Chapter 1

INTRODUCTION

1.1 Motivation

Economic management of commercially exploited natural renewable resources has been very extensively studied in the past. The natural resources are being subject to human harvesting and are also subject to biomass uncertainty in their growth resulting from the random fluctuations in the availability of nutrients and stochastic changes in the environmental factors. The biological and economical implications of harvesting of these resources in the deterministic environment have been thoroughly investigated (see Clark[18]). Beddington and May [8] and May et al. [52] have explored the problem of harvesting in the stochastic environments. In extension of their work, several mathematical models for the description of the stochastic nature of the growth of the resources subject to harvesting have been formulated in the form of stochastic differential equations and they have been analysed by applying the theory of stochastic optimal control (see, for example, Gleit [34], [35], Ludwig [46], Ludwig and Varah [50], Boyce and Daley [10], Anderson and Sutinen [2], Ryan and Hanson [57],[58], Clark[19], Conrad and Clark [23], Braumann [12], [13].[14], Gyllenberg et al. [38], Lungu and Oksendal [48], [49], Alvarez and
Shepp [1], Freckleton et al. [31], Costello and Polasky [25] and Li and Wang [45]). Gleit [34], [35] has studied the problem of optimal harvesting of a biological population which varies continuously over time and fluctuates randomly in the growth. He has formulated a stochastic differential equation driven by a Gaussian white-noise process to describe the growth of the population and obtained explicitly the optimal harvesting policies by using the Bellman's optimality conditions. He has also applied the mean-variance criterion (see Markowitz [51]) in establishing the existence of optimal harvesting policies. Ludwig [46] has obtained the harvesting policies for different growth functions of randomly fluctuating populations. Braumann [12] has examined various harvesting strategies by proposing quite general stochastic differential equation models for the growth of populations subject to harvesting in a random environment. Ryan and Hanson [57] have investigated the optimal discounted present value of commercially exploited populations with constant effort harvesting in random environment subject to random occurrences of bonanzas and disasters as independent Poisson processes. Ryan [56] has studied the economic consequences of stochastic price fluctuations on the optimal harvesting strategy for standard resource models. Gyllenberg et al. [38] have considered a discrete-time model of a population with stochastic carrying capacity and obtained optimal harvesting policies. Lunug and Oksendal [48] have considered a population in a stochastic crowded environment and obtained the optimal strategy for harvesting. Alvarez and Shepp [1] have considered a stochastic logistic model for a stochastically fluctuating population and obtained the optimal harvesting plan to maximize the expected discounted number of individuals harvested over an infinite future horizon. Lunug and Oksendal [49] have considered the problem of optimal harvesting from interacting populations in a stochastic environment and have obtained optimal harvesting strategy. In Braumann [13], [14], variable effort harvesting models of natural renewable resources varying in random environments are considered and the existence of stationary distributions is analysed. Freckleton et al. [31] have studied how the timing of harvesting affects the growth of populations. In the recent years, Xu et al. [63], Costello and Polasky [25] and Li and Wang [45] have considered the influence of the environmental carrying capacity in the determination of optimal harvesting policies. Chen and Insley [17] have analysed a regime switching model of stochastic lumber prices in a tree harvesting
problem which is modulated by a mean reverting process. They have used Hamilton-
Jacobi-Bellman variational inequality and obtained optimal harvesting prices. Based on
the survey of the existing economic models on harvesting, we are motivated in several
directions to consider some realistic situations which have not been given due importance
in the preceding studies.

I Motivation

The influence of the environmental carrying capacity in the determination of optimal
harvesting policies has been highlighted in Xu et al. [63], Costello and Polasky [25] and
Li and Wang [45]. They have assumed that the carrying capacity is either a constant or a
deterministic continuous periodic function of time. However, in natural circumstances, it is
not so. For example, due to sudden changes in the environment, the carrying capacity
switches from one level to another level instantaneously. Gyllenberg et al. [38] have
taken this aspect into account and formulated discrete-time model in the determination
of optimal harvesting policies. Accordingly, this situation triggers us to take up the project
of investigation of optimal harvesting policies in a continuous-time population subject to
stochastic carrying capacity.

II Motivation

Imhof and Walcher [39] have first introduced and analysed a stochastic model of the
growth and removal of a microbe in a chemostat by setting up and analyzing a stochastic
differential equation which takes into account the influence of random fluctuations in the
concentration of the nutrient. Using a comparison principle, they have established that their
model predicts persistence of any species growing in the medium. This paper has triggered
our interest in extending their work by proposing a stochastic model to describe the
dynamics of two interconnected and inhibited competing microbial populations growing
in a random environment.
III Motivation

In the work of Collier and Krementz [21], the individuals of the population subject to harvesting are classified as reproductive adults and young ones. The juveniles are not harvested since they are the potential individuals for the future growth of the population. Collier and Krementz [21] have not considered the investigation of optimal harvesting policies. This gap warrants investigation on the problem of optimal selective harvesting of age-structured population.

IV Motivation

In the harvesting of medicinal plants, branching nature of plants plays a dominant role. Recently, Ghimire et al. [33] have considered demographic variation and population viability in a threatened Himalayan medicinal and aromatic herb and studied harvesting effects in the species by a matrix population model. The fundamental principle in a matrix population model is based on the branching aspect of the individuals of the population. This aspect has not so far been considered in the investigation of optimal harvesting policies for renewable resources. Accordingly, it is considered worthwhile to formulate the model of Ghimire et al. [33] in terms of a Galton-Watson branching process and obtain optimal harvesting policy for the plant population.

V Motivation

In the work of Boyce et al. [11], it is observed that the fluctuations in the population and the ultimate yield are governed by the intrinsic growth rate and the environmental variations, and the growth rate experiences a bath-tub like cyclical variation (i.e., decreasing, fighting for survival and increasing). This situation forces us to focus on the problem of optimal harvesting of a stochastic organisms which is affected by the cyclical fluctuations.
1.2 The Scope of the Thesis

In this thesis, we consider five different situations which arise in harvesting of natural/renewable resources. These situations have not been considered so far in literature. We justify below how our results are not only generalizations of the previous results but also new in concepts.

(a) Gyllenberg et al. [38] have considered a discrete-time model of a population with stochastic carrying capacity. No work on continuous-time model of a population subject to a stochastically varying carrying capacity could be found in literature. We are the first to consider the carrying capacity $K$ as a stochastic process in the problem of investigation of optimal harvesting policy for continuous time population model. To be specific, we model $K$ by the equation

$$K(t) = \alpha(1 - \epsilon(-1)^{N(t)} \cos \omega t)^{-1},$$

where $N(t)$ is a Poisson process with rate $\lambda$. If we set $\epsilon = 0$, then we get back, as a particular case, the model of constant carrying capacity which has been very extensively studied. The complexity in the analysis of our model stems from the fact that the biomass $X(t)$ of the population is no longer Markov. Identifying the Markov property of the joint process $(X(t), N(t))$, we are able to analytically solve the optimal harvesting problem. Our model has the assumption that the carrying capacity fluctuates between two deterministic barriers $\alpha(1 - \epsilon \cos \omega t)^{-1}$ and $\alpha(1 + \epsilon \cos \omega t)^{-1}$. This restriction paves the way for further research in that $K(t)$ may randomly switch to different functional forms and the switching times may occur according to a renewal process.

(b) Ryan and Hanson [57],[58] have considered a single species in a random environment subject to harvesting with the occurrence of bonanzas and disasters. Imhof and Walcher [39] have first introduced and analysed a stochastic model of the growth and removal of a microbe in a chemostat which takes into account the influence of random fluctuations in the concentration of the nutrient. In this thesis, we have extended their
model to the case of two species growing in a random chemostat subject to harvesting with the occurrence of bonanzas and disasters. The primary interest in the formulation of this model is to apply the model in the study of the positive growth of several microbes in a chemostat. The net effect of bonanzas and disasters is described by a pooled Poisson process $P(t)$ with rate $\lambda$ and consequently the bio-masses $X_1(t)$ and $X_2(t)$ of the two species at time $t$ are governed by the system of stochastic differential equations

$$dX_1(t) = (r_1 - E_1)X_1(t)dt - a_1X_2(t)dt + b_1X_1(t)dP(t),$$

$$dX_2(t) = (r_2 - E_2)X_2(t)dt - a_2X_1(t)dt + b_2X_2(t)dP(t),$$

where $E_i, i = 1, 2$ be the constant harvesting effort per unit mass per unit time expended on the $i$-th population. We have succeeded in obtaining a partial differential equation for the characteristic function $\phi(\theta_1, \theta_2, t)$ of the joint process $(X_1(t), X_2(t))$. The novelty in this model is that the parameters $a_1, a_2, b_1$ and $b_2$ can be controlled and the populations can be made to grow exponentially irrespective of the fact that they compete to live in the same environment. For future studies, the model can be extended by taking into consideration the occurrence of bonanzas and disasters separately.

(c) Collier and Krementz [21] have considered a population subject to harvesting where the individuals of the population are classified as reproductive adults and young ones. The juveniles are not harvested since they are the potential individuals for the future growth of the population. Collier and Krementz [21] have not considered the investigation of optimal harvesting policies. In this thesis, as an innovation, we have distinguished the individuals as juveniles and adults and considered selective harvesting in the population of adults only. To accomplish this, we have split the habitat into two contiguous regions, one region occupied by young ones and the other by adults. Some of the new concepts in our model are (i) the two regions are inter-connected by birth-death-migration by maturity-harvesting, (ii) we have considered stochastic noise in the adult region only and (iii) we have introduced a cost of protecting young ones in the optimal harvesting analysis. For our future study, we
can analyse a model in which the recruitment of young-ones from outside the population is 
made to maintain the size of population of the young-ones.

(d) Matrix population models have been employed in studying harvesting effects in a 
population subject to demographic variation and population variability. Matrix population 
models are based on the branching aspect of the individuals of the population. Branching 
processes have not been used in formulating a stochastic model of a population subject to 
demographic variation and population viability. Our model in this thesis is a Galton-Watson 
branching process and we have newly introduced a concept called binomial harvesting. We 
have obtained an optimal harvest policy for the renewable resource. For future study, we 
can consider an age-dependent branching process and investigate the problem of optimal 
harvesting.

(e) There are situations where a population of species performs a cyclical bath-tub like 
motion (i.e. a downward growth, then followed by a recovery period, and then followed by 
an upward growth and then repeating the cycle). That is, the organism performs a random 
motion with three constant velocities cyclically over random intervals of time. In this thesis, 
we are the first to identify the intrinsic growth rate $\mu(t)$ by the equation

$$\mu(t) = \frac{1}{3} (\mu_1 + \mu_2 + \mu_3) + \frac{1}{3} (\mu_1 + \mu_2 \omega^2 + \mu_3 \omega)\omega^{N(t)} + \frac{1}{3} (\mu_1 + \mu_2 \omega + \mu_3 \omega^2)\omega^{2N(t)},$$

where $\omega$ is the imaginary cube root of unity and $N(t)$ is a Poisson process. Based on this, 
we have found an optimal harvesting policy for the population. For future study, we can 
take up the situation where the growth rate performs a random motion with $n$ velocities 
($n > 3$) and study an optimal harvesting problem.
1.3 Common Notation & Methodology

Notation

We use the following notation throughout the thesis.

\( \mathcal{P} \) stands for probability function.

\( \mathcal{E} \) stands for mathematical expectation.

\( E \) stands for the harvesting effort per unit mass per unit time.

\( \delta \) stands for the discount rate.

Methodology

The resource or population size is modeled either as a discrete or continuous variable. For example, if the resource under study is the fish population, then the biomass of the population is taken as the size of the population. On the other hand, if we study the population of deers in a forest, then the number of deers in the population is taken as the size of the population. Due to several unpredictable factors such as climatic conditions, competition for food and carrying capacity, the size of the resource is randomly changing with respect to time. If \( X(t) \) denotes the size of the population/resource subject to harvesting, then \( X(t) \) is usually described by a stochastic differential equation of the form

\[
\frac{dX(t)}{dt} = [r(X(t), t) - h(t)]dt + \sigma(X(t), t)dW(t),
\]

where \( r(X(t), t) \) is called the intrinsic growth rate, \( h(t) \) is the harvest rate, \( \sigma(X(t), t) \) is the dispersion coefficient and \( W(t) \) is a stochastic process representing the random noise. There are various types of harvest rate. In this thesis, we consider the most frequently used
linear law:

\[ h(t) = EX(t), \]  \hspace{1cm} (1.3.2)

where \( E \) is the constant rate of harvesting effort. This is in fact a limitation in our study. This form has been chosen for the purpose of numerical illustration. For nonlinear law, we can give existence and uniqueness theorems. The effort \( E \) can be called the control on the harvest rate \( h(t) \) since by choosing \( E \) differently, we get different harvest rate \( h(t) \). We assume that the initial biomass level is \( x_0 \). Then \( X(0) = x_0 \). We consider the total discounted net revenue \( V \). The objective is to maximize \( V \) over all the admissible values of \( E \). The value of \( E \) which maximizes \( V \) is called the optimal harvesting policy. The methodology for obtaining the optimal \( E \) is given below:

Let \( p \) be the price per unit of the harvested resource and \( c \) be the cost of a unit effort. Let \( R(X_E(t), E)\Delta t \) be the net economic revenue produced by exercising an amount \( E\Delta t \) of effort. Then

\[ R(X_E(t), E)\Delta t = (p h(t) - cE)\Delta t = (pX_E(t), E - c, E)\Delta t. \]  \hspace{1cm} (1.3.3)

If \( \delta > 0 \) is a constant rate of discount, then \( V \) is given by

\[ V = \int_0^T e^{-\delta t}(pX_E(t) - c)Edt, \]  \hspace{1cm} (1.3.4)

where \( T \) is the time-duration of the harvesting process. In real situations, \( T \) is usually a positive constant and is called the time horizon. Eliminating \( E \) between (1.3.1) and (1.3.3), we get a stochastic functional

\[ V = \int_0^T e^{-\delta t} \left( p - \frac{c}{X_E(t)} \right) [r(X_E(t), t)dt + \sigma(X(t), t)dW(t) - dX_E(t)]. \]  \hspace{1cm} (1.3.4)

The objective of harvesting is to maximize the expected total net revenue \( V \) by choosing a suitable harvest rate \( h(t) \). Equivalently, the objective of harvesting is to maximize the expected total net revenue \( V \) in (1.3.4) by choosing a suitable \( X_E(t) \). The function \( X_E(t) \) that maximizes \( V \) is called the stochastic equilibrium level of the population, denoted by \( X^*(t) \). The corresponding value \( E^* \) of \( E \) is the optimal harvesting policy.
There are several approaches for finding the optimal harvesting policies. In this thesis, we adopt two different approaches, namely, (a) Numerical search technique and (b) Mean-variance criterion.

(a) Numerical Search Technique: In this technique, we first find the mean value of $V$ with $E$ as a parameter. Then we compute the mean value of $V$ by varying $E$ over the admissible region. Among the computed mean values of $V$, we locate the maximum value and the corresponding value of $E$ gives the optimal harvesting policy $E^*$.

(b) Mean-variance criterion: This technique has been developed by Markowitz [51] for finding optimal portfolio selection in financial markets. In this technique, we fix the mean-value of $V$ and consider the deterministic integral

$$
\int_0^T \mathbb{E} [u(t) - m]^2 \, dt,
$$

where $\mathbb{E}$ stands for the mathematical expectation operator, $u(t) = e^{-\delta t} R(X_E(t), E)$ and

$$
\mathbb{E} \left[ \int_0^T e^{-\delta t} R(X_E(t), E) \, dt \right] = m.
$$

Applying the mean-variance rule, the optimal harvesting policy $E^*$ is obtained as the harvesting rule $E$ which minimizes the integral (1.3.5).

1.4 Organization of the Thesis

**Chapter 1** is introductory in style and structure.

**Chapter 2** discusses a continuous-time harvesting model of a population growth with stochastic variations in the carrying capacity. We investigate an optimal harvesting
policy in this chapter. The constant-effort harvesting model of a logistic population with stochastic carrying capacity is formulated. The carrying capacity is modeled as a two-state continuous-time Markov process. Using a two-dimensional stochastic process, a partial differential equation for the joint probability density function is formulated. An optimal harvesting policy is investigated and an illustrative numerical study is provided.

**Chapter 3** considers a stochastic model to describe the dynamics of two interconnected and inhibited competing microbial populations growing in a random environment and obtains certain economic analysis of the populations. Our model exhibits persistence in the mean. We provide the bio-economic model of the interconnected populations and obtain the forward equation for the probability distribution of an underlying Markov process. We obtain the mean values of the bio-masses of the two populations and discuss a procedure of obtaining the optimal harvesting policy by considering the expected net revenue realized through harvesting. A numerical study is carried out to show that our investigations are amenable of numerical treatment.

**Chapter 4** deals with the selective harvesting model in the study of the economics of the commercial exploitation of plant and animal populations. In this model, we consider a population where age plays a decisive role both in the reproductive process and the harvesting strategy. We provide the bio-economic model of the interconnected compartments of the populations. We derive the partial differential equation for the joint moment generating function of the biomasses of the compartments and also obtain the moments of the sizes. We also discuss the procedure of obtaining the optimal harvesting policy by undertaking an economic point of view of the model. We have investigated our results through a numerical study.

**Chapter 5** analyses the problem of optimal harvesting of a branching type of renewable resource. We propose and analyse two models. The first model is based on Galton-Watson branching process and the second model is based on a Markov branching process. For our first model of harvesting of a natural renewable resource, we provide the model description and obtain the equation for the probability generating function of the size of the $n$-th
1.5 **The Salient Features of the Thesis**

The salient features of the thesis which go to improve the state- of- the- art of the study of optimal harvesting models of renewable resources are given below:

(a) Consideration of a logistic population growth with a stochastic carrying capacity modulated by a Markov process.

(b) Consideration of an optimal harvesting problem pertaining to two interconnected and competing microbial populations growing in a random environment.

(c) Effective use of Bellman’s dynamic programming technique to derive the optimal harvesting policy.
(d) Consideration of interacting renewable resources and the discussion of the problem of coexistence subject to harvesting.

(e) Consideration of selective harvesting model in the study of the economics of the commercial exploitation of plant and animal populations.

(f) Effective usage of stochastic differential equations in the description of temporal fluctuations of the biomass of the renewable resources.

(g) Consideration of a problem of optimal harvesting of a branching type of renewable resource.

(h) Consideration of the optimal harvesting of an organism which evolves with a 3-component velocity vector.

(i) Usage of Markowitz's mean-variance criterion in establishing the existence of optimal controls.

(j) Extensive usage of numerical analysis.