Chapter 3

Monotone Method For System of Riemann-Liouville Fractional Differential Equations

The content of this chapter is published in the following paper.

3.1 Introduction

In the previous chapter we studied Riemann-Liouville fractional differential equation (2.2.1) with integral boundary condition (2.2.2) when the function on the right hand side is sum of nondecreasing and nonincreasing functions and developed monotone method for the Riemann-Liouville fractional integral boundary value problem (2.2.1)−(2.2.2). In this chapter we extend these results to the system of Riemann-Liouville fractional differential equations with integral boundary conditions under the condition that $f_1, f_2$ is quasimonotone nondecreasing.

The weakly coupled system of Riemann-Liouville fractional differential equations with integral boundary conditions is given by

\[ D^q u_1(t) = f_1(t, u_1(t), u_2(t)), \]
\[ u_1(0) = \int_0^T u_1(s)ds + d \]  
\[ D^q u_2(t) = f_2(t, u_1(t), u_2(t)), \]
\[ u_2(0) = \int_0^T u_2(s)ds + d \]  

(3.1.1)

where $d \in \mathbb{R}, \quad t \in J \quad f_1, f_2 \in C(J \times \mathbb{R}^2, \mathbb{R})$. Here $f_1, f_2$ are quasimonotone nondecreasing functions and $0 < q < 1$. 

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This is called system of Riemann-Liouville fractional integral boundary value problem.

The plan of the chapter is as follows:

Section 3.2 is concerned with the statement of the problem, basic definitions and basic results. Section 3.3 is devoted for the construction of two monotone sequences possessing monotone property. Monotone method is developed in this way and as an application of the monotone method qualitative properties such as existence and uniqueness of solution of the problem are obtained. In the last section, concluding remarks are given.

3.2 Basic Definitions and Lemmas

In this section we consider some basic definitions and basic results required to develop monotone method for the system of Riemann-Liouville fractional integral boundary value problem (3.1.1). The following theorem play an important role in further discussion. We state it as follows.
Theorem 3.2.1. [16] Let the functions $v(t), w(t) \in C_p([t_0, T], \mathbb{R}) \quad f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and

(i) $D^q v(t) \leq f(t, v(t))$ and

(ii) $D^q w(t) \geq f(t, w(t)), \quad t_0 < t \leq T.$

There exists $M \in (0, \frac{1}{(1-q)T^q})$ such that $f(t, u(t))$ satisfy one sided Lipschitz condition

$$f(t, u(t)) - f(t, v(t)) \leq M(u - v), \quad u \geq v, M > 0.$$ 

Then $v^0 < w^0$, implies $v(t) \leq w(t), t \in [t_0, T]$, where

$v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$.

Definition 3.2.1. A pair of functions $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $C_p(J, \mathbb{R})$ are said to be ordered lower and upper solutions $(v_1, v_2) \leq (w_1, w_2)$ of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) if

$$D^q v_i(t) \leq f_i(t, v_1(t), v_2(t)), \quad v_i(0) \leq \int_0^T v_i(s)ds + d$$

and

$$D^q w_i(t) \geq f_i(t, w_1(t), w_2(t)), \quad w_i(0) \geq \int_0^T w_i(s)ds + d.$$
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**Definition 3.2.2.** A function \( f_i = f_i(t, u_1, u_2) \) in \( C(J \times \mathbb{R}^2, \mathbb{R}) \) is said to be quasimonotone nondecreasing if

\[
f_i(t, u_1(t), u_2(t)) \leq f_i(t, v_1(t), v_2(t)) \text{ if } u_i = v_i \text{ and } u_j \leq v_j, \quad i \neq j, \ i = j = 1, 2.
\]

**Definition 3.2.3.** A function \( f_i = f_i(t, u_1, u_2) \) in \( C(J \times \mathbb{R}^2, \mathbb{R}) \) is said to be quasimonotone nonincreasing if

\[
f_i(t, u_1(t), u_2(t)) \geq f_i(t, v_1(t), v_2(t)) \text{ if } u_i = v_i \text{ and } u_j \leq v_j, \quad i \neq j, \ i = j = 1, 2.
\]

### 3.3 Monotone Method and Applications

In this section we develop monotone method for weakly coupled system of Riemann-Liouville fractional differential equations with integral boundary conditions and obtain existence and uniqueness of solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1).
Theorem 3.3.1. Assume that

(i) \( f_i = f_i(t, u_1, u_2), i = 1, 2 \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone non-decreasing

(ii) \( v^0(t) = (v_1^0, v_2^0) \) and \( w^0(t) = (w_1^0, w_2^0) \) in \( C_p(J, \mathbb{R}) \) are ordered lower and upper solutions of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) such that
\[
v_i^0(0) \leq w_i^0(0) \text{ on } J
\]

(iii) There exists \( M_i \in (0, \frac{1}{\Gamma(1-q)T^q}) \) such that \( f_i \equiv f_i(t, u_1, u_2) \) satisfies one-sided Lipschitz condition,
\[
f_i(t, u_1, u_2) - f_i(t, u_i^*, u_i^*) \geq -M_i[u_i - u_i^*], \text{ for } u_i^* \leq u_i, M_i \geq 0.
\]

Then there exist monotone sequences \( \{v_n(t)\} \) and \( \{w_n(t)\} \) such that
\[
\{v_n(t)\} \rightarrow v(t) = (v_1, v_2) \quad \text{and} \quad \{w_n(t)\} \rightarrow w(t) = (w_1, w_2) \text{ as } n \rightarrow \infty
\]

where the functions \( v(t) \) and \( w(t) \) are minimal and maximal solutions of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1).
\textbf{Proof.} For any $\eta(t) = (\eta_1, \eta_2)$ in $C_p(J, \mathbb{R})$ such that for $\eta^0_i(0) \leq \eta_i$ on $J$, consider the following system of linear fractional differential equations

\begin{equation}
\begin{aligned}
D^q u_i(t) &= f_i(t, \eta_1(t), \eta_2(t)) - M_i[u_i(t) - \eta_i(t)], \\
u_i(0) &= \int_0^T u_i(s)ds + d. \tag{3.3.1}
\end{aligned}
\end{equation}

It is clear that for every $\eta(t)$ there exists a unique solution $u(t) = (u_1(t), u_2(t))$ of the problem (3.3.1) on $J$.

For each $\eta(t)$ and $\mu(t)$ in $C_p(J, \mathbb{R})$ such that $\eta^0_i(0) \leq \eta_i(t)$, $\mu^0_i(0) \leq \mu_i(t)$, define a mapping $A$ by $A[\eta, \mu] = u(t)$, where $u(t)$ is the unique solution of the problem (3.3.1).

This mapping defines the sequences $\{v^n(t)\}$ and $\{w^n(t)\}$.

Firstly, we prove that

(A$_1$) $v^0 \leq A[v^0, w^0]$, $w^0 \geq A[w^0, v^0]$

(A$_2$) $A$ possesses the monotone property on the segment

$[v^0, w^0] = \left\{(u_1, u_2) \in C(J, \mathbb{R}) : \eta^0_i \leq u_i \leq \mu^0_i \right\}$.

Set $A[v^0, w^0] = v^1(t)$, where $v^1(t) = (v^1_1, v^1_2)$ is the unique solution of system (3.3.1) with $\eta_i = \eta^0_i(0)$.

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Setting $p_i(t) = v_i^0(t) - v_i^1(t)$ we see that

\[
D^q p_i(t) = D^q v_i^0(t) - D^q v_i^1(t)
\]

\[
\leq f_i(t, v_0^1, v_2^0) - \left\{ f_i(t, v_1^1, v_2^1) - M_i(v_i^1 - v_i^0) \right\}
\]

\[
\leq M_i(v_i^1 - v_i^0)
\]

\[
\leq -M_i p_i(t)
\]

and $p_i(t) \leq 0$.

Thus we have $D^q p_i(t) \leq -M_i p_i(t)$

and $p_i(t) \leq 0$.

Applying Theorem 3.2.1, we get $p_i(t) \leq 0$ on $0 \leq t \leq T$ and hence $v_i^0(t) - v_i^1(t) \leq 0$ which implies $v_i^0 \leq A[v^0, w^0]$.

Set $A[v^0, w^0] = w^1(t)$, where $w^1(t) = (w_1^1, w_2^1)$ is the unique solution of the problem (3.3.1) with $\mu_i = w_i^0(t)$. Setting $p_i(t) = w_i^0(t) - w_i^1(t)$ we see that

\[
D^q p_i(t) = D^q w_i^0(t) - D^q w_i^1(t)
\]

\[
= D^q w_i^0(t) - D^q w_i^1(t)
\]
\[ \geq f_i(t, w_i^0, w_i^0) - \left\{ f_i(t, w_i^1, w_i^1) - M_i(w_i^1 - w_i^0) \right\} \]
\[ \geq M_i(w_i^1 - w_i^0) \]
\[ \geq -M_i p_i(t) \]
and \( p_i(t) \geq 0 \).

Thus we have \( D^q p_i(t) \geq -M_i p_i(t) \)
and \( p_i(t) \geq 0 \).

Applying Theorem 3.2.1, we have \( w_i^0 \geq w_i^1 \). Hence \( w^0 \geq A[w^0, v^0] \).

This proves \( A_1 \).

Let \( \eta(t), \beta(t), \mu(t) \in [v^0, w^0] \) with \( \eta(t) \leq \beta(t) \). Suppose that \( A[\eta, \mu] = u(t) \) \( A[\beta, \mu] = v(t) \). Then setting \( p_i(t) = u_i(t) - v_i(t) \) we find that

\[
D^q p_i(t) = D^q(u_i(t) - v_i(t))
\]
\[ = f_i(t, \eta_1, \eta_2) - f_i(t, \beta_1, \beta_2) - M_i(u_i - \eta_i) + M_i(v_i - \beta_i) \]
\[ \leq -M_i(\eta_i - \beta_i) - M_i(u_i - \eta_i) + M_i(v_i - \beta_i) \]
\[ \leq -M_i(u_i - v_i) \]
\[ \leq -M_i p_i(t) \]
and \( p_i(t) \leq 0. \)

Thus we have \( D^q p_i(t) \leq -M_i p_i(t) \)
and \( p_i(t) \leq 0. \)

As before in \((A_1)\), we have \( A[\eta,\mu] \leq A[\beta,\mu]. \)

Similarly, if \( \eta(t), \gamma(t), \mu(t) \in [v^0, w^0] \) be such that \( \gamma(t) \leq \mu(t). \)

Suppose that \( A[\eta, \gamma] = u(t), A[\eta, \mu] = v(t) \) we can prove that \( A[\eta, \gamma] \geq A[\eta, \mu]. \) Thus it follows that the mapping \( A \) possesses monotone property on the segment \([v^0, w^0]. \)

Now in view of \((A_1)\) and \((A_2)\), define the sequences \( v^n_i(t) = A[v_i^{n-1}, w_i^{n-1}] \)
\( w^n_i(t) = A[w_i^{n-1}, v_i^{n-1}] \) on the segment \([v^0, w^0] \) by

\[ D^q v_i^n(t) = f_i(t, v_i^{n-1}, v_2^{n-1}) - M_i[v_i^n - v_i^{n-1}], \]
\[ v_i^n(0) = \int_0^T v_i^{n-1}(s)ds + d \]
\[ D^q w_i^n(t) = f_i(t, w_i^{n-1}, w_2^{n-1}) - M_i[w_i^n - w_i^{n-1}], \]
\[ w_i^n(0) = \int_0^T w_i^{n-1}(s)ds + d. \]

From \((A_1)\), we have \( v_i^0(t) \leq v_i^1(t), \quad w_i^0(t) \geq w_i^1(t). \)
Assume that $v_{i}^{k-1}(t) \leq v_{i}^{k}(t), \quad w_{i}^{k-1}(t) \geq w_{i}^{k}(t)$.

To prove $v_{i}^{k}(t) \leq v_{i}^{k+1}(t), \quad w_{i}^{k}(t) \geq w_{i}^{k+1}(t)$ and $v_{i}^{k}(t) \geq w_{i}^{k}(t)$, define $p_{i}(t) = v_{i}^{k}(t) - v_{i}^{k+1}(t)$. Thus

\[
D^{q}p_{i}(t) = D^{q}v_{i}^{k}(t) - D^{q}v_{i}^{k+1}(t) \\
= f_{i}(t, v_{1}^{k-1}, v_{2}^{k-1}) - M_{i}(v_{1}^{k} - v_{1}^{k-1}) - \\
\left\{ f_{i}(t, v_{1}^{k}, v_{2}^{k}) - M_{i}(v_{1}^{k+1} - v_{1}^{k}) \right\} \\
\leq -M_{i}(v_{1}^{k-1} - v_{1}^{k}) - M_{i}(v_{1}^{k} - v_{1}^{k-1}) + \\
M_{i}(v_{1}^{k+1} - v_{1}^{k}) \\
\leq -M_{i}(v_{i}^{k}(t) - v_{i}^{k+1}(t)) \\
\leq -M_{i}p_{i}(t) \\
\text{and} \quad p_{i}(t) \leq 0.
\]

Thus we have $D^{q}p_{i}(t) \leq -M_{i}p_{i}(t)$

and $p_{i}(t) \leq 0$.

Apply Theorem 3.2.1, we obtain $p_{i}(t) \leq 0$, which gives

$v_{i}^{k}(t) \leq v_{i}^{k+1}(t)$. Similarly we can prove $w_{i}^{k}(t) \geq w_{i}^{k+1}(t)$.

Define $p_{i}(t) = w_{i}^{k+1}(t) - w_{i}^{k}(t)$ and consider

\[
D^{q}p_{i}(t) = D^{q}(w_{i}^{k+1}(t) - w_{i}^{k}(t))
\]
\[ = D^q w_i^{k+1}(t) - D^q w_i^k(t) \]
\[ = f_i(t, w_1^k(t), w_2^k(t)) - M_i(w_i^{k+1}(t) - w_i^k(t)) - \]
\[ \{ f_i(t, w_1^{k-1}, w_2^{k-1}) - M_i(w_i^k - w_i^{k-1}) \} \]
\[ \leq M_i(w_i^{k-1} - w_i^k) - M_i(w_i^{k+1} - w_i^k) + \]
\[ M_i(w_i^k - w_i^{k-1}) \]
\[ \leq M_i(w_i^k(t) - w_i^{k+1}(t)) \]
\[ \leq -M_i p_i(t) \]

and \( p_i(t) \leq 0 \).

Thus we have \( D^q p_i(t) \leq -M_i p_i(t) \)

and \( p_i(t) \leq 0 \).

Thus we have \( w_i^k(t) \geq w_i^{k+1}(t) \). Also we prove \( v_i^k(t) \geq w_i^k(t) \).

Define \( p_i(t) = v_i^k(t) - w_i^k(t) \). Consider

\[ D^q p_i(t) = f_i(t, v_1^{k+1}, v_2^{k+1}) - M_i[v_i^k - v_i^{k+1}] - \]
\[ \{ f_i(t, w_1^{k+1}, w_2^{k+1}) - M_i[w_i^k - w_i^{k+1}] \} \]
\[ \leq -M_i[v_i^{k+1} - w_i^{k+1}] - M_i[v_i^k - v_i^{k+1}] + \]
\[ M_i[w_i^k(t) - w_i^{k+1}(t)] \]
\[ \leq -M_i[v_i^k(t) - v_i^{k+1}(t)] \]
\[ \leq -M_i p_i(t) \]

and \( p_i(t) \leq 0. \)

Thus we have \( D^q p_i(t) \leq -M_i p_i(t) \)

and \( p_i(t) \leq 0. \)

Hence we get \( v_i^k(t) \geq w_i^k(t) \). By principle of mathematical induction, it follows that
\[ v_i^0(t) \leq v_i^1(t) \leq v_i^2(t) \leq \ldots \leq v_i^n(t) \leq w_i^n(t) \leq \ldots \leq w_i^1(t) \leq w_i^0(t). \]

Thus the sequence \( \{v_i^n(t)\} \) is monotonically nondecreasing and bounded below. Also the sequence \( \{w_i^n(t)\} \) is monotonically nonincreasing and bounded above on \( J \). Hence pointwise limit exist and are given by

\[ \lim_{n \to \infty} v_i^n(t) = v_i(t), \quad \lim_{n \to \infty} w_i^n(t) = w_i(t) \text{ on } J. \]

Using corresponding Volterra fractional integral equations

\[ v_i^n(t) = v_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, v_i^n(s), v_i^2(s)) - M_i[v_i^n - v_i^{n-1}] \right\} ds \]
\begin{equation}
    w^n_i(t) = w^0_i + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, w^n_1(s), w^n_2(s)) - M_i[w^n_i - w^n_{i-1}] \right\} ds
\end{equation}

it follows that \( v(t) \) and \( w(t) \) are solutions of system (3.3.1).

Lastly we prove that \( v(t) \) and \( w(t) \) are the minimal and maximal solutions of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1). Let \( u(t) = (u_1, u_2) \) be any solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) different from \( v(t) \) and \( w(t) \), so that there exists \( k \) such that \( v^k_i(t) \leq u_i(t) \leq w^k_i(t) \) on \( J \) and set \( p_i(t) = v^{k+1}_i(t) - u_i(t) \) so that

\[
    D^q p_i(t) = D^q v^{n+1}_i(t) - D^q u_i(t) \\
    = f_i(t, v^k_1, v^k_2) - M_i[v^{k+1}_i - v^k_i] - f_i(t, u_1, u_2) \\
    \geq -M_i[v^k_i - u_i] - M_i[v^{k+1}_i - v^k_i] \\
    \geq -M_i[v^{k+1}_i - u_i] \\
    \geq -M_i p_i(t)
\]

and \( p_i(t) \geq 0 \).

Thus we have \( D^q p_i(t) \geq -M_i p_i(t) \)

and \( p_i(t) \geq 0 \).
Thus $v_{i}^{k+1}(t) \leq u_{i}(t)$ on $J$. Since $v_{i}^{0}(t) \leq u_{i}(t)$ on $J$, by principle of mathematical induction it follows that $v_{i}^{k}(t) \leq u_{i}(t)$ for all $k$.

Similarly, we can prove $u_{i} \leq w_{i}^{k}$ for all $k$ on $J$.

Hence $v_{i}^{k}(t) \leq u_{i}(t) \leq w_{i}^{k}(t)$ on $J$. Taking limit as $n \to \infty$, it follows that $v_{i}(t) \leq u_{i}(t) \leq w_{i}(t)$ on $J$. This completes the proof.

Above result can be proved when the function $f_{i} = f_{i}(t, u_{1}, u_{2})$ in $C[J \times \mathbb{R}^2, \mathbb{R}]$ is quasimonotone nonincreasing with minor changes and the result is stated as under.

**Theorem 3.3.2.** Assume that (ii) – (iii) as in Theorem 3.3.1 and if $f_{i} = f_{i}(t, u_{1}, u_{2}), i = 1, 2$ in $C[J \times \mathbb{R}^2, \mathbb{R}]$ is quasimonotone nonincreasing. Then there exists monotone sequences $\{v^{n}(t)\}$ and $\{w^{n}(t)\}$ such that

$$\{v^{n}(t)\} \to v(t) \quad \text{and} \quad \{w^{n}(t)\} \to w(t) \quad \text{as} \quad n \to \infty$$

where the functions $v(t)$ and $w(t)$ are minimal and maximal solutions of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1).

**Proof.** It is similar to above theorem. We omit the details.
Now, we obtain the uniqueness of solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) in the following result

**Theorem 3.3.3.** Assume that

(i) \( f_i = f_i(t, u_1, u_2) \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone nondecreasing,

(ii) \( v = (v_1, v_2) \) and \( w = (w_1, w_2) \) in \( C_p(J, \mathbb{R}) \) are ordered lower and upper solutions of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) on \( J \)

(iii) There exists \( M_i \in (0, \frac{1}{(1-q)T^q}) \) such that \( f_i = f_i(t, u_1, u_2) \) satisfies Lipschitz condition,

\[
|f_i(t, u_1, u_2) - f_i(t, u_1^*, u_2^*)| \leq M_i |u_i - u_i^*|, \quad M_i \geq 0
\]

(iv) \[
\lim_{n \to \infty} \|w^n - v^n\| = 0 \quad \text{where} \quad \|f\| = \int_0^T |f(s)| ds.
\]

then the solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) is unique.
Proof. Since \( v(t) \leq w(t) \), it is enough to prove \( v(t) \geq w(t) \). Define \( p_i(t) = w_i(t) - v_i(t) \). Consider

\[
D^q p_i(t) = D^q w_i(t) - D^q v_i(t) 
= f_i(t, w_1(t), w_2(t)) - f_i(t, v_1(t), v_2(t)) 
\leq -M_i p_i(t) 
\]

and \( p_i(t) \leq 0 \).

Thus we have \( D^q p_i(t) \leq -M_i p_i(t) \)
and \( p_i(t) \leq 0 \).

Apply Theorem 3.2.1, we have \( w_i(t) \leq v_i(t) \). This proves the uniqueness of the solution. Above result can be proved when the function \( f_i = f_i(t, u_1, u_2) \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone nonincreasing with minor changes and the result is stated as under.

**Theorem 3.3.4.** Assume that (ii) – (iv) as in Theorem 3.3.3 and if \( f_i = f_i(t, u_1, u_2) \) in \( C[J \times \mathbb{R}^2, \mathbb{R}] \) is quasimonotone nonincreasing, then the solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) is unique.

**Proof.** It is similar to above theorem. We omit the details.
3.4 Concluding Remarks

Remark 3.4.1. Choosing initial iterations $v_i^0(t), w_i^0(t)$, two sequences of iterations $\{v^n(t)\}$ and $\{w^n(t)\}$ are obtained when the function $(f_1, f_2)$ is quasimonotone nondecreasing (nonincreasing). They possess monotone property.

Remark 3.4.2. We have developed monotone method for the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) in which function $(f_1, f_2)$ is quasimonotone nondecreasing (nonincreasing).

Remark 3.4.3. As an application of this method, existence and uniqueness of solution of the system of Riemann-Liouville fractional integral boundary value problem (3.1.1) are obtained when the function $(f_1, f_2)$ is quasimonotone nondecreasing (nonincreasing).