Chapter 3

Some Modular Relations
Analogues to the Ramanujan’s Forty Identities with Its Applications to Partitions

Reference [2] is based on this chapter
3.1 Introduction

In his lost notebook [72], Ramanujan recorded forty beautiful modular relations involving the Rogers-Ramanujan functions without proof. The forty identities were first brought before the mathematical world by Birch [36]. Many of these identities have been established by Rogers [76], Watson [84], Bressoud [41, 42] and Biagioli [35]. Recently, Berndt et al. [27] offered proofs of 35 of the 40 identities. Most likely, these proofs might have been given by Ramanujan himself. A number of mathematicians tried to find new identities for the Rogers-Ramanujan functions similar to those that have been found by Ramanujan [72], including Berndt and Yesilyurt [25], Robins [73] and Gugg [58].

Two beautiful analogues to the Rogers-Ramanujan functions are the Göllnitz-Gordon functions, which are defined as

\[
S(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2} = \frac{1}{(q, q^4, q^7; q^8)_\infty}
\]  

(3.1.1)

and

\[
T(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+2n} = \frac{1}{(q^3, q^4, q^5; q^8)_\infty}.
\]  

(3.1.2)

Identities (3.1.1) and (3.1.2) can be found in Slater’s list [79]. Huang [65] has established a number of modular relations for the Göllnitz-Gordon functions. Chen and Huang [48] have derived some new modular relations for the Göllnitz-Gordon functions. Baruah, Bora and Saikia [19] offered new proofs of many of these identities by using Schröter’s formulas and some theta-function identities found in Ramanujan’s notebooks, as well as establishing some new relations. Gugg [58] found new proofs of modular relations, which involve only \(S(q)\) and \(T(q)\). Xia and Yao [86] offered new proofs of some modular relations established by Huang [65] and Chen and Huang [48]. They also established some new relations that involve only Göllnitz-Gordon functions.
Hahn [59, 60] defined the septic analogues of the Rogers-Ramanujan functions as

\[ A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2; q^2)_n (-q; q)_{2n}} = \frac{(q^7, q^3, q^4; q^7)_\infty}{(q^2; q^2)_\infty}, \tag{3.1.3} \]

\[ B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_n (-q; q)_{2n}} = \frac{(q^7, q^5, q^7; q^7)_\infty}{(q^2; q^2)_\infty} \tag{3.1.4} \]

and

\[ C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_n (-q; q)_{2n+1}} = \frac{(q^7, q^6, q^7; q^7)_\infty}{(q^2; q^2)_\infty}. \tag{3.1.5} \]

Identities (3.1.3), (3.1.4) and (3.1.5) are due to Rogers [75]. Later, Slater [79] offered different proofs of these identities. Hahn [59, 60] discovered and established several modular relations involving only \( A(q) \), \( B(q) \) and \( C(q) \), as well as relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.

Baruah and Bora [18] considered the following nonic analogues of the Rogers-Ramanujan functions:

\[ D(q) := \sum_{n=0}^{\infty} \frac{(q; q)_3 q^{3n^2}}{(q^3; q^3)_n (q^3; q^3)_{2n}} = \frac{(q^4, q^5, q^9; q^9)_\infty}{(q^3; q^3)_\infty}, \tag{3.1.6} \]

\[ E(q) := \sum_{n=0}^{\infty} \frac{(q; q)_{3n} (1 - q^{3n+2}) q^{3n(n+1)}}{(q^3; q^3)_n (q^3; q^3)_{2n+1}} = \frac{(q^2, q^7, q^9; q^9)_\infty}{(q^3; q^3)_\infty} \tag{3.1.7} \]

and

\[ F(q) := \sum_{n=0}^{\infty} \frac{(q; q)_{3n+1} q^{3n(n+1)}}{(q^3; q^3)_n (q^3; q^3)_{2n+1}} = \frac{(q, q^8, q^9; q^9)_\infty}{(q^3; q^3)_\infty}. \tag{3.1.8} \]

Identities (3.1.6), (3.1.7) and (3.1.8) are due to W. N. Bailey [16]. Baruah and Bora [18] established several modular relations involving \( D(q) \), \( E(q) \) and \( F(q) \). They also established some modular identities involving quotients of these functions, as well as relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.
Adiga, Vasuki and Bhaskar [8] established several modular relations for the following cubic functions:

\[ P(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2n}}{(q^4; q^4)_n} = \frac{(-q; q^2)_\infty (q^6, q, q^5; q^6)_\infty}{(q^2; q^2)_\infty} \]  

(3.1.9)

and

\[ Q(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2; q^2)_n} = \frac{(-q; q^2)_\infty (q^2, q^6; q^6)_\infty}{(q^2; q^2)_\infty}. \]  

(3.1.10)

The identities (3.1.9) and (3.1.10) can be found in [10].

Vasuki, Sharath and Rajanna [83], studied two different cubic functions defined by

\[ L(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^4; q^4)_n} = \frac{(-q; q^2)_\infty (q, q^5, q^6; q^6)_\infty}{(q^2; q^2)_\infty} \]  

(3.1.11)

and

\[ M(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^4; q^4)_n} = \frac{(-q; q^2)_\infty (q^3, q^6; q^6)_\infty}{(q^2; q^2)_\infty}. \]  

(3.1.12)

The identities (3.1.11) and (3.1.12) are due to Andrews [10] and Slater [79], respectively. Vasuki, Sharath and Rajanna [83] derived some modular relations involving \( L(q) \) and \( M(q) \).

Vasuki and Guruprasad [82] considered the following Rogers-Ramanujan type functions \( U(q) \) and \( V(q) \) of order twelve and established modular relations involving them:

\[ U(q) := \sum_{n=0}^{\infty} \frac{(q; q^2)^2_{2n} q^{4n^2}}{(q^4; q^4)_{2n}} = \frac{(q^5, q^7, q^{12}; q^{12})_\infty}{(q^4; q^4)_\infty} \]  

(3.1.13)

and

\[ V(q) := \sum_{n=0}^{\infty} \frac{(q; q^2)^2_{2n+1} q^{4n(n+1)}}{(q^4; q^4)_{2n+1}} = \frac{(q, q^{11}, q^{12}; q^{12})_\infty}{(q^4; q^4)_\infty}. \]  

(3.1.14)
The latter equalities in (3.1.13) and (3.1.14) are due to Slater [79].

Adiga, Vasuki and Srivatsa Kumar [9] established modular relations involving the functions $S_1(q)$ and $T_1(q)$ defined by

\[ S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^4)_\infty} \]  

and

\[ T_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2;q^4)_\infty}{(q,q^3;q^4)_\infty}. \]

Baruah and Bora [17] considered the following two functions, which are analogous to the Rogers-Ramanujan functions:

\[ X(q) := \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n(1 - q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q, q^{11}, q^{12}; q^{12})_\infty}{(q;q)_\infty} \]

and

\[ Y(q) := \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n-1}(1 + q^n)q^{n^2}}{(q;q)_{2n}} = \frac{(q^5, q^7, q^{12}; q^{12})_\infty}{(q;q)_\infty}, \]

where the later equalities are also due to Slater [79] and are called dodecic analogues of the Rogers-Ramanujan functions. Robins [73] in his Ph.D. thesis has established four modular relations for dodecic analogues functions using modular forms. Later, Baruah and Bora [17] established many modular relations involving some combinations of $X(q)$ and $Y(q)$, as well as relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordon functions, septic analogues and with nonic analogues functions. More recently, in his Ph.D. thesis [58], Gugg has given alternative proofs of Robins’s [73] four identities.

In Chapter 2, we have established a large class of modular relations for the functions defined by

\[ J(q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(q^2;q^2)_{n+1}(q;q)_n} = \frac{(-q; q)_\infty (q^3, q^7, q^{10}; q^{10})_\infty}{(q; q)_\infty} \]  

(3.1.19)
and

\[ K(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+3)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty (q,q^9,q^{10};q^{10})_\infty}{(q;q)_\infty}. \]  

The identities (3.1.19) and (3.1.20) are due to Rogers [75]. In Section 3.3 of this chapter, we establish modular relations connecting \( J(q) \) and \( K(q) \) with \( G(q) \) and \( H(q) \). In Section 3.4, we establish modular relations connecting \( J(q) \) and \( K(q) \) with \( S(q) \) and \( T(q) \). In Section 3.5, we establish modular relations connecting \( J(q) \) and \( K(q) \) with \( P(q) \) and \( Q(q) \). In Section 3.6, we give partition theoretic interpretations for some of our modular relations.
3.2 Some Preliminary Results

It is easy to verify that

\begin{align*}
G(q) &= \frac{f(-q^2, -q^3)}{f_1}, & H(q) &= \frac{f(-q, -q^4)}{f_1}, \\
J(q) &= \frac{f(-q^3, -q^7)}{\varphi(-q)}, & K(q) &= \frac{f(-q, -q^9)}{\varphi(-q)}, \\
S(-q) &= \frac{f(q^3, q^5)}{\psi(q)}, & T(-q) &= \frac{f(q, q^7)}{\psi(q)}, \\
P(q) &= \frac{f(-q, -q^5)}{\psi(-q)}, & Q(q) &= \frac{f(-q^2, -q^4)}{\psi(-q)}, \\
G(q)H(q) &= \frac{f_5}{f_1} \quad \text{and} \quad J(q)K(q) &= \frac{f_2 f_3^2}{f_1^2 f_5}. \quad (3.2.5)
\end{align*}

Lemma 3.2.1.

\[ f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5). \quad (3.2.6) \]

Identity (3.2.6) can be found in [1] as a corollary of Entry 28.

The following two identities follow from (2.2.16) by setting \( k = 2, a = q, q^2 \) and \( b = q^4, q^3 \), respectively:

\begin{align*}
&f(q, q^4) = f(q^7, q^{13}) + q f(q^3, q^{17}), & \quad (3.2.7) \\
&f(q^2, q^3) = f(q^9, q^{11}) + q^2 f(q, q^{19}). & \quad (3.2.8)
\end{align*}

To prove some of our results, we need the following two Schröter’s formulas, which can be found in [1]. We assume that \( \mu \) and \( \nu \) are integers, such that \( \mu > \nu \geq 0 \).

Lemma 3.2.2.

\begin{align*}
2 \psi(q^{2\mu+2\nu}) \psi(q^{2\mu-2\nu}) \\
= \sum_{m=0}^{\mu-1} q^{2m^2 + 2m} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)}) f(q^{4\mu+4\nu m}, q^{-4\nu m}). \quad (3.2.9)
\end{align*}
Lemma 3.2.3. If $\mu$ is odd, then

\[
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = q^{\mu^3/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)})f(q^{\mu+\mu\nu}q^{\mu-\mu\nu}) \\
+ \frac{(\mu-3)/2}{\sum_{m=0}^{(\mu-3)/2}} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) \\
\times f(q^{\mu+2\nu m}, q^{\mu-2\nu m}).
\]
3.3 Identities Connecting $J(q)$ and $K(q)$ with Rogers-Ramanujan Functions $G(q)$ and $H(q)$

In this section, we present some modular relations that are connecting $J(q)$ and $K(q)$ with Rogers-Ramanujan functions $G(q)$ and $H(q)$.

**Theorem 3.3.1.** We have

$$\frac{J(q)J^2(q^2) + q^3 K(q)K^2(q^2)}{K(q)G^2(q^4) + qJ(q)H^2(q^4)} = \frac{f_4^4}{f_2^2}. \quad (3.3.1)$$

**Proof.** Putting $a = -q^3$, $b = -q^7$ and $c = d = q^5$ in (2.2.21), we obtain

$$f(-q^3, -q^7)f(q^5, q^5) = f^2(-q^8, -q^{12}) - q^3 f^2(-q^2, -q^{18}). \quad (3.3.2)$$

Dividing (3.3.2) throughout by $\varphi^2(-q^2)$, employing (3.2.1) and (3.2.2) and then using the Lemma 2.2.1, we obtain

$$\frac{f_4^2 f_1^2}{f_2^2} J(q)\varphi(q^5) = \frac{f_4^4}{f_2^4} G^2(q^4) - q^3 K^2(q^2). \quad (3.3.3)$$

Setting $a = -q$, $b = -q^9$ and $c = d = q^5$ in (2.2.21) and after simplifications, we obtain

$$\frac{f_4^2 f_1^2}{f_2^2} K(q)\varphi(q^5) = J^2(q^2) - q \frac{f_4^4}{f_2^4} H^2(q^4). \quad (3.3.4)$$

Now, dividing (3.3.3) by (3.3.4), we deduce

$$\frac{J(q)}{K(q)} = \frac{\frac{f_4^4}{f_2^4} G^2(q^4) - q^3 K^2(q^2)}{J^2(q^2) - q \frac{f_4^4}{f_2^4} H^2(q^4)}, \quad (3.3.5)$$

from which we obtain (3.3.1). \hfill \Box

**Theorem 3.3.2.** We have

(i) \quad $G^2(q)J(-q) + qH^2(q)K(-q) = \frac{f_5}{f_1} = G(q)H(q)$,

(ii) \quad $G^2(q)J(-q) - qH^2(q)K(-q) = \frac{f_1 f_2 f_5}{f_2 f_5 f_2}$. 
Proof. We recall the following identity stated by Ramanujan [72] and proved by Rogers [76], Watson [84] and Berndt et al. [27]:

\[ G(q)G(q^4) + qH(q)H(q^4) = \chi^2(q). \]  

(3.3.6)

We can write (3.3.6) in the form

\[ \frac{G(q)G(q^4)}{H(q)H(q^4)} = \frac{\chi^2(q)}{H(q)H(q^4)} - q. \]  

(3.3.7)

Now, setting \( m = 2 \) and \( r = 2 \) in (2.2.23) and (2.2.24) and multiplying the resulting equations by \( G(q) \) and \( H(q) \), respectively, we obtain

\[ G(q)G(q^4) = G^2(q) \frac{f(-q^5)f(-q^{10})}{f(q^2,q^8)} \]  

(3.3.8)

and

\[ H(q)H(q^4) = H^2(q) \frac{f(-q^5)f(-q^{10})}{f(q^2,q^3)f(q^4,q^6)}. \]  

(3.3.9)

Dividing (3.3.8) by (3.3.9) and then employing (3.3.7), we find that

\[ \frac{\chi^2(q)}{H(q)H(q^4)} - q = \frac{G^2(q)f(q^2,q^3)f(q^4,q^6)}{H^2(q)f(q^4,q^6)f(q^2,q^8)}. \]  

(3.3.10)

Now, we show that

\[ \frac{f(q^2,q^3)f(q^4,q^6)}{f(q^4,q^6)f(q^2,q^8)} = \frac{J(-q)}{K(-q)}. \]  

(3.3.11)

By (1.2.2), we have

\[ \frac{f(q^2,q^3)f(q^4,q^6)}{f(q^4,q^6)f(q^2,q^8)} = \frac{(-q^2,-q^3,-q^5;q^5)_\infty(-q^4,-q^6,-q^{10};q^{10})_\infty}{(-q,-q^4,q^5;q^5)_\infty(-q^2,-q^3,q^5,q^6,q^7,-q,\ldots,q^9,-q^{10};q^{10})_\infty} \]

\[ \times \frac{(-q^3,-q^7;q^{10})_\infty}{(-q,-q^9;q^{10})_\infty} \]

\[ = \frac{f(q^3,q^7)}{f(q^9)} = \frac{J(-q)}{K(-q)}. \]
Now, using (3.3.11) in (3.3.10), we obtain
\[ G^2(q)J(-q) + qH^2(q)K(-q) = \frac{\chi^2(q)H(q)K(-q)}{H(q^4)}. \] (3.3.12)

It remains for us to show that
\[ \frac{\chi^2(q)H(q)K(-q)}{H(q^4)} = \frac{f_5}{f_1} = G(q)H(q). \] (3.3.13)

Using (2.1.2), (3.2.2) and Lemma 2.2.1, we see that
\[ \frac{\chi^2(q)H(q)K(-q)}{H(q^4)} = \frac{f_{10}}{f_2} \frac{(-q, -q^9; q^{10})_{\infty}(q^8, q^{12}; q^{20})_{\infty}}{(q, -q, q^4, q^7, q^9, -q^9; q^{10})_{\infty}(q^8, q^{12}; q^{20})_{\infty}} \]
\[ = \frac{f_{10} \chi(-q^5)}{f_2 \chi(-q)} = \frac{f_5}{f_1} = G(q)H(q). \]

This completes the proof of (i).

To prove (ii), we need the following identity stated by Ramanujan [72], the proof of which can be found in [84] and [27, Entry 3.3]:
\[ G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{f(-q^2)}. \] (3.3.14)

We can write (3.3.14) in the form
\[ \frac{G(q)G(q^4)}{H(q)H(q^4)} - q = \frac{\varphi(q^5)}{f(-q^2)H(q)H(q^4)}. \] (3.3.15)

Now, employing (3.3.8) and (3.3.9) in (3.3.15) and then use (3.3.11) to obtain
\[ G^2(q)J(-q) - qH^2(q)K(-q) = \frac{\varphi(q^5)H(q)K(-q)}{f_2 H(q^4)}. \] (3.3.16)

On employing (3.3.13) in (3.3.16) and then using Lemma 2.2.1, we obtain (ii). This completes the proof of the theorem.

\[ \square \]

**Theorem 3.3.3.** We have

(i) \[ K(q)G(q)G(q^2) - J(q)H(q)H(q^2) = 0, \]

(ii) \[ K(q)G(q)G(q^2) + J(q)H(q)H(q^2) = 2 \frac{f_{10}^2}{f_1^2}. \]
Proof. Using (1.2.2), we have

\[ f(-q, -q^4) = (q; q^5)_\infty (q^4; q^5)_\infty (q^5; q^5)_\infty = (q; q^{10})_\infty (q^9; q^{10})_\infty (q^4; q^{10})_\infty (q^6; q^{10})_\infty (q^6; q^{10})_\infty = \frac{f(-q, -q^9)f(-q^4, -q^6)}{f_{10}^2}. \]

Now employing (3.2.1) and (3.2.2) in the last equality, we obtain

\[ \frac{f_{10}^2}{f_{11}^2} = K(q)G(q^2)G(q). \] \hspace{1cm} (3.3.17)

Similarly, we can show that

\[ \frac{f_{10}^2}{f_{11}^2} = J(q)H(q^2)H(q). \] \hspace{1cm} (3.3.18)

Now, (i) and (ii) easily follow from (3.3.17) and (3.3.18). \qed

We prove the following theorem using ideas similar to those of Watson \cite{84}.

**Theorem 3.3.4.** Let \( J_*(q) := H(q)G(q^2) \) and \( K_*(q) := G(q)H(q^2) \), then

\begin{align*}
J(-q^3)J_*(q) + q^2 K(-q^3)K_*(q) &= \frac{f_{12}^2 f_{10} f_{12}}{f_2 f_5 f_6^4} \left\{ 2 \frac{f_{12}^2 f_6^2}{f_2 f_3^2 f_6^2} - \frac{f_{10}^2 f_{30}}{f_5 f_{15}^2 f_{60}^2} \right\}, \\
J(-q^8)J_*(q) + q^5 K(-q^8)K_*(q) &= \frac{f_{12}^2 f_{10} f_{12}^2}{q f_2 f_5 f_5^5} \left\{ \frac{f_{12}^2 f_5^2}{f_2 f_5 f_1^2} - \frac{f_{18}^2 f_{80}}{f_5 f_{20}^2 f_{160}^2} \right\}, \\
J(-q^2)K_*(q) + q K(-q^2)J_*(q) &= \frac{f_{12}^2 f_{12}^2}{f_2 f_5 f_{10}^2} \left\{ \frac{f_{12}^2 f_5^2}{f_2 f_5 f_4} - \frac{f_{20}^2}{f_5 f_{20}^2} \right\}, \\
J(q)J_*(q^{12}) - q^3 K(q)K_*(q^{12}) &= \frac{f_{12} f_{12} f_{60}}{f_{11} f_{24} f_{60}} \left\{ \frac{f_{12}^2 f_5^2}{f_6 f_4} - q^7 \frac{f_{16} f_{120}}{f_5 f_{20} f_{60}} \right\}. \tag{3.3.22}
\end{align*}

Proof. Using (3.2.1), (3.2.2), (3.2.6) and Lemma 2.2.1, we can write (3.3.19) in the form

\begin{align*}
f(q^9, q^{21})f(-q, -q^4)f(-q^4, -q^6) + q^2 f(q^3, q^{27})f(-q^2, -q^3)f(-q^2, -q^8) \\
= \frac{f(-q^2, -q^3)f(-q, -q^4)}{\varphi(-q^5)} \left\{ 2 \psi(q^2)\psi(q^3) - \psi(q^5)\varphi(q^{15}) \right\}. \tag{3.3.23}
\end{align*}
Setting \( a = q, q^2 \) and \( b = q^4, q^3 \), respectively, in (2.2.22) and then employing the resulting identities in (3.3.23), we obtain

\[
f(q^2, q^3)f(q^9, q^21) + q^2 f(q, q^4)f(q^3, q^{27}) = 2\psi(q^2)\psi(q^3) - \psi(q^5)\varphi(q^{15}). \tag{3.3.24}
\]

Thus, it suffices to establish the identity (3.3.24). Using (2.2.2) and (1.2.4), we have

\[
4\psi(q^3)\psi(q^2) = f(1, q^3)f(1, q^2) = \sum_{m,n=-\infty}^{\infty} q^{(3m^2 + 3m + 2n^2 + 2n)/2}. \tag{3.3.25}
\]

In this representation, we make the change of indices by setting

\[
3m - 2n = 5M + a \quad \text{and} \quad m + n = 5N + b,
\]

where \( a \) and \( b \) have values selected from the set \( \{0, \pm 1, \pm 2\} \). Then

\[
m = M + 2N + (a + 2b)/5 \quad \text{and} \quad n = -M + 3N + (3b - a)/5.
\]

It follows that values of \( a \) and \( b \) are associated, as in the following table:

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<th>±1</th>
<th>±2</th>
</tr>
</thead>
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<td>±1</td>
<td>±2</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>±2</td>
<td>±1</td>
</tr>
<tr>
<td>m</td>
<td>M + 2N</td>
<td>M + 2N ± 1</td>
<td>M + 2N</td>
</tr>
<tr>
<td>n</td>
<td>−M + 3N</td>
<td>−M + 3N ± 1</td>
<td>−M + 3N ± 1</td>
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</tbody>
</table>

When \( a \) assumes the values \(-2, -1, 0, 1\) and \( 2 \) in succession, it is easy to see that the corresponding values of \( 3m^2 + 3m + 2n^2 + 2n \) are, respectively,

\[
5M^2 - 3M + 30N^2 + 24N + 4,
\]

\[
5M^2 - M + 30N^2 - 12N,
\]

\[
5M^2 + M + 30N^2 + 12N,
\]

\[
5M^2 + 3M + 30N^2 + 36N + 10,
\]

\[
5M^2 + 5M + 30N^2.
\]
It is evident, from the equations connecting \( m \) and \( n \) with \( M \) and \( N \) that, there is a one-one correspondence between all pairs of integers \((m, n)\) and all sets of integers \((M, N, a)\). From this correspondence, we can write (3.3.25) as

\[
4\psi(q^3)\psi(q^2) = q^2 \sum_{M,N=-\infty}^{\infty} q^{(5M^2-3M+30N^2+24N)/2}
+ \sum_{M,N=-\infty}^{\infty} q^{(5M^2-M+30N^2-12N)/2} + \sum_{M,N=-\infty}^{\infty} q^{(5M^2+M+30N^2+12N)/2}
+ q^5 \sum_{M,N=-\infty}^{\infty} q^{(5M^2+3M+30N^2+36N)/2} + \sum_{M,N=-\infty}^{\infty} q^{(5M^2+5M+30N^2)/2}
\]

\[
= q^2 f(q, q^4) f(q^3, q^{27}) + f(q^2, q^3) f(q^0, q^{21}) + f(q^2, q^3) f(q^0, q^{21})
+ q^5 f(q, q^4) f(q^{-3}, q^{33}) + f(q^0, q^5) f(q^{15}, q^{15}).
\]

Upon using Lemma 2.2.2 and after some simplifications, we get (3.3.24). This completes the proof of (3.3.19).

Using (3.2.1), (3.2.2), (3.2.6) and Lemma 2.2.1, we find that (3.3.20) is equivalent to the identity

\[
\psi(q) = f(q^2, q^3) f(q^4, q^{12}) = \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+8n^2+4n}.
\]
\[ m + 4n = 5M + a \quad \text{and} \quad m - n = 5N + b, \]

where \( a \) and \( b \) have values selected from the set \( \{0, \pm 1, \pm 2\} \). Then

\[ m = M + 4N + (a + 4b)/5 \quad \text{and} \quad n = M - N + (a - b)/5. \]

It follows easily that \( a = b \), and so \( m = M + 4N + a \) and \( n = M - N \), where \(-2 \leq a \leq 2\). Thus, there is one-to-one correspondence between the set of all pairs of integers \((m, n)\), \(-\infty < m, n < \infty \) and triples of integers \((M, N, a)\), \(-\infty < M, N < \infty, \ -2 \leq a \leq 2\).

From (3.3.28), we find that

\[
\psi(q)\psi(q^4) = \sum_{a=-2}^{2} q^{2a^2+a} \sum_{M=-\infty}^{\infty} q^{10M^2+(4a+5)M} \sum_{N=-\infty}^{\infty} q^{40N^2+16aN}
\]

\[
= \sum_{a=-2}^{2} q^{2a^2+a} f(q^{15+4a}, q^{5-4a}) f(q^{40+16a}, q^{40-16a})
\]

\[
= q^6 f(q^8, q^{72}) \{ f(q^7, q^{13}) + q f(q^3, q^{17}) \}
\]

\[
+ q f(q^{24}, q^{56}) \{ f(q^9, q^{11}) + q^2 f(q, q^{10}) \} + \psi(q^5)\varphi(q^{40}). \tag{3.3.29}
\]

Employing (3.2.7) and (3.2.8) in (3.3.29), we obtain (3.3.27). The proofs of (3.3.21) and (3.3.22) follow similarly.

\[ \square \]

**Theorem 3.3.5.** We have

\[ J(q^2)K_\ast(q) + qK(q^2)J_\ast(q) = \frac{f_2^2 f_{20}}{f_1^2 f_4} \tag{3.3.30} \]

and

\[ J(q^2)K_\ast(q) - qK(q^2)J_\ast(q) = \frac{f_1 f_{10}^5}{f_2^3 f_5^2 f_{20}}, \tag{3.3.31} \]

where \( J_\ast(q) \) and \( K_\ast(q) \) are as defined in theorem 3.3.4.

**Proof.** Using (2.1.1), we have

\[
G(q) = \frac{1}{(q; q^{10})_\infty(q^4; q^{10})_\infty(q^6; q^{10})_\infty(q^9; q^{10})_\infty} = \frac{f_{10} H(q^2)}{\varphi(-q)K(q)}. \tag{3.3.32}
\]
Identity (3.3.32) can be written in the form

\[ H(q^2) = G(q)K(q)\frac{\varphi(-q)}{f_{10}}. \]  \hfill (3.3.33)

Similarly, we have

\[ G(q^2) = H(q)J(q)\frac{\varphi(-q)}{f_{10}}. \]  \hfill (3.3.34)

Replacing \( q \) by \( q^2 \) in (3.3.33) and (3.3.34) and then employing the resulting identities in (3.3.6) and (3.3.14), we get (3.3.30) and (3.3.31), respectively.

\[ \text{Proof.} \] Using (3.2.1), (3.2.2), (3.2.6) and Lemma 2.2.1, we find that (3.3.35) is equivalent to the identity

\[ f(-q^3, -q^7)f(-q^2, -q^8)f(-q^8, -q^{12}) - qf(-q, -q^9)f(-q^4, -q^6)f(-q^4, -q^{16}) \]

\[ = \frac{f(-q^2, -q^8)f(-q^4, -q^6)}{\varphi(-q^{10})}\{\varphi(-q)\psi(q^2) + q\varphi(-q^5)\psi(q^{10})\}. \]  \hfill (3.3.37)

Putting \( a = q^2, q^4 \) and \( b = q^8, q^6 \), respectively, in (2.2.22) and then using the resulting identities in (3.3.37), we obtain

\[ f(-q^3, -q^7)f(q^4, q^6) - qf(-q, -q^9)f(q^2, q^8) = \varphi(-q)\psi(q^2) + q\varphi(-q^5)\psi(q^{10}). \]  \hfill (3.3.38)

Thus (3.3.38) is equivalent to (3.3.35). To prove (3.3.38), we employ Theorem 2.2.8 with the parameters \( a = b = q, c = q^2, d = q^6, \epsilon_1 = 1, \epsilon_2 = 0, \alpha = 1, \beta = 4 \) and
\( m = 5 \) and then using Lemma 2.2.2, we get

\[
\varphi(-q)\psi(q^2) = f(-q^3, -q^7)f(q^{18}, q^{22}) + q^2f(-q^{-1}, -q^{11})f(q^{14}, q^{26})
\]

\[
+ q^{12}f(-q^{-9}, -q^{19})f(q^{6}, q^{34}) + q^{30}f(-q^{-17}, -q^{27})f(q^{-2}, q^{42})
\]

\[
+ q^{56}f(-q^{-25}, -q^{35})f(q^{-10}, q^{50})
\]

\[
= f(-q^3, -q^7)\{f(q^{18}, q^{22}) + q^4f(q^{2}, q^{38})\}
\]

\[-qf(-q, -q^9)\{f(q^{14}, q^{26}) + q^2f(q^{6}, q^{34})\} - q\varphi(-q^5)\psi(q^{10}). \quad (3.3.39)
\]

Changing \( q \) to \( q^2 \) in (3.2.7) and (3.2.8) and then employing the resulting identities in (3.3.39), we obtain (3.3.38). The proof of (3.3.36) follows similarly.

**Theorem 3.3.7.** We have

\[
J(-q^{12})K_*(q) + q^7K(-q^{12})J_*(q) = \frac{f_{10}f_{12}f_{18}}{q^2f_2f_5f_{24}} \left\{ \frac{f_4f_6^2}{f_2f_3} - \frac{f_{10}f_{12}^5}{f_5f_{20}f_{240}} \right\}, \quad (3.3.40)
\]

\[
J(q^{21})J_*(q) - q^{13}K(q^{21})K_*(q^2) = \frac{f_{20}f_{42}}{q^3f_4f_{10}f_{21}} \left\{ \frac{f_{20}f_{105}^2}{f_{210}f_{10}} - \frac{f_3f_7f_{12}f_{28}}{f_6f_{14}} \right\}, \quad (3.3.41)
\]

\[
J(q^9)K_*(q^2) - q^5K(q^9)J_*(q^2) = \frac{f_{2}f_{18}}{q^4f_5f_{10}} \left\{ \frac{f_{20}f_{105}^2}{f_{10}f_{90}} - \frac{f_1f_4f_{9}f_{36}}{f_2f_{18}} \right\}, \quad (3.3.42)
\]

\[
J(q)J_*(q^{42}) - q^9K(q)K_*(q^{42}) = \frac{f_{2}f_{120}}{q^4f_{10}f_{210}} \left\{ \frac{f_3f_7f_{12}f_{28}}{f_6f_{14}} + q^{25}f_{25}^2f_{240}^2 \right\}, \quad (3.3.43)
\]

\[
J(q)K_*(q^{18}) - q^{-3}K(q)J_*(q^{18}) = \frac{f_{2}f_{180}}{q^3f_2f_{36}f_{90}} \left\{ q^{10}f_{10}^2f_{180}^2 - \frac{f_1f_4f_{9}f_{36}}{f_2f_{18}} \right\}. \quad (3.3.44)
\]

**Proof.** Using (3.2.1), (3.2.2), (3.2.6) and Lemma 2.2.1, one can write (3.3.40) in the form

\[
f(-q^2, -q^3)f(-q^2, -q^8)f(q^{36}, q^{84}) + q^7f(-q, -q^4)f(-q^4, -q^6)f(q^{12}, q^{108})
\]

\[
= \frac{f(-q, -q^4)f(-q^2, -q^3)}{q^2\varphi(-q^5)} \{\psi(q^2)\psi(q^3) - \psi(q^5)\varphi(q^{60})\}. \quad (3.3.45)
\]

Employing (2.2.22) with \( a = q, q^2 \) and \( b = q^4, q^3 \), respectively, in (3.3.45), we find that

\[
\psi(q^2)\psi(q^3) - \varphi(q^{60})\psi(q^5) = q^2f(q, q^4)f(q^{36}, q^{84}) + q^0f(q^2, q^3)f(q^{12}, q^{108}). \quad (3.3.46)
\]
Thus, (3.3.40) is equivalent to (3.3.46). However, identity (3.3.46) can be verified easily using (3.2.9) with setting $\mu = 5$ and $\nu = 1$ and then changing $q^4$ to $q$ in the resulting identity. The proofs of (3.3.41) and (3.3.42) follow similarly using (3.2.9) with setting $\mu = 5$, $\nu = 2$ and $\nu = 4$, respectively. In a similar way, identities (3.3.43) and (3.3.44) can be established using (3.2.10) with setting $\mu = 5$, $\nu = 2$ and $\nu = 4$, respectively.

**Observation 3.3.8.** In most of the above identities, the functions $J(q)$, $K(q)$, $J_*(q)$ and $K_*(q)$ occur in combinations,

\[
J(q^r)J_*(q^s) - q^{(3r+s)/5}K(q^r)K_*(q^s), \quad \text{where } 3r + s \equiv 0(\text{mod } 5), \tag{3.3.47}
\]

\[
J(q^r)K_*(q^s) + q^{(3r-s)/5}K(q^r)J_*(q^s), \quad \text{where } 3r - s \equiv 0(\text{mod } 5), \tag{3.3.48}
\]

or when one or both of $q^r$ and $q^s$ are replaced by $-q^r$ and $-q^s$, respectively, in either (3.3.47) or (3.3.48).
3.4 Identities Connecting $J(q)$ and $K(q)$ with Göllnitz-Gordon Functions $S(q)$ and $T(q)$

In this section, we present relations involving some combinations of $J(q)$ and $K(q)$ with the Göllnitz-Gordon functions $S(q)$ and $T(q)$.

**Theorem 3.4.1.** Define

\[
U(\alpha, \beta) := J(q^\alpha)J(q^\beta) + q^{3(\alpha+\beta)/5}K(q^\alpha)K(q^\beta),
\]

\[
U^*(\alpha, \beta) := K(q^\alpha)J(q^\beta) + q^{3(-\alpha+\beta)/5}J(q^\alpha)K(q^\beta),
\]

\[
V(\alpha, \beta) := S(-q^\alpha)S(-q^\beta) + q^{(\alpha+\beta)/2}T(-q^\alpha)T(-q^\beta),
\]

\[
V^*(\alpha, \beta) := T(-q^\alpha)S(-q^\beta) + q^{(-\alpha+\beta)/2}S(-q^\alpha)T(-q^\beta).
\]

Then

\[
U(1, 39) + \frac{f_2 f_4 f_5 f_{156} f_{390}}{q^3 f_1^2 f_78 f_{195}} V(2, 78)
\]

\[
= \frac{f_2 f_{390}}{2q^8 f_1^2 f_{195}} \left\{ \frac{f_5^5 f_{16}^6}{f_3^2 f_3^2 f_{12}^2 f_{1248}} - \frac{f_5^5 f_{195}^2}{f_{10} f_{390}} + 2q^{20} \frac{f_5^2 f_{12}^2}{f_4 f_{156}} + 4q^8 \frac{f_5^2 f_{1248}}{f_{16} f_{624}} \right\}, \quad (3.4.1)
\]

\[
U(7, 33) + q^{21} \frac{f_4^2 f_{66} f_{32}^2 f_{24}}{f_2^2 f_3^2 f_{462}^2} V(2, 462)
\]

\[
= \frac{f_4^4 f_{66}}{2q^8 f_1^2 f_{53}^2} \left\{ \frac{f_5^5 f_{16}^6}{f_3^2 f_{1848} f_{392}} - \frac{f_5^5 f_{165}^2}{f_{70} f_{330}} + 2q^{116} \frac{f_5 f_{1848}}{f_4 f_{924}} + 4q^{96} \frac{f_5^2 f_{32} f_{392}}{f_{16} f_{3696}} \right\}, \quad (3.4.2)
\]

\[
U^*(1, 31) - \frac{f_2 f_{62} f_{62} f_{124}}{q^2 f_1^2 f_{31}^2} V^*(2, 62)
\]

\[
= - \frac{f_2 f_{155}^2}{2q^2 f_1^2 f_{31}^2} \left\{ \frac{f_5^5 f_{155}}{f_3^2 f_{248} f_{992}} - \frac{f_5^2 f_{155}^2}{f_{10} f_{310}} + 2q^{156} \frac{f_5 f_{248}}{f_4 f_{124}} + 4q^{64} \frac{f_5^2 f_{992}}{f_{16} f_{496}} \right\}, \quad (3.4.3)
\]

\[
U(13, 27) + q^{37} \frac{f_4^2 f_{54} f_{54}^2 f_{1404}}{f_2 f_{13}^2 f_{27}^2 f_{702}} V^*(2, 702)
\]

\[
= \frac{f_4^2 f_{54}}{2q^8 f_{13}^2 f_{27}^2} \left\{ \frac{f_5^5 f_{516}^2}{f_3^2 f_{2808} f_{11232}} - \frac{f_5^5 f_{135}^2}{f_{130} f_{270}} + 2q^{176} \frac{f_5 f_{2808}}{f_4 f_{1404}} + 4q^{704} \frac{f_5^2 f_{11232}}{f_{16} f_{5616}} \right\}, \quad (3.4.4)
\]
Using (3.2.2), (3.2.3) and Lemma 2.2.1, we can write (3.4.1) in the alternative form

\[
U^\ast(3,13) = -\frac{f_6f_{26}}{2q^5f_3^2f_7^3} \left\{ \frac{f_6^5f_{156}}{f_8^3f_3^2f_7^2f_212} - \frac{f_6^5f_{156}}{f_30f_{130}} + 2q^{20}f_2^2f_3^2_1f_{156} + 4q^{80}f_3^2f_{1248} \right\}, \tag{3.4.5}
\]

\[
U(17,23) + q^{41}f_4^2f_{34}f_{16}f_{1564}^2 V(2,782)
= -\frac{f_{34}f_{46}}{2q^8f_{17}f_2^3} \left\{ \frac{f_6^5f_{156}}{f_32f_2^2f_7^2f_212} - \frac{f_6^5f_{156}}{f_{170}f_{230}} + 2q^{196}f_2^2f_3^2_1f_{1564} + 4q^{784}f_3^2f_{12512} \right\}, \tag{3.4.6}
\]

\[
U(19,21) + q^{43}f_4^2f_{38}f_{42}f_{1566}^2 V(2,798)
= -\frac{f_{38}f_{42}}{2q^8f_{19}f_2^3} \left\{ \frac{f_6^5f_{156}}{f_32f_2^2f_7^2f_212} - \frac{f_6^5f_{156}}{f_{190}f_{110}} + 2q^{200}f_2^2f_3^2_1f_{1566} + 4q^{800}f_3^2f_{12768} \right\}, \tag{3.4.7}
\]

\[
U(3,37) + q^{7}f_2^2f_6f_2f_{444}^2 V^\ast(2,222)
= \frac{f_6f_{72}}{2q^8f_3^2f_7^3} \left\{ \frac{f_6^5f_{176}}{f_8^3f_2^2f_7^2f_222} - \frac{f_6^5f_{176}}{f_{30}f_{370}} + 2q^{56}f_2^2f_3^2_1f_{444} + 4q^{224}f_3^2f_{3552} \right\}, \tag{3.4.8}
\]

\[
U(9,31) + q^{27}f_2^2f_6f_{62}f_{116}^2 V(2,558)
= -\frac{f_{18}f_{62}}{2q^8f_3^2f_3^3} \left\{ \frac{f_6^5f_{444}}{f_8^3f_3^2f_222} - \frac{f_6^5f_{444}}{f_{90}f_{410}} + 2q^{142}f_2^2f_3^2_1f_{116} + 4q^{562}f_3^2f_{8928} \right\}. \tag{3.4.9}
\]

**Proof.** Using (3.2.2), (3.2.3) and Lemma 2.2.1, we can write (3.4.1) in the alternative form

\[
\phi(-q^5)\phi(-q^{195}) + 2q^8f(-q^3,-q^7)f(-q^{17},-q^{273})
+ 2q^{32}f(-q,-q^9)f(-q^{351},-q^{39})
\]

\[
= \phi(q^8)\phi(q^{312}) - 2q^5f(q^6,q^{10})f(q^{234},q^{390}) + 2q^{20}\psi(q^4)\psi(q^{156})
- 2q^{45}f(q^2,q^{14})f(q^{78},q^{546}) + 4q^8\psi(q^{16})\psi(q^{624}). \tag{3.4.10}
\]
Identity (3.4.10) can be easily verified using Lemma 2.2.12, (2.2.30) and Lemma 2.2.2 with the following sets of choice of parameters:

\[ R(0, 1, 0, 0, 1, 39, 1, 5, 8, 1, 1) = R(0, 1, 0, 0, 1, 39, 1, 8, 5, 1, 1). \]

This completes the proof of (3.4.1). The proofs of (3.4.2)–(3.4.9) follow in a similar way. \qed
3.5 Identities Connecting $J(q)$ and $K(q)$ with Cubic Functions $P(q)$ and $Q(q)$

In this section, we present relations involving some combinations of $J(q)$ and $K(q)$ with the cubic functions $P(q)$ and $Q(q)$.

**Theorem 3.5.1.** Define

$$U(\alpha, \beta) := \varphi(-q^\alpha)\varphi(q^\beta) \left\{ J(q^\alpha)J(-q^\beta) - q^{3(\alpha+\beta)/5}K(q^\alpha)K(-q^\beta) \right\},$$

$$V(\alpha, r) := \psi(-q^2)\psi(-q^\alpha) \left\{ Q(q^2)Q(q^\alpha) + (-1)^r q^{(2+\alpha)/4}P(q^2)P(q^\alpha) \right\}.$$

Then,

$$U(7, 23) + q^{21}V(322, 0) = \frac{1}{2q^6} \left\{ \varphi(-q^{35})\varphi(q^{115}) - \varphi(-q^6)\varphi(-q^{966}) \right\}, \quad (3.5.1)$$

$$U(1, 29) - \frac{1}{q} V(58, 1) = \frac{1}{2q^6} \left\{ \varphi(-q^5)\varphi(q^{145}) - \varphi(-q^6)\varphi(-q^{174}) \right\}, \quad (3.5.2)$$

$$U(11, 19) - q^{29}V(418, 1) = \frac{1}{2q^6} \left\{ \varphi(-q^{55})\varphi(q^{85}) - \varphi(-q^6)\varphi(-q^{1254}) \right\}, \quad (3.5.3)$$

$$U(13, 17) - q^{31}V(442, 1) = \frac{1}{2q^6} \left\{ \varphi(-q^{65})\varphi(q^{95}) - \varphi(-q^6)\varphi(-q^{1326}) \right\}. \quad (3.5.4)$$

**Proof.** Using (3.2.2) and (3.2.4), we can write (3.5.1) in the form

$$\varphi(-q^{35})\varphi(q^{115}) - 2q^6 f(-q^{21}, -q^{49}) f(q^{69}, q^{161}) + 2q^{24} f(-q^7, -q^{63}) f(q^{23}, q^{207})$$

$$= \varphi(-q^6)\varphi(-q^{966}) + 2q^{27} f(-q^4, -q^8) f(-q^{44}, -q^{1288})$$

$$+ 2q^{108} f(-q^2, -q^{10}) f(-q^{322}, -q^{1610}). \quad (3.5.5)$$

Identity (3.5.5) can be easily verified using Lemma 2.2.12, (2.2.30) and Lemma 2.2.2 with the following sets of choice of parameters:

$$R(1, 1, 0, 0, 7, 23, 1, 5, 6, 1, 1) = R(1, 1, 0, 0, 1, 161, 7, 6, 35, 1, 1).$$

This completes the proof of (3.5.1). The proofs of (3.5.2)–(3.5.4) follow similarly. □
3.6 Applications to the Theory of Partitions

Some of our modular relations yield theorems in the theory of partitions. In this section, we present partition theoretic interpretations of the Theorem 3.3.2 and the identities (3.3.1), (3.3.21) and (3.3.35). In this section, we use the following alternative definitions of \( J(q) \) and \( K(q) \):

\[
J(q) := \frac{(q^{10}; q^{10})_{\infty}}{(q, q^5, q^9; q^{10})_{\infty}(q; q)_{\infty}} \tag{3.6.1}
\]

and

\[
K(q) := \frac{(q^{10}; q^{10})_{\infty}}{(q^3, q^5, q^7; q^{10})_{\infty}(q; q)_{\infty}}. \tag{3.6.2}
\]

**Theorem 3.6.1.** Let \( p_1(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 2, \pm 4, \pm 8, \pm 9 \pmod{20} \) with \( \pm 2, \pm 4 \) and \( \pm 8 \pmod{20} \) having two colors. Let \( p_2(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 3, \pm 4, \pm 6, \pm 7 \pmod{20} \) with \( \pm 4, \pm 6 \) and \( \pm 8 \pmod{20} \) having two colors. Let \( p_3(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 2, \pm 3, \pm 4, \pm 6 \) and \( \pm 7 \pmod{20} \) with \( \pm 2, \pm 4 \) and \( \pm 6 \pmod{20} \) having two colors. Let \( p_4(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 2, \pm 6 \) and \( \pm 8 \pmod{20} \) with \( \pm 2, \pm 6 \) and \( \pm 8 \pmod{20} \) having two colors. Then, for any positive integer \( n \geq 3 \),

\[
p_1(n) + p_2(n - 3) = p_3(n) + p_4(n - 1).
\]

**Proof.** Using (2.1.1), (2.1.2), (3.6.1) and (3.6.2), it is easy to verify that the identity (3.3.1) is equivalent to

\[
\frac{(q^{20}; q^{20})^2}{(q, q^5, q^9; q^{10})_{\infty}(q^2, q^{10}, q^{18}; q^{20})^2_{\infty}(q^4; q^4)_{\infty}^4} + q^3 \frac{(q^3, q^5, q^7; q^{10})_{\infty}(q^6, q^{10}, q^{14}; q^{20})^2_{\infty}(q^4; q^4)_{\infty}^4}{(q^{20}; q^{20})^2_{\infty}} = 1
\]

\[
+ \frac{q}{(q^3, q^5, q^7; q^{10})_{\infty}(q^{4}, q^{16}, q^{20})^2_{\infty}(q^2; q^2)_{\infty}^2} + \frac{q}{(q, q^5, q^9; q^{10})_{\infty}(q^8, q^{18}; q^{20})^2_{\infty}(q^2; q^2)_{\infty}^2}. \tag{3.6.3}
\]
Now, rewrite all the products on both sides of (3.6.3) subject to the common base $q^{20}$ to obtain

\[
\frac{1}{(q^{1\pm}, q^{9\pm}; q^{20})_\infty (q^{2\pm}, q^{4\pm}, q^{8\pm}; q^{20})} + \frac{q^3}{(q^{3\pm}, q^{7\pm}; q^{20})_\infty (q^{4\pm}, q^{6\pm}, q^{8\pm}; q^{20})_\infty}
\]

\[
= \frac{1}{(q^{3\pm}, q^{7\pm}; q^{20})_\infty (q^{2\pm}, q^{4\pm}, q^{6\pm}; q^{20})} + \frac{q}{(q^{1\pm}, q^{9\pm}; q^{20})_\infty (q^{2\pm}, q^{6\pm}, q^{8\pm}; q^{20})_\infty}.
\]

The four quotients of (3.6.4) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$, respectively. Hence, (3.6.4) is equivalent to

\[
\sum_{n=0}^{\infty} p_1(n)q^n + q^3\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n + q\sum_{n=0}^{\infty} p_4(n)q^n,
\]

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Equating coefficients of $q^n$ ($n \geq 3$) on both sides yields the desired result. \hfill \Box

Example 3.6.2. The following table illustrates the case $n = 5$ in Theorem 3.6.1

<table>
<thead>
<tr>
<th>$p_1(5) = 8$</th>
<th>$p_2(2) = 0$</th>
<th>$p_3(5) = 2$</th>
<th>$p_4(4) = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4_r + 1, 4_g + 1, 2_r + 2_r + 1$</td>
<td>$3 + 2_r, 2_r + 2_g$</td>
<td>$2_r + 2_r, 2_r + 2_g$</td>
<td>$2_r + 2_r, 2_r + 2_g$</td>
</tr>
<tr>
<td>$2_r + 2_g + 1, 2_g + 2_g + 1$</td>
<td>$3 + 2_g$</td>
<td>$3 + 2_g$</td>
<td>$3 + 2_g$</td>
</tr>
<tr>
<td>$2_r + 1 + 1 + 1, 2_g + 1 + 1 + 1$</td>
<td>$2_g + 1 + 1$</td>
<td>$2_g + 1 + 1$</td>
<td>$2_g + 1 + 1$</td>
</tr>
<tr>
<td>$1 + 1 + 1 + 1 + 1$</td>
<td>$1 + 1 + 1 + 1$</td>
<td>$1 + 1 + 1 + 1$</td>
<td>$1 + 1 + 1 + 1$</td>
</tr>
</tbody>
</table>

Theorem 3.6.3. Let $p_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1$, $\pm 2$, $\pm 4$, $\pm 6$, $\pm 9$ and $10 \pmod{20}$ with $\pm 2$, $\pm 4$ and $10 \pmod{20}$ having two colors and $\pm 6 \pmod{20}$ having three colors. Let $p_2(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 2$, $\pm 3$, $\pm 6$, $\pm 7$, $\pm 8$ and $10 \pmod{20}$ with $\pm 6$, $\pm 8$ and $10 \pmod{20}$ having two colors and $\pm 2 \pmod{20}$ having three colors. Let $p_3(n)$ denote the number of partitions of $n$ into odd parts having two colors. Then, for any positive integer $n \geq 1$,

\[
p_1(n) + p_2(n - 1) = p_3(n).
\]
Proof. Using (3.2.2), (2.2.22), (1.2.3), (2.1.1), (2.1.2) and (1.2.2), we find that the Theorem 3.3.2(i) is equivalent to

$$\frac{1}{q} \frac{(q, q^4; q^5)_\infty (q^3, q^7, q^{10}; q^{10})_\infty (q^2, q^{18}, q^{20}; q^{20})_\infty}{(q^2; q^5)_\infty (q, q^6, q^{10}; q^{10})_\infty (q^6, q^{14}, q^{20}; q^{20})_\infty} + \frac{q}{(q^2, q^5)_\infty (q, q^6, q^{10}; q^{10})_\infty (q^5; q^5)_\infty} = \frac{(q; q)_\infty (q^{10}; q^{20})_\infty^2 (q^{20}; q^{20})_\infty (q^2, q^{18}, q^{20}; q^{20})_\infty (q^6, q^{14}, q^{20}; q^{20})_\infty}{(q^2; q^5)_\infty (q; q^2; q^2; q^2; q^2; q^5)_\infty}.$$

(3.6.5)

Now, rewrite all the products on both sides of (3.6.5) subject to the common base $q^{20}$ to obtain

$$\frac{1}{q} \frac{(q^{1\pm}, q^{9\pm}; q^{20})_\infty (q^{2\pm} q^{4\pm}, q^{10}; q^{10})_\infty (q^{2\pm}; q^{20})_\infty^3}{(q^{3\pm}, q^{7\pm}; q^{20})_\infty (q^{6\pm}; q^{6\pm}; q^{8\pm}, q^{10}; q^{10})_\infty (q^{2\pm}; q^{20})_\infty^3} + \frac{q}{(q^{3\pm}, q^{5\pm}; q^{7\pm}; q^{20})_\infty} = \frac{1}{(q; q^2)_\infty^2}.$$

(3.6.6)

The three quotients of (3.6.6) represent the generating functions for $p_1(n)$, $p_2(n)$ and $p_3(n)$, respectively. Hence, (3.6.6) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n) q^n + q \sum_{n=0}^{\infty} p_2(n) q^n = \sum_{n=0}^{\infty} p_3(n) q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of $q^n (n \geq 1)$ on both sides yields the desired result.

Example 3.6.4. The following table illustrates the case $n = 4$ in Theorem 3.6.3

<table>
<thead>
<tr>
<th>$p_1(4)$</th>
<th>$p_2(3)$</th>
<th>$p_3(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>4$_r$, 4$_g$, 2$_r$ + 2$_r$, 2$_r$ + 2$_g$</td>
<td>3</td>
<td>3$_r$ + 1$_r$, 3$_r$ + 1$_g$, 3$_g$ + 1$_r$, 3$_g$ + 1$_g$</td>
</tr>
<tr>
<td>2$_g$ + 2$_g$, 2$_r$ + 1 + 1</td>
<td>1$_r$ + 1$_r$ + 1$_r$, 1$_r$ + 1$_r$ + 1$_r$ + 1$_g$</td>
<td></td>
</tr>
<tr>
<td>2$_g$ + 1 + 1, 1 + 1 + 1 + 1</td>
<td>1$_r$ + 1$_r$ + 1$_g$ + 1$_g$, 1$_r$ + 1$_g$ + 1$_g$ + 1$_g$</td>
<td></td>
</tr>
</tbody>
</table>

| 1$_r$ + 1$_r$ + 1$_g$ + 1$_g$ | 1$_g$ + 1$_g$ + 1$_g$ + 1$_g$ |
Theorem 3.6.5. Let \( p_1(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 4, \pm 6, \pm 9 \) and \( 10 \pmod{20} \) having two colors. Let \( p_2(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 2, \pm 3, \pm 7, \pm 8 \) and \( 10 \pmod{20} \) with \( \pm 8 \) and \( 10 \pmod{20} \) having two colors. Let \( p_3(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 2, \pm 4, \pm 5, \pm 6 \) and \( \pm 8 \pmod{20} \) with \( \pm 5 \pmod{20} \) having two colors. Then, for any positive integer \( n \geq 1 \), we have

\[
p_1(n) - p_2(n-1) = p_3(n).
\]

Proof. In a similar way, as in Theorem 3.6.3, the Theorem 3.3.2(ii) is equivalent to

\[
\frac{1}{(q^{1\pm}, q^{6\pm}, q^{9\pm}, q^{20})_\infty (q^{4\pm}, q^{10}, q^{20})_\infty} - \frac{q}{(q^{2\pm}, q^{3\pm}, q^{7\pm}, q^{20})_\infty (q^{8\pm}, q^{10}, q^{20})_\infty} = \frac{1}{(q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{6\pm}, q^{8\pm}, q^{20})_\infty}.
\]

(3.6.7)

The three quotients of (3.6.7) represent the generating functions for \( p_1(n) \), \( p_2(n) \) and \( p_3(n) \), respectively. Hence, (3.6.7) is equivalent to

\[
\sum_{n=0}^{\infty} p_1(n)q^n - q \sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n
\]

where we set \( p_1(0) = p_2(0) = p_3(0) = 1 \). Equating coefficients of \( q^n (n \geq 1) \) on both sides yields the desired result. \( \square \)

Example 3.6.6. The following table illustrates the case of \( n = 8 \) in Theorem 3.6.5

<table>
<thead>
<tr>
<th>( p_1(8) = 7 )</th>
<th>( p_2(7) = 2 )</th>
<th>( p_3(8) = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6 + 1 + 1, 4_r + 4, 4_r + 4_g, 4_g + 4_g )</td>
<td>( 7 )</td>
<td>( 8, 6 + 2, 4 + 4, )</td>
</tr>
<tr>
<td>( 4_r + 1 + 1 + 1 + 1, 4_g + 1 + 1 + 1 + 1 )</td>
<td>( 3 + 2 + 2 )</td>
<td>( 4 + 2 + 2, 2 + 2 + 2 + 2 )</td>
</tr>
<tr>
<td>( 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Theorem 3.6.7. Let \( p_1(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 4, \pm 6, \pm 8, \pm 9, \pm 11, \pm 14, \pm 16, \pm 19 \) and \( 20 \pmod{40} \) with \( \pm 20 \pmod{40} \) having two colors and \( \pm 4, \pm 6, \pm 14 \) and \( \pm 16 \pmod{40} \) having three colors. Let \( p_2(n) \)
denote the number of partitions of \( n \) into parts congruent to \( \pm 2, \pm 3, \pm 7, \pm 8, \pm 12, \pm 13, \pm 16, \pm 17, \pm 18 \) and \( 20 \) (mod 40) with \( 20 \) (mod 40) having two colors and \( \pm 2, \pm 8, \pm 12 \) and \( \pm 18 \) (mod 40) having three colors. Let \( p_3(n) \) denote the number of partitions of \( n \) into parts not congruent to \( \pm 2, \pm 6, \pm 8, \pm 10, \pm 14, \pm 16 \) and \( \pm 18 \) (mod 40) with \( \pm 4, \pm 5, \pm 12 \pm 15 \) (mod 20) having two colors and \( 20 \) (mod 40) having four colors. Let \( p_4(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 2, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 15, \pm 16, \pm 18 \) (mod 40) with \( \pm 4, \pm 5, \pm 8, \pm 10, \pm 12, \pm 15 \) and \( \pm 16 \) (mod 40) having two colors. Then, for any positive integer \( n \geq 1 \),

\[
p_1(n) + p_2(n - 1) = 2p_3(n) - p_4(n).
\]

**Proof.** Using (3.2.2), (2.2.22), (1.2.3), (2.1.1), (2.1.2) and (1.2.2), we deduce that the identity (3.3.21) is equivalent to

\[
\frac{1}{(q^2; q^2)_\infty (q^5; q^5)_\infty (q^8; q^8)_\infty (q^{20}; q^{20})^5_\infty (q, q^4; q^5)_\infty (q^4, q^6; q^{10})_\infty}
\times \frac{1}{(q^6 q^{14} q^{20}; q^{20})_\infty (q^4, q^{26}; q^{40}; q^{40})_\infty}
\]

\[
+ \frac{q}{(q^2; q^2)_\infty (q^5; q^5)_\infty (q^8; q^8)_\infty (q^{20}; q^{20})^5_\infty (q^2, q^3; q^5)_\infty (q^2, q^8; q^{10})_\infty}
\times \frac{1}{(q^2 q^{18} q^{20}; q^{20})_\infty (q^{12}, q^{28}; q^{40}; q^{40})_\infty}
\]

\[
= \frac{1}{(q; q)_\infty (q^4; q^4)_\infty (q^{20}; q^{20})^5_\infty (q^5, q^{10}; q^{10})_\infty (q^{20}, q^{20}; q^{40}; q^{40})_\infty}
\times \frac{1}{(q^4, q^{36}; q^{40}; q^{40})_\infty (q^{12}, q^{28}; q^{40}; q^{40})_\infty}
\]

\[
- \frac{1}{(q^2; q^2)_\infty (q^8; q^8)_\infty (q^5; q^5)_\infty (q^{40}; q^{40})^2_\infty (q^5, q^{10}; q^{10})_\infty (q^{20}, q^{20}; q^{40}; q^{40})_\infty}
\times \frac{1}{(q^4, q^{36}; q^{40}; q^{40})_\infty (q^{12}, q^{28}; q^{40}; q^{40})_\infty}
\]

(3.6.8)

Now, rewrite all the products on both sides of (3.6.8) subject to the common base \( q^{40} \).
to obtain

\[
\frac{1}{(q^1; q^{8n}; q^{9q}; q^{11q}; q^{19q}; q^{40})_\infty (q^{2n}; q^{40})_\infty (q^{4q}; q^{6q}; q^{14q}; q^{16q}; q^{40})_\infty} 
+ \frac{q}{(q^{3q}; q^{7q}; q^{13q}; q^{16q}; q^{17q}; q^{40})_\infty (q^{20n}; q^{40})_\infty (q^{2q}; q^{8q}; q^{12q}; q^{18q}; q^{40})_\infty} 
\]

\[
= \frac{1}{(q^1; q^{3q}; q^{7q}; q^{9q}; q^{11q}; q^{17q}; q^{40})_\infty} 
\times \frac{1}{(q^{4q}; q^{5q}; q^{12q}; q^{15q}; q^{20}; q^{40})_\infty} 
\]

\[
\times \frac{1}{(q^{6q}; q^{14q}; q^{18q}; q^{40})_\infty (q^{4q}; q^{5q}; q^{8q}; q^{10q}; q^{12q}; q^{15q}; q^{16q}; q^{40})_\infty} 
\]

(3.6.9)

The four quotients of (3.6.9) represent the generating functions for \( p_1(n) \), \( p_2(n) \), \( p_3(n) \) and \( p_4(n) \), respectively. Hence, (3.6.9) is equivalent to

\[
\sum_{n=0}^{\infty} p_1(n)q^n + q \sum_{n=0}^{\infty} p_2(n)q^n = 2 \sum_{n=0}^{\infty} p_3(n)q^n - \sum_{n=0}^{\infty} p_4(n)q^n
\]

where we set \( p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1 \). Equating coefficients of \( q^n \) \((n \geq 1)\) on both sides yields the desired result.

\[ \square \]

**Example 3.6.8.** The following table illustrates the case of \( n = 7 \) in Theorem 3.6.7

<table>
<thead>
<tr>
<th>( p_1(7) = 7 )</th>
<th>( p_2(6) = 11 )</th>
<th>( p_3(7) = 10 )</th>
<th>( p_4(7) = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6r + 1, 6g + 1 )</td>
<td>( 3 + 3, 2r + 2r + 2r )</td>
<td>( 7, 5r + 1 + 1 )</td>
<td>( 5r + 2 )</td>
</tr>
<tr>
<td>( 6_w + 1 )</td>
<td>( 2r + 2r + 2g, 2r + 2g + 2_g )</td>
<td>( 5g + 1 + 1, 4r + 3 )</td>
<td>( 5g + 2 )</td>
</tr>
<tr>
<td>( 4r + 1 + 1 + 1 )</td>
<td>( 2_g + 2_g + 2_g, 2_g + 2_g + 2_w )</td>
<td>( 4_g + 3, 3 + 3 + 1 )</td>
<td></td>
</tr>
<tr>
<td>( 4_g + 1 + 1 + 1 )</td>
<td>( 2_g + 2_w + 2_w, 2_w + 2_w + 2_w )</td>
<td>( 4_r + 1 + 1 + 1 )</td>
<td></td>
</tr>
<tr>
<td>( 4_g + 1 + 1 + 1 )</td>
<td>( 2_w + 2_w + 2_r, 2_w + 2_r + 2_r )</td>
<td>( 4_g + 1 + 1 + 1 )</td>
<td></td>
</tr>
<tr>
<td>( 1 + 1 + 1 + 1 )</td>
<td>( 2r + 2_g + 2_w )</td>
<td>( 3 + 1 + 1 + 1 + 1 )</td>
<td>( 1 + 1 + 1 + 1 + 1 )</td>
</tr>
</tbody>
</table>

**Theorem 3.6.9.** Let \( p_1(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8 \) and \( \pm 9 \) \((\text{mod } 20)\) with \( \pm 1, \pm 5 \) and \( \pm 9 \) \((\text{mod } 20)\)
having two colors and \( \pm 4 \pmod{20} \) having three colors. Let \( p_2(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 7, \pm 8 \) and \( \pm 9 \pmod{20} \) with \( \pm 3, \pm 5 \) and \( \pm 7 \pmod{20} \) having two colors and \( \pm 8 \pmod{20} \) having three colors. Let \( p_3(n) \) denote the number of partitions of \( n \) into parts congruent to \( 10 \pmod{20} \) with two colors. Let \( p_4(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1, \pm 3, \pm 4, \pm 7, \pm 8 \) and \( \pm 9, 10 \pmod{20} \) with two colors. Then, for any positive integer \( n \geq 1 \),

\[
p_1(n) - p_2(n - 1) = p_3(n) + p_4(n - 1).
\]

**Proof.** Using (2.1.1), (2.1.2), (3.6.1) and (3.6.2) and then rewriting all the products subject to the common base \( q^{20} \), we find that the identity (3.3.35) is equivalent to

\[
\frac{1}{(q^{4\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}; q^{20})_\infty (q^{1\pm}, q^{5\pm}, q^{9\pm}, q^{20})_\infty (q^{4\pm}; q^{20})^3_\infty} - \frac{q}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{9\pm}, q^{20})_\infty (q^{3\pm}, q^{5\pm}, q^{7\pm}, q^{20})^2_\infty (q^{8\pm}; q^{20})^3_\infty} = \frac{1}{(q^{10}; q^{20})^2_\infty} + \frac{q}{(q^{1\pm}, q^{3\pm}, q^{4\pm}, q^{7\pm}, q^{8\pm}, q^{9\pm}, q^{10}; q^{20})^2_\infty} \tag{3.6.10}
\]

The four quotients of (3.6.10) represent the generating functions for \( p_1(n) \), \( p_2(n) \), \( p_3(n) \) and \( p_4(n) \), respectively. Hence, (3.6.10) is equivalent to

\[
\sum_{n=0}^{\infty} p_1(n)q^n - q \sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n + q \sum_{n=0}^{\infty} p_4(n)q^n
\]

where we set \( p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1 \). Equating coefficients of \( q^n \ (n \geq 1) \) on both sides yields the desired result.

**Example 3.6.10.** We illustrate Theorem 3.6.9 in the case of \( n = 10 \), and we can easily verify that \( p_1(10) = 135 \), \( p_2(9) = 51 \), \( p_3(10) = 2 \) and \( p_4(9) = 82 \).