Chapter 1

Introduction
1.1 Ramanujan’s Forty Identities

The Indian mathematical genius Srinivasa Ramanujan, who was one of the greatest mathematicians that India has produced, was born on 22nd December, 1887. He died on 26th April, 1920, and left behind three notebooks [70], a “lost notebook”, other manuscripts [72] and published papers [71]. He made substantial contributions to the analytical number theory and worked on elliptic functions, continued fractions and infinite series.

Throughout the thesis, we use the customary notations

\[
(a; q)_0 := 1, \\
(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1, \\
(a; q)_\infty := \lim_{n \to \infty} (a; q)_n, \quad |q| < 1
\]

and

\[
(a_1, a_2, \ldots, a_n; q)_\infty := \prod_{i=1}^{n} (a_i; q)_\infty.
\]

One of the most significant contributions made by S. Ramanujan to the area of the number theory are his forty beautiful modular relations for the Rogers-Ramanujan functions.

The well-known Rogers-Ramanujan functions [68, 71, 74], which play an important role in our work, are defined, for \(|q| < 1\), by

\[
G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4, q^5)_\infty} \tag{1.1.1}
\]

and

\[
H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2, q^3, q^5)_\infty}. \tag{1.1.2}
\]
On March 13, 1919, at a meeting of the London Mathematical Society, Professor G. H. Hardy communicated an abstract by Srinivasa Ramanujan entitled “Algebraic Relations between Certain Infinite Products” [69], where Ramanujan remarks “I have now found an algebraic relation between \( G(q) \) and \( H(q) \), \( \text{viz.} \):

\[
H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6. \tag{1.1.3}
\]

Another noteworthy formula is

\[
H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1. \tag{1.1.4}
\]

Each of these formulae is the simplest of a large class.” Unfortunately, Ramanujan had died a year later, without revealing either his proofs of these relationships or the “large class” of which they were representative.

In [54], H. B. C. Darling furnished proofs of certain identities and congruences which have been enunciated by Ramanujan, but of which Ramanujan has not yet furnished complete demonstrations. In this paper, Darling was the first to publish a proof of (1.1.3).

In February of 1919, some of these forty modular relations were communicated privately by Ramanujan to L. J. Rogers, and the latter has published his proofs of ten of them [76], including the proofs of (1.1.3) and (1.1.4), the method used by Rogers is substantially H. Schröter’s method of constructing modular equations and it is a modification of the method by which Jacobi proved the fundamental formulæ of theta functions. In Section 9 of his paper [76], Rogers remarks that “As I understand that Ramanujan has left no proof, I suggest the proof given in this section”. Rogers did not indicate if further identities were included in Ramanujan’s communication to him.

In 1933, when the Silver Jubilee of the foundation of the Indian Mathematical Club was celebrated, G. N. Watson [84], proved eight identities, six of them are members of the set of forty which have hitherto been unpublished, and to which
Schröter's method is not obviously applicable, two of the identities proved by Rogers are added, since they are required in the course of proving the six new identities, and he gave proofs whose main differences from the proofs given by Rogers are in matters of arrangement. In [84], Watson remarks “Among the formulae contained in the manuscripts left by Ramanujan is a set of about forty which involve functions of the types $G(q)$ and $H(q)$; the beauty of these formulae seems to me to be comparable with that of the Rogers-Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely; if my belief is well-founded, the undivided credit for the discovery of these formulae is due to Ramanujan.”

Nothing further was done about these identities until 1975 when B. J. Birch rediscovered them in Watson’s transcription and communicated the entire list to the Cambridge Philosophical Society [36].

In 1977, D. Bressoud [41, 42] by borrowing ideas and approaches from both Rogers and Watson, succeeded in proving fifteen of the twenty-four identities which had been unproved. In addition, he was able to make certain generalizations which carry some of the identities beyond the Rogers-Ramanujan functions. Most likely these proofs might have been given by Ramanujan himself. After the work of Rogers, Watson and Bressoud, nine remained to be proved.

In 1989, A. J. F. Biagioli [35], proved eight identities by invoking the theory of modular forms. This method of modular forms can also be used to establish the last identity. In 2007, B. C. Berndt et al. [27] authored a beautiful memoir on the 40 identities in which they offered proofs of thirty-five of the identities in the spirit of Ramanujan. In [89], H. Yesilyurt developed a significant generalization of the ideas of Rogers and Bressoud, and consequently proved two further identities. Finally, Yesilyurt [90] employed his primary theorems from [89] to establish the remaining three identities, including the one that had never been proved by any means. Recently, in Chapter 8 of their book [15], G. E. Andrews and Berndt collected proofs for all
forty identities which Ramanujan himself might have given. Many mathematicians tried to find new identities for the Rogers-Ramanujan functions similar to those which have been found by Ramanujan [72], including Berndt and Yesilyurt [25], Yesilyurt [89], S. Robins [73] and C. Gugg [58].

Motivated by Ramanujan’s forty identities, many functions of the Rogers-Ramanujan type were studied by many mathematicians. S.-S. Huang [65] and S.-L. Chen and Huang [48] have studied two functions of order eight which are called the Göllnitz-Gordon functions and derived a list of new modular relations. These two functions were restudied by N. D. Baruah, J. Bora and N. Saikia [19] offered new proofs of many of these identities. E. X. W. Xia and X. M. Yao [86] offered new proofs of some modular relations established by Huang [65] and Chen and Huang [48]. They also established some new relations which involve only Göllnitz-Gordon functions. H. Hahn [59, 60] has established several modular relations for two functions of order seven which are called the septic analogues of the Rogers-Ramanujan functions furthermore in her thesis [60] has established relations that are connecting these functions with the Rogers-Ramanujan and Göllnitz-Gordon functions. Similarly Baruah and Bora [18] have obtained modular relations for the nonic analogues of the Rogers-Ramanujan functions. C. Adiga, K. R. Vasuki and N. Bhaskar [8] established several modular relations for the cubic functions. Vasuki, G. Sharath and K. R. Rajanna [83], studied two different cubic functions. Baruah and Bora [17] considered two functions of order twelve which are analogous of the Rogers-Ramanujan functions. Vasuki and P. S. Guruprasad [82] considered the Rogers-Ramanujan type functions of order twelve and established modular relations involving them. Adiga, Vasuki and B. R. Srivatsa Kumar [9] established modular relations involving two functions of Rogers-Ramanujan type. Almost all of these functions which have been studied so far are due to Rogers [75] and L. J. Slater [79].
In Chapter 2, we consider Rogers-Ramanujan type functions of order ten and establish several modular relations involving these functions, which are analogues to the Ramanujan’s forty identities for Rogers-Ramanujan functions. Furthermore, we give partition theoretic interpretations of some of our modular relations.

In Chapter 3, we establish further modular relations connecting those two functions, which have been studied in Chapter 2, with Rogers-Ramanujan functions, Göllnitz-Gordon functions and cubic functions, which are analogues to the Ramanujan’s forty identities for Rogers-Ramanujan functions. Furthermore, we give partition theoretic interpretations of some of our modular relations.
1.2 Products of Theta Functions

Ramanujan’s general theta function is defined by

\[ f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.2.1} \]

The Jacobi triple product identity [1, Entry 19] in Ramanujan’s notation is

\[ f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{1.2.2} \]

Ramanujan has defined the following three special cases of (1.2.1) [1, Entry 22]:

\[ \varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}(q^2; q^2)_{\infty}, \tag{1.2.3} \]

\[ \psi(q) := f(q, q^2) = \sum_{n=-\infty}^{\infty} q^{n(2n+1)} = (q^2; q^2)_{\infty} (q; q^2)_{\infty} \tag{1.2.4} \]

and

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty}. \tag{1.2.5} \]

Also, after Ramanujan, define

\[ \chi(q) := (-q; q^2)_{\infty}. \tag{1.2.6} \]

Theta functions are important in several areas, including the theory of elliptic functions, modular forms, analytic number theory and Riemann’s surfaces. Theta functions also appear in physics in the partition function of strings and two dimensional conformal field theories. For these and other applications, theta function identities are very useful. Ramanujan’s general theta function \( f(a,b) \) satisfies a beautiful addition formula [1, Entry 31]. One can write a product of two theta functions as a linear combination of other products of theta functions. In 1854, Schröter in his dissertation [78], has established very general and useful formulas for the products of
two theta functions. These formulas have been used later by several mathematicians to establish many of Ramanujan’s modular equations. Rogers developed this theory to prove some of Ramanujan’s forty identities [76]. Rogers’ approach has been generalized by many mathematicians including Bressoud [41, 42], Yesilyurt [89] and more recently by L.-C. Zhang [91]. Many formulas, for the products of two theta functions, can be found in the literature which we will use to prove some modular relations for Rogers-Ramanujan type functions. For example, a theorem proved by R. Blecksmith, J. Brillhart and I. Gerst [37], provides a representation for a product of two theta functions as a sum of $m$ products of pairs of theta functions, under certain conditions. This theorem generalizes formulas of Schröter which can be found in [1, 21]. Blecksmith, Brillhart and Gerst theorem [37], has been used in [27] to prove some of the Ramanujan’s forty identities. Chen and Huang [49] derived two theorems for certain products of theta functions using Watson’s technique [84] which was used to establish some of the Ramanujan’s forty identities. Q. Yan [87] has used the technique of L. Carlitz and M. V. Subbarao [45], to establish a general identity for expanding the product of two theta functions, found in [51]. Recently, Z. Cao [44] gave a general theorem to write a product of $n$ theta functions as linear combinations of other products of theta functions.

In Chapter 4, we derive new general formulas to express the product of two theta functions as linear combinations of other products of theta functions. Several known theta function identities follow immediately from our formulas. In fact, we derive many identities for products of $\varphi(q)$, $\psi(q)$ and $f(-q)$ which are defined by Ramanujan and also we establish several modular relations for Rogers-Ramanujan functions, septic, nonic and dodecic analogues of the Rogers-Ramanujan functions.

One of the contributions of Ramanujan is the theory modular equations which takes a large space of his notebook [70], Chapters 19 and 20 contain many modular
equations for several degrees. Many approaches have been used by Berndt and his collaborators to prove these modular relations. One of the main methods which was used is the theory of theta functions and frequently employs Schröter's formulas. Berndt [20], remarked that "Ramanujan has not explicitly stated Schröter's formulas in any of his published papers, notebooks, or unpublished manuscripts. But, it seems clear, from the theory of theta functions that he did develop, that Ramanujan must have been aware of these formulas or at least of the principles that yield the many special cases that Ramanujan doubtless used. However, the majority of Ramanujan's modular equations do not appear to be direct results of Schröter's formulas. We conjecture that Ramanujan knew other general formulas involving theta functions, which are still unknown to us, and which he used to derive further modular equations. In particular, we think that Ramanujan derived an unknown general formula involving quotients of theta functions".

In Chapter 5, we employ the formulas presented in Chapter 4 to establish new identities that are analogous to the famous Schröter's formulas. As applications of these formulas, we establish several new modular relations for Rogers-Ramanujan functions, Göllnitz-Gordon, septic, nonic and dodecic analogues of the Rogers-Ramanujan functions.
1.3 The Borweins’ Cubic Theta Functions

In 1991, J. M. Borwein and P. B. Borwein [39] introduced and studied the following three functions:

\[ a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \]

\[ b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \]

and

\[ c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}, \]

where \( \omega = \exp(2\pi i/3) \). The functions defined in (1.3.1)–(1.3.3) are called the “cubic” theta-functions or Borweins’ cubic theta functions also called one-variable cubic theta functions. In 1994, the Borweins and F. Garvan [40] have developed the cubic functions theory and established many interesting results. In [40], they have provided an elementary proof of the identity

\[ a(q)^3 = b(q)^3 + c(q)^3, \]

which is a beautiful cubic analogue of Jacobi’s fundamental theta function identity

\[ \theta_3(q)^4 = \theta_2(q)^4 + \theta_4(q)^4. \]

In 1999, H. H. Chan and Y. L. Ong [46] have established few results pointing toward the beginnings of a theory of signature 7. They defined the analogue of \( \varphi(q) \) and \( a(q) \), in the septic theory, by

\[ \sigma(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}. \]

Chan and Ong [46] established some identities that are analogues of some identities found in [40] and many other identities related to modular equations.
The cubic theta functions defined by (1.3.1)–(1.3.3) play an important role in establishing one of Ramanujan’s alternative theories of elliptic functions (see [26]) and in the theory of representations of integers (see [24]). These functions have been studied and generalized by many mathematicians. In 1993, M. D. Hirschhorn, Garvan and Borwein [64], have introduced two-variable analogues of the cubic theta functions and they generalized many identities found in [40], as well as they established many new identities that are connecting these functions and theta functions. Two-variable cubic theta functions have been studied later by S. Cooper [53] who established some new results which are analogues of some results found in [64]. In 1995, S. Bhargava [31] introduced three-variable cubic theta functions and established several properties of them. Bhargava et al. [32, 33, 34] have developed this theory and generalized some identities found in [53]. More recently, in 2010, X. M. Yang [88], introduced four-variable cubic theta functions and established several identities, found in the literature, for the products of three theta functions.

In Chapter 6, we define three new functions which are generalizations of the cubic theta functions defined in (1.3.1)–(1.3.3) and we establish generalizations for some identities found in [40]. The main purpose of this chapter is to establish general formulas that are connecting our functions and Ramanujan’s general theta function. Many identities found in the literature follow as a special case of our identities, for example the main results of [33]. We also derive general formulas for certain products of theta functions.
1.4 \(q\)-Continued Fractions

Ramanujan has made substantial contributions to many areas of the mathematics. One of his main contributions is in the theory of \(q\)-continued fractions. The well-known continued fraction of Ramanujan is the Rogers-Ramanujan continued fraction

\[
R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots, \quad |q| < 1,
\]  

(1.4.1)

which first appeared in a paper by Rogers [74] in 1894. This continued fraction has many representations, for example it can be expressed in terms of infinite products as follows:

\[
R(q) = q^{1/5} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}.
\]  

(1.4.2)

Identity (1.4.2) has been proved by both Rogers [74] and Ramanujan [70, Vol. II, Chapter 16, Section 15], [1]. Page 56 of Ramanujan’s lost notebook is devoted to identities of theta functions and the continued fraction \(R(q)\). On page 365 of his Lost Notebook [72], Ramanujan wrote down five identities which show the relations between \(R(q)\) and the five continued fractions \(R(-q)\), \(R(q^2)\), \(R(q^3)\), \(R(q^4)\) and \(R(q^5)\).

In his first letter to G. H. Hardy [71, pp. xxvii], Ramanujan gave the first non-elementary evaluations of \(R(q)\), namely \(R(e^{-2\pi})\) and \(R(-e^{-\pi})\). Proofs of these can be found in [22]. On page 366 of his lost notebook [72], Ramanujan investigated the continued fraction

\[
G(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \cdots, \quad |q| < 1,
\]  

(1.4.3)

which is known as Ramanujan’s cubic continued fraction. He claimed that there are many results of \(G(q)\) which are analogous to those of \(R(q)\).

Ramanujan-Göllnitz-Gordon continued fraction \(H(q)\) is defined by

\[
H(q) := \frac{q^{1/2}}{1 + q} + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \frac{q^6}{1 + q^7} + \cdots, \quad |q| < 1.
\]  

(1.4.4)

In his notebook [70, p. 229], Ramanujan offered many identities for \(H(q)\).
The above three $q$-continued fractions have been studied by many mathematicians who have found many modular relations involving them and they represented in many forms, for example in terms of infinite products, theta functions, $n$-dissections for some positive integer $n$ and integrals.

Motivated by Ramanujan theory for these continued fractions, many other continued fractions have been discovered and studied.

In Chapter 7, we establish two $q$-series representations of Ramanujan’s continued fraction found in his lost notebook. As a special case, we obtain the following continued fraction:

$$N(q) := \frac{1}{1 - \frac{2q}{1} - \frac{q^2 - q}{1} - \frac{q^3 + q^2}{1} - \frac{q^4 - q^2}{1} - \ldots}, \quad |q| < 1. \quad (1.4.5)$$

We obtain the infinite products representation of $N(q)$, also we establish three equivalent integral representations of $N(q)$ and some modular equations for the continued fraction $N(q)$.

If $q = e^{-\pi \sqrt{n}}$ where $n$ is a positive integer, two class invariants $G_n$ and $g_n$ [22] are defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q) \quad \text{and} \quad g_n := 2^{-1/4} q^{-1/24} \chi(-q), \quad (1.4.6)$$

which are called Ramanujan-Weber class invariants. In Chapter 7, we also find continued fraction representations for the Ramanujan-Weber class invariants $g_n$ and $G_n$ and obtain several relations between them. Furthermore, we derive relations between our continued fraction $A(q) := N(q) + 1$ with the Ramanujan-Göllnitz-Gordon and Ramanujan’s cubic continued fractions.