Chapter 4

Monotone Method For System of Caputo Fractional Differential Equations

The content of this chapter is published in the following paper.

4.1 Introduction

The purpose of lubrication is to separate two surfaces sliding past each other with a film of gas or liquid material which can be sheared without causing any damage to the surfaces. The lubricant film thickness and dimensions of the bearing surfaces are chosen to insure that no contact occurs between the opposing surfaces. This type of lubrication is known as hydrodynamics lubrication. This can be modeled as boundary value problem under various assumptions. The periodic boundary value problems for differential equations of integer order are studied by many researchers due to its wide applications in many branches of sciences. Monotone method for periodic boundary value problems for differential equations of integer order is well developed by Lakhsmikantham et al.\cite{28}. Qualitative properties of the solutions of periodic boundary value problems for differential equations of integer order are studied using monotone method. The same approach was used by McRae to periodic boundary value problem for fractional differential equations. The first order periodic boundary value problem (PBVP) was considered by McRae \cite{38} and developed monotone iterative technique for PBVP. This motivates us to develop monotone iterative technique for weakly coupled system of
periodic boundary value problem (PBVP).

The purpose of this chapter is to develop the monotone method for weakly coupled system of Caputo fractional differential equations with periodic boundary conditions. As an application of the monotone method existence and uniqueness of solution of weakly coupled system of Caputo fractional differential equations with periodic boundary are obtained.

The plan of the chapter is as under:

In section 4.2, preliminary definitions and some basic results are given. Section 4.3 is devoted for the development of monotone method for weakly coupled system of PBVP with Caputo fractional differential equation. Existence and uniqueness of solution of weakly coupled system of PBVP are obtained. Concluding remarks are given in the last section.
4.2 Preliminaries

The initial value problem for Caputo fractional differential equation of order $q$ is

$$^cD^q u(t) = f(t, u(t)), \quad u(t_0) = u_0, \quad t_0 \leq t \leq T.$$ 

and the corresponding fractional Volterra integral equation [38] is

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, u(s)) ds. \quad (4.2.1)$$

We require the function space [37] $C_p([t_0, T], \mathbb{R})$ and is defined as

$$C_p([t_0, T], \mathbb{R}) = \left\{ u \in C([t_0, T], \mathbb{R}) \text{ and } (t-t_0)^p u(t) \in C([t_0, T], \mathbb{R}) \right\}$$

The sector is defined as

$$\Omega = \left\{ (t, u_1, u_2) \in [t_0, T] \times \mathbb{R}^2 : v_i(t) \leq u_i(t) \leq w_i(t) \right\}$$
Consider the following weakly coupled system of PBVP for Caputo fractional differential equation of order $q$ ($0 < q < 1$)

\[
\begin{align*}
^{c}D^{q}u_{1}(t) &= f_{1}(t, u_{1}(t), u_{2}(t)), \quad u_{1}(0) = u_{1}(2\pi) \\
^{c}D^{q}u_{2}(t) &= f_{2}(t, u_{1}(t), u_{2}(t)), \quad u_{2}(0) = u_{2}(2\pi)
\end{align*}
\]  

(4.2.2)

where $f_{1}, f_{2} \in C(J \times \mathbb{R}^{2}, \mathbb{R})$, $J = [0, 2\pi]$.

**Definition 4.2.1.** Suppose a pair of functions $(v(t), w(t))$ in $C^{1}(J, \mathbb{R})$ and $v(t) \leq w(t)$ are called ordered lower and upper solutions of PBVP (4.2.2) if, for $i = 1, 2$

\[
^{c}D^{q}v_{i}(t) \leq f_{i}(t, v_{1}(t), v_{2}(t)), \quad v_{i}(0) \leq v_{i}(2\pi)
\]

and

\[
^{c}D^{q}w_{i}(t) \geq f_{i}(t, w_{1}(t), w_{2}(t)), \quad w_{i}(0) \geq w_{i}(2\pi)
\]

**Definition 4.2.2.** Let $f_{i}(t, u_{1}(t), u_{2}(t))$ be real valued continuous function defined on $J \times \mathbb{R}^{2}$. We say that $f_{i}(t, u_{1}(t), u_{2}(t))$ satisfies Lipschitz condition if there exists $M_{i} \geq 0$ such that

\[
|f_{i}(t, u_{1}(t), u_{2}(t)) - f_{i}(t, u_{1}^{*}(t), u_{2}^{*}(t))| \leq M_{i}|u_{i} - u_{i}^{*}|, \quad i = 1, 2
\]

for all $(t, u_{1}, u_{2})$ and $(t, u_{1}^{*}, u_{2}^{*}) \in J \times \mathbb{R}^{2}$.
To develop monotone technique for PBVP (4.2.2), consider the explicit solution of the nonhomogeneous linear Caputo fractional differential equation [38] with initial condition

\[ ^cD^q u(t) = \lambda u(t) + f(t), \quad u(t_0) = u_0 \]

where \( f \in C([t_0, T], \mathbb{R}) \). The corresponding fractional Volterra integral equation is

\[ u(t) = u_0 + \frac{\lambda}{\Gamma(q)} \int_{t_0}^{t} \frac{u(s)}{(t-s)^{1-q}} ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-q}} ds. \]

Using the method of successive approximations [38], the solution of the fractional Volterra integral equation (4.2.1) is

\[ u(t) = u_0 E_q[\lambda(t-t_0)^q] + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s) ds \]

where

\[ E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk + 1)} \quad \text{and} \quad E_{q,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk + q)} \]

are Mittag-Leffler functions of one and two parameters respectively.
**Theorem 4.2.1.** Let $f(t, u(t)) \in C(J \times \mathbb{R}, \mathbb{R})$ and $v(t), w(t) \in C^1(J, \mathbb{R})$. Assume that for $0 \leq t \leq 2\pi$,

\[
^cD^q v(t) \leq f(t, v(t)), \quad v(0) \leq v(2\pi)
\]
\[
^cD^q w(t) \geq f(t, w(t)), \quad w(0) \geq w(2\pi).
\]

Further suppose $f(t, u)$ is strictly decreasing in $u$ for each $t$. Then

\[
v(t) \leq w(t), \quad 0 \leq t \leq 2\pi.
\]

**Proof:** It is by contradiction. Suppose that inequality (4.2.4) is not true, then, $v(t) > w(t)$. Thus there exists an $\epsilon > 0$ and $t_0 \in J$ such that $v(t_0) = w(t_0) + \epsilon$ and $v(t) < w(t) + \epsilon, 0 \leq t \leq 2\pi$. Setting $m(t) = v(t) - w(t) - \epsilon$, we find that, if $t_0 \in J, m(t_0) = 0$ and $m(t) < 0$, implies $m(t) < 0, 0 \leq t \leq t_0 \leq 2\pi$. If $t_0 = 0$ we have, $v(2\pi) \geq w(2\pi) + \epsilon$. Hence in all cases we have $m(t_0) = 0$ and $m(t) < 0$ for $0 \leq t \leq t_0 \leq 2\pi$. By Lemma 1.3.5, we have $^cD^q m(t_0) \geq 0$. Hence

\[
^cD^q v(t_0) \geq ^cD^q w(t_0).
\]

Using inequalities (4.2.3), we have

\[
f(t_0, v(t_0)) \geq ^cD^q v(t_0) \geq ^cD^q w(t_0) \geq f(t_0, w(t_0)) > f(t_0, v(t_0))
\]

which is a contradiction. Hence the theorem.
Corollary 4.2.2. Let \( m : J \rightarrow \mathbb{R} \) be continuous and satisfy

\[
c^{D}_{q}m(t) \leq -Mm(t), \quad 0 \leq t \leq 2\pi, \quad m(0) \leq m(2\pi), \quad M > 0,
\]

then \( m(t) \leq 0, \quad 0 \leq t \leq 2\pi. \)

Next consider the linear nonhomogeneous fractional differential equation with periodic boundary conditions

\[
c^{D}_{q}u(t) = -Mu(t) + h(t), \quad u(0) = u(2\pi), \quad (4.2.5)
\]

where \( M > 0, \quad h \in C(J, \mathbb{R}). \)

In order to prove the existence of a solution of the linear PBVP (4.2.5), we begin with the solution of initial value problem,

\[
c^{D}_{q}u(t) = -Mu(t) + h(t), \quad u(0) = u^{0}, \quad (4.2.6)
\]

given by

\[
u(t) = u^{0}E_{q}[Mt^{q}] + \int_{0}^{t} (t - s)^{q-1}E_{q,q}[M(t - s)^{q}]h(s)ds
\]

Putting \( t = 2\pi \), we obtain

\[
u(2\pi) = u^{0}E_{q}[M(2\pi)^{q}] + \int_{0}^{2\pi} (2\pi - s)^{q-1}E_{q,q}[M(2\pi - s)^{q}]h(s)ds
\]
Since \( u(2\pi) = u(0) = u^0 \) we have,

\[
u^0 = u^0 E_q[ Mt^q ] + \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}[ M(2\pi - s)^q ] h(s) ds
\]

\[
u^0 = \frac{1}{1 - E_q[ M(t-s)^q ]} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}[ M(2\pi - s)^q ] h(s) ds.
\]

The solution of the linear PBVP (4.2.5) is

\[
u(t) = \frac{E_q[ Mt^q ]}{1 - E_q[ Mt^q ]} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}[ M(2\pi - s)^q ] h(s) ds
\]

\[
+ \int_0^t (t-s)^{q-1} E_{q,q}[ M(t-s)^q ] h(s) ds.
\]  

\[ (4.2.7) \]

### 4.3 Monotone Method and Applications

In this section, the monotone method for weakly coupled system of Caputo fractional differential equations with periodic boundary conditions is developed. Using the lower and upper solutions, we construct two monotone convergent sequences which converges to minimal and maximal solutions. We obtain the existence and uniqueness of solution of PBVP using monotone method in the following

**Theorem 4.3.1.** Suppose that

1. \( v^0(t) = (v^0_1(t), v^0_2(t)) \) and \( w^0(t) = (w^0_1(t), w^0_2(t)) \) in \( C^1(J, \mathbb{R}) \) are ordered lower and upper solutions of PBVP (4.2.2) such that...
\begin{equation}
\begin{aligned}
v_i^0(0) \leq w_i^0(0) \text{ on } J
\end{aligned}
\end{equation}

(ii) \( f_i = f_i(t, u_1(t), u_2(t)) \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone nondecreasing

(iii) \( f_i \equiv f_i(t, u_1(t), u_2(t)) \) satisfies one-sided Lipschitz condition

\begin{equation}
\begin{aligned}
f_i(t, u_1, u_2) - f_i(t, u_1^*, u_2^*) \geq -M_i(u_i - u_i^*)
\end{aligned}
\end{equation}

whenever \( v_i^0 \leq u_i^* \leq u_i \leq w_i^0, \quad M_i \geq 0, \quad i = 1, 2. \)

Then there exist monotone sequences

\( \{v^n(t)\} = (v^n_1(t), v^n_2(t)) \) and \( \{w^n(t)\} = (w^n_1(t), w^n_2(t)) \) such that

\( \{v^n(t)\} \rightarrow v(t) = (v_1(t), v_2(t)) \text{ and } \{w^n(t)\} \rightarrow w(t) = (w_1(t), w_2(t)) \)

as \( n \rightarrow \infty \) and \( v(t), w(t) \) are extremal solutions of PBVP (4.2.2).

Proof. For any \( \eta = (\eta_1, \eta_2) \) in \( C^1(J, \mathbb{R}) \) such that for \( v_i^0 \leq \eta_i \leq w_i^0, \)
consider the following system of linear fractional differential equations with periodic boundary conditions

\begin{equation}
\begin{aligned}
cD^q u_i = f_i(t, \eta_1, \eta_2) - M_i(u_i - \eta_i), \quad u_i(0) = u_i(2\pi).
\end{aligned}
\end{equation}
This linear PBVP has a solution. Next we show that the linear PBVP (4.3.3) has a unique solution \( u(t) = (u_1(t), u_2(t)) \). If not, suppose \( x(t) = (x_1(t), x_2(t)) \) and \( y(t) = (y_1(t), y_2(t)) \) are two different solutions of linear PBVP (4.3.3).

Define \( p_i(t) = x_i(t) - y_i(t) \). Consider

\[
{^cD^q}p_i(t) = {^cD^q}x_i(t) - {^cD^q}y_i(t) \\
= f_i(t, \eta_1, \eta_2) - f_i(t, \eta_1, \eta_2) - M_i(x_i - \eta_i) + M_i(y_i - \eta_i) \\
= -M_i(x_i - y_i) \\
= -M_i p_i(t) \\
\text{and} \quad p_i(0) = p_i(2\pi).
\]

Apply Corollary 4.2.2, to the linear problem

\[
{^cD^q}p_i(t) = -M_i p_i(t) \\
\text{and} \quad p_i(0) = p_i(2\pi),
\]

we get \( p_i(t) \equiv 0 \). It follows that \( x(t) \equiv y(t) \) and uniqueness is proved.

Define a mapping \( A \) by \( A[\eta, \mu] = u(t) \), where \( u(t) \) is the unique solution of linear PBVP (4.3.3). Now we prove the following statements...
(N1): \( v^0 \leq A[v^0, w^0], \ w^0 \geq A[w^0, v^0] \)

(N2): \( A \) possesses the monotone property on the segment

\[
[v^0, w^0] = \left\{ u(t) \in C^1(J, \mathbb{R}) : v^0 \leq u \leq w^0 \right\}.
\]

First we prove (N1). Set \( A[v^0, w^0] = v^1(t) \), where \( v^1(t) = (v^1_1, v^1_2) \) is the unique solution of linear PBVP (4.3.3) and \( v^0 \) is a lower solution of PBVB (4.2.2).

Set \( p_i(t) = v^0_i(t) - v^1_i(t) \) with \( \eta_i = v^0_i \), we observe that

\[
{c}D^q p_i(t) = {c}D^q v^0_i(t) - {c}D^q v^1_i(t)
\]
\[
\leq f_i(t, v^0_1, v^0_2) - f_i(t, v^0_1, v^0_2) + M_i(v^1_i - v^0_i)
\]
\[
\leq -M_i(v^0_i - v^1_i)
\]
\[
\leq -M_i p_i(t)
\]

and \( p_i(0) \leq p_i(2\pi) \).

Apply Corollary 4.2.2 to the linear problem

\[
{c}D^q p_i(t) \leq -M_i p_i(t)
\]

and \( p_i(0) \leq p_i(2\pi) \),
we have $p_i(t) \leq 0$ for $0 \leq t \leq 2\pi$ implies $v^0 \leq A[v^0, w^0]$.

Next we prove that $w^0 \geq A[w^0, v^0]$.

Set $A[w^0, v^0] = w^1$, where $w^1 = (w^1_1, w^1_2)$ is the unique solution of linear PBVP (4.3.3) and $w^0$ is a upper solution of PBVP (4.2.2).

Set $p_i = w^0_i - w^1_i$ with $\mu_i = w^0_i$ we see that

$$
cD^q p_i(t) = cD^q w^0_i(t) - cD^q w^1_i(t)
\geq f_i(t, w^0_1, w^0_2) - f_i(t, w^1_1, w^1_2) + M_i(w^1_i - w^0_i)
\geq -M_i(w^0_i - w^1_i)
\geq -M_i p_i(t)
$$

and $p_i(0) \geq p_i(2\pi)$.

Apply Corollary 4.2.2, to the linear problem

$$
cD^q p_i(t) \geq -M_i p_i(t)
$$

and $p_i(0) \geq p_i(2\pi)$,

we have $p_i(t) \geq 0$ for $0 \leq t \leq 2\pi$.

Thus $w^0_i \geq w^1_i$ implies $w^0 \geq A[w^0, v^0]$.

To prove $(N_2)$, let $\eta = (\eta_1, \eta_2)$ and $\mu = (\mu_1, \mu_2)$ in $[v^0, w^0]$ be such that $\eta_i \leq \mu_i$. Suppose that $A[\eta_i, \mu] = u_i = (u^1_i, u^2_i)$ and
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$A[\eta, \mu_i] = v_i = (v_i^1, v_i^2)$. Consider $p_i(t) = u_i(t) - v_i(t)$, using Lipschitz condition we obtain

$$c^D q p_i(t) = c^D q (u_i(t) - v_i(t))$$

$$= c^D q u_i(t) - c^D q v_i(t)$$

$$= f_i(t, \eta_1, \eta_2) - f_i(t, \mu_1, \mu_2) - M_i(u_i - \eta_i) + M_i(v_i - \mu_i)$$

$$\leq -M_i(\eta_i - \mu_i) - M_i(u_i - \eta_i) + M_i(v_i - \mu_i)$$

$$\leq -M_i(u_i - v_i)$$

$$\leq -M_i p_i(t)$$

and $p_i(0) = p_i(2\pi)$.

Apply Corollary 4.2.2, to the linear problem

$$c^D q p_i(t) \leq -M_i p_i(t)$$

and $p_i(0) = p_i(2\pi)$,

we have $u_i(t) \leq v_i(t)$. Hence $A[\eta_i, \mu] \leq A[\eta, \mu_i]$. Thus operator $A$ possesses monotone property on $[v^0, w^0]$.

Define the sequences $\{v_i^n(t)\}$ and $\{w_i^n(t)\}$ by $v_i^n = A[v_i^{n-1}, w_i^{n-1}]$ and
\[ w^n_i = A[w^{n-1}_i, v^{n-1}_i] \] respectively. Using \((N_1)\) and \((N_2)\), we obtain

\[ v^0_i(t) \leq v^1_i(t) \leq ... \leq v^n_i(t) \leq w^n_i(t) \leq ... \leq w^1_i(t) \leq w^0_i(t), \quad i = 1, 2. \]

Clearly the sequences \(\{v^n_i(t)\}\) and \(\{w^n_i(t)\}\) are uniformly bounded and convergent. Also the sequences \(\{cD^q v^n_i(t)\}\) and \(\{cD^q w^n_i(t)\}\) are uniformly bounded on \(J\) in view of the fact

\[ cD^q v^{n+1}_i(t) = f_i(t, v^n_1, v^n_2) - M_i(v^{n+1}_i - v^n_i), \quad v^{n+1}_i(0) = v^{n+1}_i(2\pi) \]

\[ cD^q w^{n+1}_i(t) = f_i(t, w^n_1, w^n_2) - M_i(w^{n+1}_i - w^n_i), \quad w^{n+1}_i(0) = w^{n+1}_i(2\pi). \]

Then by Lemma 1.3.6, it follows that the sequences \(\{v^n_i(t)\}\) and \(\{w^n_i(t)\}\) are equicontinuous on \(J\). Applying Ascoli-Arzela Theorem, it follows that, as \(n \to \infty\)

\[ v^n_i(t) \to v_i(t) \text{ and } w^n_i(t) \to w_i(t) \text{ where} \]

\[ \lim_{n \to \infty} v^n_i(t) = v_i(t) \quad \text{and} \quad \lim_{n \to \infty} w^n_i(t) = w_i(t) \quad \text{on } J. \]

From the corresponding fractional Volterra integral equations

\[ v^n_i(t) = v^0_i + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \left\{ f_i(s, v^n_1, v^n_2) - M_i(v^n_i - \eta_i) \right\} ds \]

\[ w^n_i(t) = w^0_i + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \left\{ f_i(s, w^n_1, w^n_2) - M_i(w^n_i - \mu_i) \right\} ds \]
or equation (4.2.7), it follows that \( v(t) = (v_1(t), v_2(t)) \) and \( w(t) = (w_1(t), w_2(t)) \) are the solutions of PBVP (4.2.2).

Our next result is to show that \( v(t) \) and \( w(t) \) are the minimal and maximal solutions of PBVP (4.2.2). If \( u(t) = (u_1(t), u_2(t)) \) is any solution of PBVP (4.2.2) such that \( v_0^i \leq u_i \leq w_0^i \) then

\[
\begin{align*}
v_0^i &\leq v_i \leq u_i \leq w_0^i.
\end{align*}
\]

To see this, suppose that for some \( k > 0 \), \( v_i^k \leq u_i \leq w_i^k \) on \( J \) and prove that \( v_i^{k+1} \leq u_i \leq w_i^{k+1} \) on \( J \).

Firstly we prove \( v_i^{k+1}(t) \leq u_i(t) \). Setting \( p_i(t) = v_i^{k+1}(t) - u_i(t) \) and, we observe that

\[
\begin{align*}
^cD^q p_i(t) &= ^cD^q(v_i^{k+1}(t) - u_i(t)) \\
&= ^cD^q v_i^{k+1}(t) - ^cD^q u_i(t) \\
&= f_i(t, v_1^k, v_2^k) - M_v(v_i^{k+1} - v_i^k) - f_i(t, u_1, u_2)
\end{align*}
\]

By Lipschitz condition

\[
\begin{align*}
&\leq -M_v(v_i^k - u_i) - M_v(v_i^{k+1} - v_i^k) \\
&\leq -M_v(v_i^{k+1} - u_i) \\
&\leq -M_v p_i(t)
\end{align*}
\]

and \( p_i(0) = p_i(2\pi) \).
Apply Corollary 4.2.2, to the linear problem

\[ c^Dq p_i(t) \leq -M_i p_i(t) \]

and \( p_i(0) = p_i(2\pi) \),

we get \( v_i^{k+1}(t) \leq u_i(t) \). Now we prove that \( u_i(t) \leq w_i^{k+1}(t) \) on \( J \).

Define \( p_i(t) = u_i(t) - w_i^{k+1}(t) \). Consider

\[ c^Dq p_i(t) = c^Dq u_i(t) - c^Dq w_i^{k+1}(t) \]

\[ = f_i(t, u_1, u_2) - f_i(t, w_1^k, w_2^k) + M_i(w_i^{k+1} - w_i^k) \]

\[ \leq -M_i(u_i - w_i^k) + M_i(w_i^{k+1} - w_i^k) \]

By Lipschitz condition

\[ \leq -M_i(u_i - w_i^{k+1}) \]

\[ \leq -M_i p_i(t) \]

and \( p_i(0) = p_i(2\pi) \).

Apply Corollary 4.2.2, to the linear problem

\[ c^Dq p_i(t) \leq -M_i p_i(t) \]

and \( p_i(0) = p_i(2\pi) \),

it follows that \( u_i(t) \leq w_i^{k+1}(t) \) on \( J \). Thus we have \( v_i^{k+1} \leq u_i \leq w_i^{k+1} \)
on $J$. Since $v_i^0(t) \leq u_i(t) \leq w_i^0(t)$ on $J$, by principle of mathematical induction it follows that $v_i^n(t) \leq u_i(t) \leq w_i^n(t)$ for all $n$. Taking limit as $n \to \infty$, we get

$$v_i(t) \leq u_i(t) \leq w_i(t)$$
on $J$.

Hence $v(t)$ and $w(t)$ are extremal solutions of PBVP (4.2.2).

Above theorem can be proved when the function $f_i = f_i(t, u_1, u_2)$ in $C[J \times \mathbb{R}^2, \mathbb{R}]$ is quasimonotone nonincreasing with minor changes and the result is stated as under.

**Theorem 4.3.2.** Suppose that

(i) $v^0(t) = (v_1^0(t), v_2^0(t))$ and $w^0(t) = (w_1^0(t), w_2^0(t))$ in $C^1(J, \mathbb{R})$ are ordered lower and upper solutions of PBVP (4.2.2) such that

$$v_i^0(0) \leq w_i^0(0)$$
on $J$

(ii) $f_i = f_i(t, u_1(t), u_2(t))$ in $C[J \times \mathbb{R}^2, \mathbb{R}]$ is quasimonotone nonincreasing

(iii) $f_i \equiv f_i(t, u_1(t), u_2(t))$ satisfies one-sided Lipschitz condition

$$f_i(t, u_1, u_2) - f_i(t, u_1^*, u_2^*) \geq -M_i(u_i - u_i^*)$$

(4.3.3)

whenever $v_i^0 \leq u_i^* \leq u_i \leq w_i^0$, $M_i \geq 0$, $i = 1, 2$. 

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Then there exist monotone sequences
\[ \{v^n(t)\} = (v^n_1(t), v^n_2(t)) \] and \[ \{w^n(t)\} = (w^n_1(t), w^n_2(t)) \] such that
\[ \{v^n(t)\} \to v(t) = (v_1(t), v_2(t)) \quad \text{and} \quad \{w^n(t)\} \to w(t) = (w_1(t), w_2(t)) \]
as \( n \to \infty \) and \( v(t), w(t) \) are extremal solutions of PBVP (4.2.2).

**Proof.** It is similar to above theorem. We omit the details.

**Theorem 4.3.3.** Suppose that assumptions (i)-(iii) in Theorem 4.3.1 hold and if, for \( 0 < N_i < M_i, \quad i = 1, 2 \)

\[ f_i(t, u_1, u_2) - f_i(t, u^*_1, u^*_2) \leq -N_i(u_i - u^*_i) \quad v^0_i \leq u^*_i \leq u_i \leq w^0_i, \]

then the PBVP (4.2.2) has unique solution.

**Proof.** To prove uniqueness it is enough to prove that \( v_i(t) \geq w_i(t) \).

Set \( p_i(t) = w_i(t) - v_i(t) \), we observe that

\[ ^cD^q p_i(t) = ^cD^q w_i(t) - ^cD^q v_i(t) \]

\[ = f_i(t, w_1, w_2) - f_i(t, v_1, v_2) \]

\[ \leq -M_i(w_i - v_i) \quad \text{By Lipschitz condition} \]

\[ \leq -M_i p_i(t) \]

and \( p_i(0) = p_i(2\pi) \).
Apply Corollary 4.2.2, to the linear problem

\[ ^cD^q p_i(t) \leq -M_i p_i(t) \]

and \( p_i(0) = p_i(2\pi) \),

we have \( p_i(t) \leq 0 \) on \( J \) implies \( v_i(t) \geq w_i(t) \). Hence the result.

Above theorem can be proved when the function \( f_i = f_i(t, u_1, u_2) \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone nonincreasing with minor changes and the result is stated as under.

**Theorem 4.3.4.** Suppose that assumptions (i)-(iii) in Theorem 4.3.2 hold and if, for \( 0 < N_i < M_i, \quad i = 1, 2 \),

\[ f_i(t, u_1, u_2) - f_i(t, u_i^*, u_i^*) \leq -N_i(u_i - u_i^*) \quad v_i^0 \leq u_i^* \leq u_i \leq w_i^0, \]

then the PBVP (4.2.4) has unique solution.

**Proof.** It is similar to above theorem. We omit the details.
4.4 Concluding Remarks

Remark 4.4.1. Choosing initial iterations \(v_i^0(t), w_i^0(t)\), two sequences of iterations \(\{v^n(t)\}\) and \(\{w^n(t)\}\) are obtained when the function \((f_1, f_2)\) is quasimonotone nondecreasing (nonincreasing). They possesses monotone property.

Remark 4.4.2. We have developed monotone method for the system of Caputo fractional periodic boundary value problem (4.2.4) in which function \((f_1, f_2)\) is quasimonotone nondecreasing (nonincreasing).

Remark 4.4.3. As an application of the monotone method, existence and uniqueness of solution of the system of Caputo fractional periodic boundary value problem (4.2.4) are obtained when the function \((f_1, f_2)\) is quasimonotone nondecreasing (nonincreasing).