Chapter 7

On a Continued Fraction of Ramanujan and Ramanujan-Weber Class Invariants

Reference [5] is based on this chapter
7.1 Introduction

The Rogers-Ramanujan continued fraction \( R(q) \) is defined by

\[
R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}} , \quad |q| < 1 ,
\]

which first appeared in a paper by Rogers [74] in 1894. This continued fraction has many representations, for example it can be expressed in terms of infinite products as follows:

\[
R(q) = q^{1/5} \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}} .
\]

Identity (7.1.2) has been proved by both Rogers [74] and also discovered by Ramanujan [70, Vol. II, Chapter 16, Section 15], [1]. Ramanujan [72, p. 50] also gave 2- and 5-dissections of this continued fraction and its reciprocal which were first proved by Andrews [12] and Hirschhorn [62], respectively. Furthermore, in his ‘lost’ notebook [72, p. 36], Ramanujan stated four \( q \)-series representations for \( R(q) \) (see [13, Entry 4.5.1, p. 121]).

On page 46 in his ‘lost’ notebook [72], Ramanujan claims that

\[
R(q) = \frac{\sqrt{5} - 1}{2} \exp \left( -1/5 \int_{q}^{1} \frac{(1-t)^5(1-t^2)^5 \cdots \cdot dt}{(1-t^5)(1-t^{10}) \cdots t} \right) ,
\]

where \( 0 < q < 1 \). Identity (7.1.3) was proved by Andrews [12]. Adiga, T. Kim, M. Naika and H. S. Madhusudhan [7] have established two integral representations for the Ramanujan’s cubic continued fraction.

The following \( q \)-series identity is Heine’s \( q \)-analogue of the Gauss \( \sb{2}F_{1} \) summation formula [61]:

\[
\sum_{n=0}^{\infty} \frac{(a; q)_{n} (b; q)_{n}}{(c; q)_{n} (q; q)_{n}} \left( \frac{c}{ab} \right)^{n} = \frac{(c/a; q)_{\infty} (c/b; q)_{\infty}}{(c; q)_{\infty} (c/(ab); q)_{\infty}} , \quad \left| \frac{c}{ab} \right| < 1 .
\]

\[
\text{(7.1.4)}
\]
In his ‘lost’ notebook [72], Ramanujan recorded many interesting continued fraction identities, for instance:

\[
G(aq, \lambda q, b; q) = \frac{1}{1 + \frac{aq + \lambda q}{1 + \frac{bq + \lambda q^2}{1 + \frac{a^2 + \lambda q^3}{1 + \frac{bq^2 + \lambda q^4}{\cdots}}}}}, \tag{7.1.5}
\]

\[
\frac{1}{1 + \frac{\lambda q - abq^2}{1 + \frac{\lambda q^2 - abq^4}{1 + \frac{\lambda q^3 - abq^6}{\cdots}}}}. \tag{7.1.6}
\]

where

\[
G(a, \lambda, b; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a; q)_n a^n}{(q; q)_n(-bq; q)_n}. \tag{7.1.7}
\]

We use the following notations:

\[
F_1(a, b, \lambda; q) := \frac{G(aq, \lambda q, b; q)}{G(a, \lambda, b; q)}, \tag{7.1.8}
\]

and

\[
F_1(a, b, -b; q) := F(a, b; q). \tag{7.1.9}
\]

For proofs of (7.1.5) and (7.1.6), see Bhargava and Adiga [28]. Identities (7.1.5) and (7.1.6) have been generalized and proved by many mathematicians including Bhargava, Adiga and D. D. Somashekara [30] and Andrews [10].

We need the following lemmas to prove our main theorems, which can be found in [14, Entry 1.3.2, p. 13] but with \(b\) replaced by \(-b\). Here, we present a slightly different proof of this lemma.

**Lemma 7.1.1.** For any complex numbers \(a\) and \(b\),

\[
\sum_{n=0}^{\infty} \frac{(b/a; q)_n a^n q^{n(n+1)/2}}{(q; q)_n(-bq; q)_n} = \frac{(-aq; q)_\infty}{(-aq/b; q)_\infty}, \quad |q| < 1. \tag{7.1.10}
\]

**Proof.** Letting \(a \to \infty\) in (7.1.4) and then replacing \(c\) by \(-cq\), we find that

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(b; q)_n (c/b)^n}{(-cq; q)_n(q; q)_n} = \frac{(-cq/b; q)_\infty}{(-cq; q)_\infty}. \tag{7.1.11}
\]

Putting \(b = c/a\) in (7.1.11) and then changing \(c\) to \(b\) in the resulting identity, we obtain (7.1.10). \(\square\)
First identity in the following lemma can be found in [56, Eq. (1.2.38), p. 6] and the second one can be verified easily.

**Lemma 7.1.2.** We have

\[(aq^{-m}; q)_n = \frac{(a; q)_n (qa^{-1}; q)_m}{(a^{-1}q^{1-n}; q)_m} q^{-mn}\] (7.1.12)

and

\[(aq^{1-n}; q)_\infty = (-a)^n q^{-n(1-n)/2} (a^{-1}; q)_n (aq; q)_\infty.\] (7.1.13)

In this chapter, we derive some \(q\)-series representations of the Ramanujan’s continued fraction \(F(a, b; q)\). In Section 7.2, we establish two \(q\)-series representations of the Ramanujan’s continued fraction \(F(a, b; q)\). Furthermore, we establish some relations between special cases of \(F(a, b; q)\). In Section 7.3, we study the special case \(N(q) := F(-1, 1; q)\) of Ramanujan’s continued fraction (7.1.5), and we establish three equivalent integral representations of \(N(q)\) and some modular equations for the continued fraction \(N(q)\). In Section 7.4, we find continued fraction representations for the Ramanujan-Weber class invariants \(g_n\) and \(G_n\) and obtain several relations between them. We also derive relations between our continued fraction \(N(q)\) with the Ramanujan-Göllnitz-Gordon and Ramanujan’s cubic continued fractions.
7.2 Main Results

In this section, we establish two new $q$-series representations of the Ramanujan’s continued fraction $F(a, b; q)$ and we use these representations to establish many relations between some particular cases of $F(a, b; q)$.

**Theorem 7.2.1.** For any complex numbers $a$ and $b \neq 0$, and $|q| < 1$, we have

\[
F(a, b; q) = -\frac{a}{b} \sum_{n=1}^{\infty} \left(\frac{aq}{b}; q\right)_n (-aq; q)_n \frac{q^n}{(-aq; q)_n} + \frac{(-bq; q)_\infty}{(-aq; q)_\infty \left(\frac{aq}{b}; q\right)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} b^n. \tag{7.2.1}
\]

**Proof.** Using (7.1.7), we may write

\[
G(aq, -bq, b; q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{(b/a; q)_n (aq)_n}{(q; q)_n (-bq; q)_n}
\]

\[
= \frac{1}{(1 - \frac{b}{aq})} \sum_{n=0}^{\infty} \left(\frac{b}{aq}; q\right)_n \frac{1 - \frac{b}{aq} q^{n-1}}{(q; q)_n (-bq; q)_n} (aq)_n
\]

\[
= -\frac{aq}{b(1 - \frac{aq}{b})} \cdot (-aq^2; q)_\infty + \frac{1}{(1 - \frac{aq}{b})} \sum_{n=0}^{\infty} \left(\frac{b}{aq}; q\right)_n \frac{q^{n(n+1)/2}}{(q; q)_n (-bq; q)_n} (aq^2)_n,
\]

where we have employed Lemma 7.1.1 in the last step with $a$ replaced by $aq$. Repeating the above manipulations $m$ times, we find that

\[
G(aq, -bq, b; q) = -\frac{a}{b(-bq; q)_\infty} \sum_{k=1}^{m} \left(\frac{aq^{k+1}}{b}; q\right)_k \frac{q^k}{(aq)_k} + \frac{1}{(\frac{aq}{b}; q)_m} \sum_{n=0}^{\infty} \left(\frac{b}{aq}; q\right)_n \frac{q^{n(n+1)/2}}{(q; q)_n (-bq; q)_n} (aq^{m+1}_n)^n.
\]

Using (7.1.12) with $a$ replaced by $\frac{b}{a}$ and after some simplifications, we obtain

\[
G(aq, -bq, b; q) = -\frac{a(-aq; q)_\infty}{b(-bq; q)_\infty} \sum_{k=1}^{m} \left(\frac{aq^k}{b}; q\right)_k \frac{q^k}{(-aq; q)_k} + \sum_{n=0}^{\infty} \left(\frac{b}{a} q^{1-n}; q\right)_m \frac{q^{n(n+1)/2}(aq)_n}{(q; q)_n (-bq; q)_n}.
\]
Letting $m \to \infty$ in the above identity, and then using (7.1.13), we deduce

$$G(aq, -bq, b; q) = \left( -aq; q \right)_\infty \left( \frac{-bq}{b}; q \right)_\infty \left( -aq; q \right)_\infty \left( \frac{-bq}{q}; q \right)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} - b^n}{(-aq; q)_n(q; q)_n}.$$ (7.2.2)

Employing Lemma 7.1.1 and (7.1.8) in (7.2.2), we complete the proof of the theorem.

To prove next our theorem, we need the following lemma, which can be found in Chapter 16 [1, Eq. (8.1), p. 8] and is called D. B. Sears’ [77, Eq. (10.1), p. 174] transformation:

**Lemma 7.2.2.** For $|de/abc|, |e/a|$ and $|q| < 1$,

$$\sum_{k=0}^{\infty} (a; q)_k(b; q)_k(c; q)_k \left( \frac{de}{abc} \right)_k = \frac{\left( e/a; q \right)_\infty \left( de/bc; q \right)_\infty}{\left( e; q \right)_\infty \left( de/abc; q \right)_\infty} \times \sum_{k=0}^{\infty} \frac{(a; q)_k(d/b; q)_k(d/c; q)_k}{(d; q)_k(de/bc; q)_k(k; q)_k} \left( \frac{e}{a} \right)_k.$$ (7.2.3)

**Theorem 7.2.3.** For any nonzero complex numbers $a$ and $b$, and $|q| < 1$, we have

$$F(a, b; q) = \frac{(-aq; q)_\infty(aq/b; q)_\infty}{(-aq; q)_\infty(aq/b; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(-aq; q)_n(q; q)_n(1+aq^{n+1})}.$$ (7.2.4)

**Proof.** Letting $a$ and $b$ tend to $\infty$ in both sides of (7.2.3) and then replacing $e, d$ and $e$ by $-aq$, $-bq$ and $-aq^2$, and after some simplifications, we deduce

$$G(aq, -bq, b; q) = \left( -aq; q \right)_\infty \sum_{k=0}^{\infty} \frac{(-b)^k q^{k(k+1)}}{(-aq; q)_k(q; q)_k(1+aq^{k+1})}.$$ (7.2.5)

Employing (7.2.5) and (7.1.8) in (7.2.2), after some simplifications, we complete the proof of the theorem.
We need the following interesting lemma:

**Lemma 7.2.4.** We have

\[
\sum_{n=0}^{\infty} q^n (-aq; q)_n (-bq; q)_n = \frac{1}{(-aq; q)_{\infty} (-bq; q)_{\infty}} \sum_{m=0}^{\infty} (-1)^{m+1} a^{-m-1} b^m q^{m(m+1)/2} \\
+ \left(1 + \frac{1}{a}\right) \sum_{m=0}^{\infty} \frac{(-1)^m a^{-m} b^m q^{m(m+1)/2}}{(-bq; q)_m}, \quad a \neq 0,
\]

(7.2.6)

\[
(-bq; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-\lambda/a; q)_n a^n q^{n(n+1)/2}}{(q; q)_n (-bq; q)_n} = (-aq; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-\lambda/b; q)_n b^n q^{n(n+1)/2}}{(q; q)_n (-aq; q)_n}.
\]

(7.2.7)

Identities (7.2.6) and (7.2.7) are due to Ramanujan [72], and their proofs can be found in [14, Entry 6.3.1, p. 115] and [13, Entry 6.2.2, p. 146], respectively. Recently, the identity (7.2.6) has been proved by Somashekara and D. Mamta [80] as a special case of more general formula. Proof of (7.2.7) can also be found in [29].

In the following interesting theorem, we establish functional equations involving the continued fraction \(F(a, b; q)\).

**Theorem 7.2.5.**

(i) If \(a \neq -1\) and \(b \neq 0\), then

\[
F(a, b; q) - \left(1 - \frac{a}{b}\right)(1 + a) F(aq^{-1}, b; q) = \frac{a}{b}.
\]

(7.2.8)

(ii) If \(a \neq -1\), \(a \neq b\) and \(b \neq 0\), then

\[
F\left(-\frac{a}{bq}, b^{-1}; q\right) + bF(aq^{-1}, b; q) = \frac{1}{(\frac{a}{b}; q)_{\infty} (-a; q)_{\infty}} \left\{ F\left(0, b^{-1}; q\right) + bF(0, b; q) \right\}.
\]

(7.2.9)
(iii) If $a \neq b$ and $b \neq 0$, then

$$
F(a, b; q) + b^{-1}(1 + a) \left(1 - \frac{a}{b}\right) F(-\frac{a}{bq}, b^{-1}; q) = \frac{a}{b} + \frac{1}{(\frac{a}{b}q; q)_{\infty}(-aq; q)_{\infty}} \left\{ F(0, b; q) + b^{-1}F(0, b^{-1}; q) \right\}.
$$

(7.2.10)

**First proof.** One can easily verify that

$$
G(aq, -bq, b; q) - \left(1 - \frac{a}{b}\right) G(a, -bq, b; q) = \frac{a}{b} G(a, -b, b; q).
$$

Dividing both sides of the above identity by $G(a, -b, b; q)$ and then using the identity

$$
\frac{G(a, b, -b; q)}{G(aq^{-1}, b, -b; q)} = \frac{1}{1 + a}, \quad a \neq -1,
$$

which follows from Lemma 7.1.1, we obtain (7.2.8).

To prove (7.2.9), we observe that, Theorem 7.2.1 can be written in the form

$$
F(a, b; q) = -\frac{a}{b} \sum_{n=1}^{\infty} \frac{q^n}{(\frac{a}{b}q; q)_n(-aq; q)_n} + \frac{F(0, b; q)}{(\frac{a}{b}q; q)_{\infty}(-aq; q)_{\infty}}.
$$

(7.2.11)

We may also rewrite Theorem 7.2.1 in the form

$$
F(-ab^{-1}, b^{-1}; q) = a \sum_{n=1}^{\infty} \frac{q^n}{(\frac{a}{b}q; q)_n(-aq; q)_n} + \frac{F(0, b^{-1}; q)}{(\frac{a}{b}q; q)_{\infty}(-aq; q)_{\infty}}.
$$

(7.2.12)

Adding (7.2.11) and (7.2.12), after some simplifications and changing $a$ by $aq^{-1}$, we deduce (7.2.9).

Identity (7.2.10) follows easily from (7.2.8) and (7.2.9) by eliminating $F(aq^{-1}, b; q)$.

**Second proof.** Replacing $a$ and $b$ by $-\frac{a}{b}$ and $a$, respectively, in (7.2.6), then changing $\beta$ by $b$ and using a special case of (7.2.7), we find that

$$
\sum_{n=0}^{\infty} \frac{q^n}{(\frac{a}{b}q; q)_n(-aq; q)_n} = \frac{b F(0, b; q)}{a(\frac{a}{b}q; q)_{\infty}(-aq; q)_{\infty}} + \left(1 - \frac{b}{a}\right) \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(-aq; q)_m}.
$$

(7.2.13)
Using (7.2.7), (7.1.8) and Lemma 7.1.1, we deduce that

\[
\sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(-aq; q)_m} = (1 + a) \binom{aq^{-1}, b; q}{a \neq -1}. \tag{7.2.14}
\]

From (7.2.11), (7.2.13) and (7.2.14), we deduce (7.2.8).

Replacing \( b \) by \( -\frac{a}{b} \) in (7.2.6) and using a special case of (7.2.7), we obtain

\[
\sum_{n=0}^{\infty} q^n \binom{aq^{-1}, b; q}{n} \frac{(-aq; q)^{n}}{(a^2; q^2)_{\infty}} = \left(1 + \frac{1}{a}\right) \sum_{m=0}^{\infty} \frac{(b^{-1})^m q^{m(m+1)/2}}{(a^2; q^2)_{\infty}}. \tag{7.2.15}
\]

Using (7.2.7), (7.1.8) and Lemma 7.1.1, we find that

\[
\sum_{m=0}^{\infty} \frac{(b^{-1})^m q^{m(m+1)/2}}{(a^2; q^2)_{\infty}} = \binom{-a b q^{-1}, b^{-1}; q}{a \neq -1}. \tag{7.2.16}
\]

From (7.2.13)-(7.2.16), we complete the proof of (7.2.9).

The proof of (7.2.10) follows easily from (7.2.11), (7.2.15) and (7.2.16).

Setting \( a = 0 \) and \( b = 1 \) in Theorem 7.2.1, and then employing \( q \)-binomial theorem in the resulting identity, we deduce that

\[
\binom{0, 1; q}{(q^2; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}. \tag{7.2.17}
\]

Using Theorem 7.2.5 and (7.2.22), it is easy to verify the following corollary:

**Corollary 7.2.6.** We have

\[
F(0, a; q) = (1 - a^2) F(aq^{-1}, 1; q) = a, \quad a \neq -1, \tag{7.2.18}
\]

\[
F(a, -1; q) - (1 + a^2) F(aq^{-1}, -1; q) = -a, \quad a \neq -1, \tag{7.2.19}
\]

\[
F(-aq^{-1}, 1; q) + F(aq^{-1}, 1; q) = \frac{2(q^2; q^2)_{\infty}}{(a^2; q^2)_{\infty}(q; q^2)_{\infty}}, \quad a \neq \pm 1, \tag{7.2.20}
\]

\[
F(a, 1; q) + (1 - a^2) F(-aq^{-1}, 1; q) = a + \frac{2(q^2; q^2)_{\infty}}{(a^2 q^2; q^2)_{\infty}(q; q^2)_{\infty}}, \quad a \neq 1. \tag{7.2.21}
\]
The following continued fraction

\[ M(q) := \frac{q^{1/8}}{1} + \frac{-q}{1} + \frac{q - q^2}{1} + \frac{-q^3}{1} + \frac{q^2 - q^4}{1} + \ldots, \quad |q| < 1, \quad (7.2.22) \]

was studied by Adiga and Kim [6]. They established an integral representation of \( M(q) \) and obtained its explicit evaluations. In addition, they derived its relationship with the Ramanujan-Göllnitz-Gordon continued fraction. Motivated by this in the next sections, we study the continued fraction

\[ N(q) := \frac{1}{1} - \frac{2q}{1} - \frac{q^2 - q}{1} - \frac{q^3 + q^2}{1} - \frac{q^4 - q^2}{1} - \ldots, \quad |q| < 1. \quad (7.2.23) \]
7.3 Integral Representations and Modular Equations for the Continued Fraction $N(q)$

In this section, we study the continued fraction $N(q)$, and we establish three equivalent integral representations and some modular equations for this continued fraction.

Our continued fraction $N(q)$ can be expressed in terms of infinite product as follows:

$$N(q) := F(-1, 1; q) = \frac{2}{(q; q^2)_\infty} - 1.$$ \hfill (7.3.1)

The above expression can be verified in two different ways, first one, by using (7.2.20) with $a = q$. The second, by using (7.2.21) with $a = -1$.

The following theorem follows easily by induction on $k$ with help of (7.2.18):

**Theorem 7.3.1.** For any positive integer $k$, we have

$$(q^2; q^2)_k \sum_{n=1}^{k} \frac{q^n}{(q^2; q^2)_n} = (q^2; q^2)_k F(-1, 1; q) - F(-q^k, 1; q).$$ \hfill (7.3.2)

In the following lemma, we establish relations involving $N(q)$ and some theta functions.

**Lemma 7.3.2.** For the continued fraction $N(q)$ defined by (7.2.23), we have

$$2N(q^2) = N(q)N(-q) + N(q) + N(-q) - 1,$$ \hfill (7.3.3)

$$2 \frac{f(-q^2)}{f(-q)} = N(q) + 1,$$ \hfill (7.3.4)

$$8 \frac{\psi(q)}{\phi(-q)} = (N(q) + 1)^2,$$ \hfill (7.3.5)

$$4 \frac{f(-q^2)}{\phi(-q)} = (N(q) + 1)^2,$$ \hfill (7.3.6)

$$\frac{\psi(q)}{\psi(-q)} = \frac{N(q) + 1}{N(-q) + 1}.$$ \hfill (7.3.7)
$$\frac{\varphi(q)}{\varphi(-q)} = \left( \frac{N(q) + 1}{N(-q) + 1} \right)^2,$$  \hspace{1cm} (7.3.8)

$$2 \frac{f(-q)}{\varphi(-q)} = N(q) + 1$$  \hspace{1cm} (7.3.9)

and

$$4 \frac{\psi(q)}{\varphi(-q^2)} = (N(q^2) + 1) (N(q) + 1).$$  \hspace{1cm} (7.3.10)

**Proof.** Identity (7.3.1) can be written in the form

$$(q; q^2)_\infty = \frac{2}{N(q) + 1}. \hspace{1cm} (7.3.11)$$

Changing $q$ to $-q$ and then multiplying the resulting identity by (7.3.11), we deduce

$$2(N(q^2) + 1) = (N(q) + 1)(N(-q) + 1), \hspace{1cm} (7.3.12)$$

which is same as (7.3.3).

In a similar way, identities (7.3.4)-(7.3.10) follow immediately from (1.2.2)–(1.2.5) and (7.3.11).

Using (7.3.4), one can easily verify the following identity by induction on $n$

$$f \left( -q^{2^n} \right) = 2^{-n} A \left( q^{2^{n-1}} \right) A \left( q^{2^{n-2}} \right) \cdots A(q^4) A(q^2) A(q), \hspace{1cm} (7.3.13)$$

where $A(q) := N(q) + 1$.

In the following theorem, we establish three equivalent integral representations for the continued fraction $N(q)$.

**Theorem 7.3.3.** For $0 < q < 1$,

$$N(q) = -1 + \exp \int \left( \frac{1}{8q} \left\{ \varphi^4(q) - 16q^2 \frac{\chi'(-q^2)}{\chi(-q^2)} - 1 \right\} \right) dq, \hspace{1cm} (7.3.14)$$

$$N(q) = -1 + \exp \int \left( \frac{1}{16q} \left\{ \varphi^4(q) + 8q \frac{\chi'(-q)}{\chi(-q)} - 16q^2 \frac{\chi'(-q^2)}{\chi(-q^2)} - 1 \right\} \right) dq \hspace{1cm} (7.3.15)$$
and

\[ N(q) = -1 + \exp \int \left( \frac{1}{12q} \left\{ \varphi^4(q) + 4q \frac{\chi'(-q)}{\chi(-q)} - 16q^2 \frac{\chi'(-q^2)}{\chi(-q^2)} - 1 \right\} \right) \, dq. \]  \hspace{1cm} (7.3.16)

where \( \varphi(q) \) and \( \chi(q) \) are as defined in (1.2.3) and (1.2.6).

Proof. Using (7.3.1), we obtain

\[ \log(N(q) + 1) = \log 2 - \sum_{n=1}^{\infty} \log(1 - q^{2n-1}). \]  \hspace{1cm} (7.3.17)

Taking the derivative of both sides, we find that

\[ \frac{d}{dq} [\log(N(q) + 1)] = \sum_{n=1}^{\infty} \frac{(2n - 1)q^{2n-2}}{1 - q^{2n-1}}. \]  \hspace{1cm} (7.3.18)

The logarithmic derivative of \( \chi(-q) = 1/(-q;q)_\infty \) is given by

\[ \frac{\chi'(-q)}{\chi(-q)} = \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1 + q^n}. \]  \hspace{1cm} (7.3.19)

Using (7.3.18) and (7.3.19) in the following identity, [24, p. 61]:

\[ \varphi^4(q) = 1 + 8 \sum_{n=1}^{\infty} \left\{ \frac{(2n - 1)q^{2n-1}}{1 - q^{2n-1}} + \frac{2nq^{2n}}{1 + q^{2n}} \right\}, \]  \hspace{1cm} (7.3.20)

we obtain

\[ \frac{d}{dq} [\log(N(q) + 1)] = \frac{1}{8q} \left\{ \varphi^4(q) - 16q^2 \frac{\chi'(-q^2)}{\chi(-q^2)} - 1 \right\}. \]  \hspace{1cm} (7.3.21)

Integrating both sides of (7.3.21) and then exponentiating, we get (7.3.14). Identities (7.3.15) and (7.3.16) follow in a similar way on using (7.3.6) and (7.3.5), respectively.

This completes the proof of the theorem. \( \square \)
Theorem 7.3.4. Define $A(q) := N(q) + 1$. Then,

\begin{align}
A^2(q) + A^2(-q) &= \frac{2A(q^4)A^3(q^2)}{A^2(-q^4)}, \quad (7.3.22) \\
A^2(q) - A^2(-q) &= 4^{-1}qA^2(q^8)A(q^4)A^3(q^2), \quad (7.3.23) \\
A^4(q) + A^4(-q) &= 8 \frac{A^6(q^2)}{A^4(-q^2)}, \quad (7.3.24) \\
A^4(q) - A^4(-q) &= 8^{-1}qA^4(q^4)A^6(q^2), \quad (7.3.25) \\
A^8(q) - A^8(-q) &= 2^{-4}qA^8(q^2). \quad (7.3.26)
\end{align}

**Proof.** Changing $q$ to $-q$ in the reciprocal of (7.3.6) and then adding the resulting identity to the reciprocal of (7.3.6), we obtain

\[ \frac{1}{4f(-q^2)} \{ \varphi(q) + \varphi(-q) \} = \frac{1}{A^2(-q)} + \frac{1}{A^2(q)}. \quad (7.3.27) \]

Using the following identity, which is due to Ramanujan [70], [1, Entry 25(i)]:

\[ \varphi(q) + \varphi(-q) = 2\varphi(q^4), \quad (7.3.28) \]

then employing (7.3.6) with $q$ replaced by $-q^4$ and (7.3.12) in (7.3.28), we find that

\[ 8 \frac{f(-q^8)}{f(-q^2)} = \left( \frac{A(-q^4)}{A(q^2)} \right)^2 \{ A^2(q) + A^2(-q) \}. \quad (7.3.29) \]

Employing (7.3.4) in the left hand-side of the above identity and after some simplifications, we deduce (7.3.22).

The proof of (7.3.23) follows in a similar way, but with using [1, Entry 25(ii)]

\[ \varphi(q) - \varphi(-q) = 4q\psi(q^8), \quad (7.3.30) \]

as well as the identities (7.3.9) and (7.3.10).

The proof of (7.3.24) follows also in a similar way, with using [1, Entry 25(vi)]

\[ \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \quad (7.3.31) \]
after, using (7.3.6) and (7.3.12).

Identity (7.3.25) follows immediately from (7.3.22) and (7.3.23) with the help of (7.3.12).

The proof of (7.3.26) follows easily from (7.3.24) and (7.3.25). This completes the proof of the theorem.

Theorem 7.3.5. Define $A(q) := N(q) + 1$. Then,

\begin{align*}
A^2(q)A^2(-q^3) + A^2(-q)A^2(q^3) &= 8 \frac{A^2(q^2)A^2(q^6)}{A(-q^2)A(-q^6)}, \\
A^2(q)A^2(-q^3) - A^2(-q)A^2(q^3) &= qA^2(q^2)A^2(q^6)A(q^4)A(q^{12}), \\
A(q)A(-q^5) + A(-q)A(q^5) &= \frac{A(q^2)A(q^4)A^2(q^{10})}{A(q^{20})}, \\
A(q)A(-q^5) - A(-q)A(q^5) &= \frac{qA^2(q^2)A(q^{10})A(q^{20})}{A(q^4)}, \\
A(q)A(-q^9) + A(-q)A(q^9) &= 4 \frac{A(q^6)f^2(-q^{12})}{f(-q^2)f(-q^{18})}, \\
A(q^8)A(-q^4) + qA(q^{24})A(-q^{12}) &= 4 \frac{A(-q^4)A(-q^{12})}{A(-q)A(q^4)}, \\
4A^2(q^4) + qA^4(q^8) &= 16 \frac{A^2(q)A^2(q^8)}{A(q^4)A^3(q^2)}, \\
\left( \frac{A(q^2)}{A(-q^5)} \right)^2 - q \left( \frac{A(q^{10})}{A(-q)} \right)^2 &= \frac{4}{A(q^5)A(q)}, \\
\left( \frac{A(q^4)}{A(-q^6)} \right)^2 + q \left( \frac{A(q^{12})}{A(-q^2)} \right)^2 &= \frac{4}{A(-q^3)A(-q)}, \\
\left( \frac{A(q^2)}{A(-q^9)} \right)^2 - q^2 \left( \frac{A(q^{18})}{A(-q)} \right)^2 &= \frac{f^2(-q^3)}{f(-q^2)f(-q^{18})}, \\
8 - qA^2(q^8)A^2(-q^4) &= 32 \frac{A^2(-q^4)}{A^2(q)A(q^2)A(q^4)}, \\
8 + qA(-q^2)A(q^4)A(-q^6)A(q^{12}) &= 32 \frac{A(-q^2)A(-q^6)}{A^2(-q)A^2(q^3)}.
\end{align*}

Proof. The following theta function identity can be found in [43]:

\begin{align*}
2 \varphi(-q^4) \varphi(-q^{12}) &= \varphi(-q^3) \varphi(q) + \varphi(q^3) \varphi(-q).
\end{align*}
Dividing both sides of (7.3.44) by $f(-q^2)f(-q^6)$ and then using (7.3.6), we obtain
\[
2 \frac{f(-q^8)f(-q^{24})}{A^2(q^4)A^2(q^{12})f(-q^2)f(-q^6)} = \frac{1}{A^2(q^3)A^2(-q)} + \frac{1}{A^2(-q^3)A^2(q)},
\]
(7.3.45)
Using (7.3.4) and (7.3.12) in the above identity, after some simplifications, we deduce (7.3.32).

The proofs of the identities (7.3.33)–(7.3.43) follow in a similar way on using Lemma 7.3.2 and the following theta function identities, respectively,

\[
4q\psi(-q^2)\psi(-q^6)\psi(-q^8) = \varphi(-q^3)\varphi(q) - \varphi(q^3)\varphi(-q),
\]
(7.3.46)
\[
2f(-q^8)f(q^{10}) = \psi(-q^5)\psi(q) + \psi(q^5)\psi(-q),
\]
(7.3.47)
\[
2qf(-q^{40})f(q^2) = \psi(-q^5)\psi(q) - \psi(q^5)\psi(-q),
\]
(7.3.48)
\[
2f(-q^{24})f(q^6) = \psi(-q^9)\psi(q) + \psi(q^9)\psi(-q),
\]
(7.3.49)
\[
\psi(-q^3)\psi(q) = \varphi(-q^{24})\psi(-q^4) + q\varphi(-q^8)\psi(-q^{12}),
\]
(7.3.50)
\[
\psi(q^8)\varphi(q) = \psi^2(q^4) + q\varphi(q^8)\psi(q^{16}),
\]
(7.3.51)
\[
\varphi(q^5)\psi(q^2) = f(-q^5)f(-q) + q\varphi(q)\psi(q^{10}),
\]
(7.3.52)
\[
\psi(q^3)\psi(q) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}),
\]
(7.3.53)
\[
\varphi(q^9)\psi(q^2) = f^2(-q^3) + q^2\varphi(q)\psi(q^{18}),
\]
(7.3.54)
\[
\varphi(-q^4)\varphi(-q) = \varphi^2(-q^8) - 2q\psi^2(-q^4),
\]
(7.3.55)
\[
\varphi(-q^3)\varphi(q) = 2q\psi(-q^6)\psi(-q^2) + \varphi(-q^{12})\varphi(-q^4),
\]
(7.3.56)

where $f(q) = f^3(-q^2)/\{f(-q)f(-q^4)\}$. Identities (7.3.46)–(7.3.56) can be found in Chapters 4 and 5. This completes the proof of the theorem.

**Theorem 7.3.6.** For any positive integers $\alpha$ and $\beta$, we have
\[
A^2(q^\alpha)A^2(q^\beta) - A^2(-q^\alpha)A^2(-q^\beta) = \frac{q^\beta}{4}B(q) \left\{ \frac{A(q^{8\beta})}{A(-q^{4\beta})} \right\}^2 + q^{\alpha-\beta} \left( \frac{A(q^{8\alpha})}{A(-q^{4\beta})} \right)^2
\]
(7.3.57)
and
\[ A^2 (q^\alpha) A^2 (q^\beta) + A^2 (-q^\alpha) A^2 (-q^\beta) = \frac{1}{32} B(q) \left\{ \left( \frac{8}{A (-q^{4\alpha}) A (-q^{4\beta})} \right)^2 + q^{\alpha+\beta} \left( A (q^{8\alpha}) A (q^{8\beta}) \right)^2 \right\}, \]

where \( B(q) := A (q^{4\alpha}) A (q^{4\beta}) A^3 (q^{2\alpha}) A^3 (q^{2\beta}). \)

**Proof.** Adding (7.3.28) and (7.3.30), we obtain
\[ \varphi(q^4) + 2q\psi(q^8) = \varphi(q). \] (7.3.59)
Changing \( q \) to \( q^\alpha \) and \( q^\beta \) in (7.3.59), and then multiplying the resulting identities, we find that
\[ \varphi(q^\alpha)\varphi(q^\beta) = \varphi(q^{4\alpha})\varphi(q^{4\beta}) + 2q^\alpha\psi(q^{8\alpha})\varphi(q^{4\beta}) + 2q^\beta\varphi(q^{4\alpha})\psi(q^{8\beta}) + 4q^{\alpha+\beta}\psi(q^{8\alpha})\psi(q^{8\beta}). \] (7.3.60)

Now, changing \( q^\alpha \) and \( q^\beta \) to \(-q^\alpha\) and \(-q^\beta\), respectively, in (7.3.60) and then subtracting the resulting identity from (7.3.60), we deduce that
\[ \varphi(q^\alpha)\varphi(q^\beta) - \varphi(-q^\alpha)\varphi(-q^\beta) = 4q^\beta \left\{ \varphi(q^{4\alpha})\psi(q^{8\beta}) + q^{\alpha-\beta}\varphi(q^{4\beta})\psi(q^{8\alpha}) \right\}. \] (7.3.61)
Dividing both sides of (7.3.61) by \( f(-q^{2\alpha}) f(-q^{2\beta}) \) and using (7.3.4), (7.3.6), (7.3.12) and simply provable identity \( 2\psi(q) = A(q)f(-q^2) \), we obtain (7.3.57).

Proof of (7.3.58) follows in a similar way, so we omit the details. \( \Box \)
7.4 Ramanujan-Weber Class invariants

In this section, we find continued fraction representations for the Ramanujan-Weber class invariants $g_n$ and $G_n$ and obtain several relations between them. Furthermore, we derive relations between our continued fraction $A(q)$ with the Ramanujan-Göllnitz-Gordon and Ramanujan’s cubic continued fractions.

If $q = e^{-\pi \sqrt{n}}$ where $n$ is a positive integer, two class invariants $G_n$ and $g_n$ [22] are defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q) \quad \text{and} \quad g_n := 2^{-1/4} q^{-1/24} \chi(-q). \quad (7.4.1)$$

In his first notebook [70], Ramanujan recorded the values for 107 class invariants. On pages 294–299 in his second notebook [70], Ramanujan gave a table of values for 77 class invariants, three of which are not found in the first notebook. By the time Ramanujan wrote his paper [67], he was aware of H. Weber’s work [85], and so his table of 46 class invariants in [67] does not contain any that are found in Weber’s book [85]. Hence Ramanujan calculated a total of 116 class invariants. In Chapter 34 [22, pp. 189-204], Berndt gave a table of all values of $G_n$ and $g_n$ found by Ramanujan.

Many other mathematicians have found several new values of $G_n$ and $g_n$.

Using (7.3.1) and (7.4.1), one can easily find that

$$A(q) = 2^{3/4} q^{-1/24} g_n^{-1} \quad \text{and} \quad A(-q) = 2^{3/4} q^{-1/24} G_n^{-1}. \quad (7.4.2)$$

Using the above two identities, it is clear that, we can find values of $A(q)$ and $A(-q)$ using known values for $g_n$ and $G_n$.

We can obtain the continued fraction representation of the Ramanujan-Weber class invariants $g_n$ and $G_n$ using (7.2.23) and (7.4.2) as follows:

$$g_n = 2^{3/4} e^{\pi \sqrt{n}/24} \frac{1}{1} + \frac{1}{1 - \frac{e^{-\pi \sqrt{n}}}{1}} - \frac{e^{-2\pi \sqrt{n}} - e^{-\pi \sqrt{n}}}{1} - \frac{e^{-3\pi \sqrt{n}} + e^{-2\pi \sqrt{n}}}{1} - \cdots, \quad (7.4.3)$$
and
\[
G_n = 2^{3/4} \frac{e^{\pi \sqrt{n}/24}}{1} + \frac{1}{1} - \frac{2 e^{-\pi \sqrt{n}}}{1} - \frac{e^{-2 \pi \sqrt{n}} + e^{-\pi \sqrt{n}}}{1} - \frac{e^{-3 \pi \sqrt{n}} + e^{-2 \pi \sqrt{n}}}{1} - \ldots.
\]  
(7.4.4)

In [67], Ramanujan stated the following two identities:

\[
g_{4n} = 2^{1/4} g_n G_n, \quad \tag{7.4.5}
\]
\[
(g_n G_n)^8 (G_n^8 - g_n^8) = \frac{1}{4}. \quad \tag{7.4.6}
\]

We can easily verify that (7.4.5) and (7.4.6) are equivalent to (7.3.12) and (7.3.26), respectively.

Many mathematicians have established several relations between \(g_n\) and \(G_n\). For example, Borwein and Borwien [38], Berndt [22] and Mahadeva Naika [66]. In the following theorem, we establish some new relations between \(G_n\) and \(g_n\).

**Theorem 7.4.1.** We have

\[
g_{9n}^2 G_n^2 + g_n^2 G_{9n}^2 = 2^{1/2} G_{4n} G_{36n}, \quad \tag{7.4.7}
\]
\[
g_{9n}^2 G_n^2 - g_n^2 G_{9n}^2 = \frac{2^{1/2}}{g_{16n} G_{144n}}, \quad \tag{7.4.8}
\]
\[
g_{25n} G_n + g_n G_{25n} = 2^{1/2} \frac{G_{100n}}{g_{16n}}, \quad \tag{7.4.9}
\]
\[
g_{25n} G_n - g_n G_{25n} = 2^{1/4} \frac{g_{16n}}{g_{16n} g_{400n}}, \quad \tag{7.4.10}
\]
\[
\frac{1}{g_n G_{81n}} + \frac{1}{g_{81n} G_n} = 2^{5/4} q^{1/6} \frac{f^2(-q^{12})}{g_{36n} f(-q^3) f(-q^{18})}, \quad \tag{7.4.11}
\]
\[
\frac{G_{144n}}{g_{64n}} + \frac{G_{16n}}{g_{576n}} = 2^{1/2} g_{9n} G_n, \quad \tag{7.4.12}
\]
\[
2^{1/2} g_{64n}^4 + g_{16n}^2 = 2 \left( \frac{g_{64n}}{g_n} \right)^2 \left( g_{4n} G_{16n} \right)^3, \quad \tag{7.4.13}
\]
\[
\frac{G_{25n}^2}{g_{4n}^2} - \frac{G_{n}^2}{g_{100n}^2} = 2^{1/2} g_{925n}, \quad (7.4.14)
\]
\[
\frac{G_{36n}^2}{g_{16n}^2} + \frac{G_{4n}^2}{g_{144n}^2} = 2^{1/2} G_{n} G_{9n}, \quad (7.4.15)
\]
\[
\frac{G_{81n}^2}{g_{4n}^2} - \frac{G_{n}^2}{g_{524n}^2} = q^{-7/12} \frac{f^2(-q^3)}{f(-q^2) f(-q^{18})}, \quad (7.4.16)
\]
\[
\frac{G_{16n}^2}{g_{64n}^2} = 2^{1/2} g_{6n}^2 G_{n}^2 G_{9n}, \quad (7.4.17)
\]
\[
(G_{n} g_{9n})^5 (G_{9n} g_{n}) - (G_{n} g_{9n}) (G_{9n} g_{n})^5 = 1, \quad (7.4.19)
\]
\[
(G_{n} g_{25n})^3 (G_{25n} g_{n}) - (G_{n} g_{25n}) (G_{25n} g_{n})^3 = 1. \quad (7.4.20)
\]

**Proof.** Identities (7.4.7) – (7.4.18) follow from (7.3.32) – (7.3.43), respectively, on using (7.4.2) and (7.4.5). Identity (7.4.19) follows by multiplying (7.4.7) by (7.4.8) and using (7.4.5). Similarly (7.4.20) follows from (7.4.9) and (7.4.10). \(\square\)

The following theorem follows from Theorem 7.3.6 with help of (7.4.2) and (7.4.5):

**Theorem 7.4.2.** We have

\[
G_{\alpha^2 n}^2 G_{\beta^2 n}^2 - g_{\alpha^2 n}^2 g_{\beta^2 n}^2 = B \left\{ \frac{G_{16\alpha^2 n}^2}{g_{64\beta^2 n}^2} + \frac{G_{16\beta^2 n}^2}{g_{64\alpha^2 n}^2} \right\} \quad (7.4.21)
\]

and

\[
G_{\alpha^2 n}^2 G_{\beta^2 n}^2 + g_{\alpha^2 n}^2 g_{\beta^2 n}^2 = B \left\{ \frac{G_{16\alpha^2 n}}{g_{64\beta^2 n}^2} G_{16\beta^2 n}^2 + \frac{1}{g_{64\alpha^2 n}^2 g_{64\beta^2 n}^2} \right\}, \quad (7.4.22)
\]

where \( B = \frac{2^{-1/2}}{G_{4\alpha^2 n}^2 G_{4\beta^2 n}^2 g_{4\alpha^2 n}^2 g_{4\beta^2 n}^2}. \)

Ramanujan-Göllnitz-Gordon continued fraction \( H(q) \) is defined by

\[
H(q) := \frac{q^{1/2}}{1 + q} + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \frac{q^6}{1 + q^7} + \cdots, \quad |q| < 1. \quad (7.4.23)
\]
In his notebook [70, p. 229], Ramanujan offered two identities for $H(q)$, namely,

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}$$  \hfill (7.4.24)

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}.$$  \hfill (7.4.25)

Using the above relations, we remark that $A(q)$ and $H(q)$ are related by the equations

$$\frac{1}{H(q)} - H(q) = \frac{64}{q^{1/2}A^4(-q^2)A^2(q^2)}$$  \hfill (7.4.26)

and

$$\frac{1}{H(q)} + H(q) = \frac{128}{q^{1/2}A^2(-q)A^3(q^2)A^2(-q^2)}.$$  \hfill (7.4.27)

From equation (7.4.26), we can compute $H(q)$, using the known values of $A(q^2)$ and $A(-q^2)$. On the other hand, we can also compute $A(q^2)$, using the known values of $H(q)$ and $A(-q^2)$.

On page 366 of his ‘lost’ notebook [72], Ramanujan recorded the continued fraction

$$G(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \ldots, \quad |q| < 1,$$  \hfill (7.4.28)

which is known as Ramanujan’s cubic continued fraction. Ramanujan [72, p. 366] offered some identities for $G(q)$ including the following identity

$$G(q) = q^{1/3} \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}.$$  

The above identity can be written in terms of our continued fraction as

$$4G(q)A(q) - q^{1/3}A^3(q^3) = 0.$$

Adiga and Kim [6] have shown that

$$M(q) = q^{1/8} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$
where $M(q)$ is given by (7.2.22). From the above identity, we remark that $M(q)$ and $A(q)$ are related by the equation

$$f(-q^2) = 2q^{-1/8} \frac{M(q)}{A(q)}, \quad (7.4.29)$$

where $f(-q)$ is as defined in (1.2.5). From equations (7.4.29), we can compute theta function $f(-q^2)$, using the known values of $A(q)$ and $M(q)$. On the other hand, we can also compute $A(q)$, using the known values of $f(-q^2)$ and $M(q)$. For example,

**Theorem 7.4.3.** [22, Entry 2, p. 326]. We have

$$f(-e^{-4\pi}) = 2^{-7/8} e^{\pi/6} \frac{\pi^{1/4}}{\Gamma(3/4)} \quad (7.4.30)$$

and

$$f(-e^{-8\pi}) = 2^{-11/8} e^{\pi/3} (\sqrt{2} - 1)^{1/2} (4 + 3\sqrt{2})^{1/8} \frac{\pi^{1/4}}{\Gamma(3/4)}, \quad (7.4.31)$$

where $\Gamma(x)$ is the classical Gamma function.

**Proof.** Employing the value $G_1 = 1$ [22, p. 189] in (7.4.5) and (7.4.6) and then using (7.4.2) we find that

$$A(e^{-2\pi}) = 2^{5/8} e^{\pi/12} \quad (7.4.32)$$

and

$$A(e^{-4\pi}) = \frac{2^{3/8} e^{\pi/6}}{(1 + \frac{3}{4} \sqrt{2})^{1/8}}. \quad (7.4.33)$$

Adiga and Kim [6] found that

$$M(e^{-2\pi}) = 2^{-5/4} \frac{\pi^{1/4}}{\Gamma(3/4)} \quad (7.4.34)$$

and

$$M(e^{-4\pi}) = 2^{-7/4} (\sqrt{2} - 1)^{1/2} \frac{\pi^{1/4}}{\Gamma(3/4)}. \quad (7.4.35)$$

Using the above four equalities in (7.4.29), we obtain the desired results.