shows that the profiles of the modes (2) and (4) are very much distorted by the helicon-phonon interaction and these modes tend to become equal at a magnetic field of $2.3 \times 10^5$ G. Experimental work for a magnetic field of this order should therefore bring about interesting results relating to the helicon-phonon interaction. Finally, in fig. (4.23), we give
FIG. 21: VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
$\text{Nb}_3\text{Sn (001) PLANE ALONG [100] AXIS}$
FIG. 4.23 VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
\( \text{Nb}_3\text{Sn}(001) \) PLANE AT 20° WITH [100] AXIS

FIG. 4.24 VARIATION OF PHASE VELOCITY WITH MAGNETIC FIELD
\( \text{Nb}_3\text{Sn}(001) \) PLANE AT 45° WITH [100] AXIS
the profiles of the modes (1), (3) and (4) and their variation with the magnetic field at an angle of 45° with the x-axis. The mode (2) is not shown here, as it becomes soft at this angle. The figure shows that the helicon-phonon interaction alters the helicon mode also, and the helicon mode shows a minimum, at about $B = 1.1 \times 10^5 \text{G}$. The velocity of this mode increases afterwards with magnetic field till $B = 1.4 \times 10^5 \text{G}$, and thereafter it starts decreasing steadily with the magnetic field to a very low value. The above results show that Nb$_3$Sn should be one of the ideal metals to investigate the helicon-phonon interaction and in this field too, this substance will not fail to enrich us, with anomalous results for magnetic fields exceeding $1 \times 10^5 \text{G}$. 
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