Chapter 3

Fourth Order Nonlinear Elliptic Equations

...the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations ...and therefore we can learn by comparing different nonlinear problems.

WERNER HEISENBERG.

The content of this chapter is published in following paper.

3.1 Introduction

Fourth order nonlinear elliptic equations have many applications in various field of science and engineering. Because of this large number of researchers are working on study of existence of solution of second order, fourth order nonlinear elliptic boundary value problems. It is well known that maximum principle and comparison principle are the two sides of the same coin. It is popular that maximum principle, comparison principle and existence results go hand in hand (cf.[14], [16], [18]).

It is well known method of defining a function on the solution of differential equation and obtain various results about the solution of the equation by means of the auxiliary function. In [46] Miranda obtains the first such result for biharmonic equation, $\Delta^2 u = 0$, by proving that the function

$$P = |\nabla u(x)|^2 - u\Delta u$$  \hspace{1cm} (3.1.1)

is subharmonic on its domain. Since, then many researchers have employed this technique on various class of fourth order partial differential equations [58, 66, 90]. Recently Dhaigude and Gosavi [15] extend a maximum principle due to Schaefer [71] for a class of fourth
order semilinear elliptic equations to a more general fourth order semilinear elliptic equation of the form
\[ \Delta^2 u + a(x, y)\Delta u + b(x, y)f(u) = 0. \]

In this chapter, we study the existence problem for fourth order nonlinear elliptic equation of the form
\[ \Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0. \]

We plan this chapter as follows. In section 3.2 we mentioned useful result and notation. We developed maximum principles for a class of fourth order nonlinear elliptic equations in section 3.3. In section 3.4 we discuss the non-existence of non-trivial solutions of the boundary value problem under consideration.

3.2 Prerequisites

Consider \( \Omega \) is a plane domain bounded with smooth boundary \( \partial \Omega \).

For simplicity, we use the summation convention and denote partial derivatives \( \frac{\partial u}{\partial x_i} \) by \( u_i \) and \( \frac{\partial^2 u}{\partial x_i^2} \) by \( u_{ii} \).

The following Lemma [83] is useful to prove our results.
Lemma 3.1. For a sufficiently smooth function $v$ the inequality

$$Nv_{,ik}v_{,ik} \geq (\Delta v)^2$$

satisfied in some domain $\Omega \subset \mathbb{R}^N$.

3.3 Maximum Principles

In this section, we proved some important theorems.

Now, we prove maximum principles for the function $P$ such that

$$P = |\nabla u(x)|^2 - u\Delta u,$$

which is the main result of this chapter.

Theorem 3.2. Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be a solution of

$$\Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0 \quad (3.3.1)$$

where $a(x, y), b(x, y)$ are bounded in $\Omega$ and

$$b(x, y)u(x, y)f(u) + a(x, y)u(x, y)g(\Delta u) \geq 0 \quad \text{in} \quad \Omega. \quad (3.3.2)$$
Then the function

\[ P = |\nabla u(x)|^2 - u \Delta u \]  (3.3.3)

assume its maximum on \( \partial \Omega \).

**Proof:** Define

\[ P = |\nabla u(x)|^2 - u \Delta u \]

Differentiate with respect to \( x_k \), we get

\[ P_{,k} = 2u_{,i}u_{,ik} - u_{,k} \Delta u - u(\Delta u)_{,k} \]  (3.3.4)

Again differentiate equation (3.3.4) with respect to \( x_k \), we get

\[ P_{,kk} = 2u_{,ik}u_{,ik} + 2u_{,i}u_{,ikk} - u_{,kk} \Delta u - u_{,k}(\Delta u)_{,k} - u_{,k}(\Delta u)_{,k} - u(\Delta u)_{,kk}. \]  (3.3.5)

Using well known identities in differential geometry

\[ \begin{align*}
  i) & \quad P_{,kk} = \Delta P, \quad u_{,kk} = \Delta u \\
  ii) & \quad u_{,ikk} = u_{,kki} = (\Delta u)_{,i} \\
  iii) & \quad (\Delta u)_{,kk} = \Delta^2 u
\end{align*} \]
Using (i, ii, iii) in equation (3.3.5), we obtain

\[ \Delta P = 2u_{,ik}u_{,ik} + 2u_{,i}(\Delta u)_{,i} - \Delta u \Delta u - 2u_{,k}(\Delta u)_{,k} - u \Delta^2 u. \] (3.3.6)

\[ \Delta P = 2u_{,ik}u_{,ik} + 2u_{,i}(\Delta u)_{,i} - (\Delta u)^2 - 2u_{,i}(\Delta u)_{,i} - u \Delta^2 u. \] (3.3.7)

[ By knowledge of differential geometry

\[ u_{,k} = u_{,i} \text{ and } (\Delta u)_{,i} = (\Delta u)_{,k}. \] ]

\[ \Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u \Delta^2 u. \] (3.3.8)

Using equation (3.3.1) in equation (3.3.8), which yields

\[ \Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u[-a g(\Delta u) - b f]. \] (3.3.9)

\[ \Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 + a u g(\Delta u) + b u f. \] (3.3.10)

Using Lemma 3.1 and assumption (3.3.2), we observe that the right hand side of equation (3.3.10) is non-negative.

Hence

\[ \Delta P \geq 0 \quad \text{in} \quad \Omega. \]

By maximum principle, the result follows.

*Remark* 3.3. If \( a = 0 \) and \( b = 0 \) in equation (3.3.1) then it gives the maximum principle due to Miranda [46].
Remark 3.4. If \( g(\Delta u) = \Delta u \) in (3.3.1) then it reduces to maximum principle due to Dhaigude and Gosavi [15].

Thus, we claim that our results are more general.

We now state and prove another maximum principle for different function.

**Theorem 3.5.** Suppose that \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) be a solution of

\[
\Delta^2 u + a(x,y)g(\Delta u) + b(x,y)f(u) = 0 \tag{3.3.11}
\]

i) \( a(x,y), b(x,y) \) are bounded with \( b > 0 \) in \( \Omega \) and

\[
b(x,y)u(x,y)f(u) + a(x,y)u(x,y)g(\Delta u) \geq 0 \quad \text{in} \quad \Omega. \tag{3.3.12}
\]

ii) \( \nabla \left( \frac{1}{b} \right) \) is bounded and \( \Delta b \leq 0 \). Then the function

\[
R = \frac{1}{b} \left[ ||\nabla u(x)||^2 - u\Delta u \right] \tag{3.3.13}
\]

assumes its nonnegative maximum on \( \partial \Omega \) unless \( R < 0 \) in \( \Omega \).

**Proof:** We have, the function

\[
R = \frac{1}{b} \left[ ||\nabla u(x)||^2 - u\Delta u \right].
\]
Differentiate with respect to $x_k$, we get

$$R_{,k} = \frac{1}{b}[\nabla(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,k}}{b^2}(|\nabla u(x)|^2 - u\Delta u) \quad (3.3.14)$$

Differentiating equation (3.3.14) with respect to $x_k$, we have

$$R_{,kk} = \frac{1}{b}[\Delta(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,k}}{b^2}[\nabla(|\nabla u(x)|^2 - u\Delta u)]$$
$$- \frac{b_{,k}}{b^2}[\nabla(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,kk}}{b^2}(|\nabla u(x)|^2 - u\Delta u)$$
$$+ \frac{2b_{,k}b_{,k}}{b^3}(|\nabla u(x)|^2 - u\Delta u). \quad (3.3.15)$$

By making use of following notations

1. $b_{,k} = \nabla b$, $b_{,k}^2 = |\nabla b|^2$
2. $R_{,kk} = \Delta R$, $b_{,kk} = \Delta b$

in equation (3.3.11) and simplify, we get

$$\Delta R = \frac{1}{b}[\Delta(|\nabla u(x)|^2 - u\Delta u)] - \frac{2b_{,k}}{b^2}[\nabla(|\nabla u(x)|^2 - u\Delta u)]$$
$$- \frac{\Delta b}{b^2}(|\nabla u(x)|^2 - u\Delta u) + \frac{2|\nabla b|^2}{b^3}(|\nabla u(x)|^2 - u\Delta u).$$

On simplifying, we obtain
\( \Delta R = \frac{1}{b} [\Delta (|\nabla u(x)|^2 - u \Delta u)] - \frac{\Delta b}{b^2} (|\nabla u(x)|^2 - u \Delta u) \\
- 2 \frac{\nabla b}{b} \left\{ - \frac{\nabla b}{b^2} [||\nabla u(x)||^2 - u \Delta u] + \frac{1}{b} [\nabla (|\nabla u(x)|^2 - u \Delta u)] \right\} \\
(3.3.16) \\

Substituting equation (3.3.13) and equation (3.3.14) in equation (3.3.16), we have

\( \Delta R = \frac{1}{b} [\Delta (|\nabla u(x)|^2 - u \Delta u)] - \frac{\Delta b}{b} R - 2 \frac{\nabla b}{b} R, k. \) (3.3.17)

\( \Delta R = \frac{1}{b} [\Delta (|\nabla u(x)|^2 - u \Delta u)] + 2 b \nabla \left( \frac{1}{b} \right) \cdot \nabla R - \frac{\Delta b}{b} R. \) (3.3.18)

\[
\begin{align*}
\therefore \frac{\nabla b}{b} & = - \frac{b, k}{b} = b \left( - \frac{b, k}{b^2} \right) = b \nabla \left( \frac{1}{b} \right)
\end{align*}
\]

Thus, we get

\( \Delta R - 2 b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R = \frac{1}{b} \left[ \Delta (|\nabla u(x)|^2 - u \Delta u) \right]. \) (3.3.19)

By using equation (3.3.8),

\( \Delta R - 2 b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R = \frac{1}{b} [2 u, k, u - (\Delta u)^2 - u \Delta^2 u]. \) (3.3.20)
Using (3.3.11), we have

\[ \Delta R - 2b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R \]
\[ = \frac{1}{b} \left[ 2u_{,ik}u_{,ik} - (\Delta u)^2 - u \left( -a g(\Delta u) - b f \right) \right]. \quad (3.3.21) \]

\[ \Delta R - 2b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R \]
\[ = \frac{1}{b} \left[ 2u_{,ik}u_{,ik} - (\Delta u)^2 + a g(\Delta u) + b f u \right]. \quad (3.3.22) \]

by Lemma 3.1, we have

\[ \Delta R - 2b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R \]
\[ \geq \frac{1}{b} \left[ (\Delta u)^2 - (\Delta u)^2 + a g(\Delta u) + b f u \right]. \quad (3.3.23) \]

\[ \Delta R - 2b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R \geq \frac{1}{b} [a g(\Delta u) + b f u]. \quad (3.3.24) \]

Therefore

\[ \Delta R - 2b \nabla \left( \frac{1}{b} \right) \nabla R + \frac{\Delta b}{b} R \geq 0. \quad (3.3.25) \]

By Hopf’s maximum principle [68], the result follows.

Here we state some important remarks.
Remark 3.6. If $a = 0$ in equation (3.3.11) then we get the maximum principle due to Sperb [83].

Remark 3.7. If $b \equiv \text{constant}$, then the condition ”unless $R < 0$ in $\Omega$” in Theorem 3.5 can be omitted.

Remark 3.8. If $g(\Delta u) = \Delta u$ in equation (3.3.11) then yields the maximum principle due to Dhaigude and Gosavi [15].

3.4 Applications

In this section, we prove non-existence of nontrivial solutions $u(x)$ of the following boundary value problems by using our maximum principle.

Problem- I: Consider the equation

$$\Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0, \quad \text{in } \Omega$$  \hspace{1cm} (3.4.1)

with boundary conditions $u(x, y) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega$.  \hspace{1cm} (3.4.2)
Problem- II: Consider the equation

\[ \Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0, \quad \text{in } \Omega \quad (3.4.3) \]

with boundary conditions \( u(x, y) = 0, \quad \Delta u = 0 \quad \text{on } \partial \Omega. \quad (3.4.4) \)

The following Lemma is useful to prove the non-existence result.

**Lemma 3.9.** [71] Suppose that

i) \( u \in C^4(\Omega) \cap C^2(\bar{\Omega}) \) is a solution of fourth order nonlinear elliptic equation

\[ \Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0 \quad \text{in } \Omega, \quad (3.4.5) \]

ii) \( u(x, y) \) satisfy

\[ b(x, y)u(x, y)f(u) + a(x, y)u(x, y)g(\Delta u) \geq 0 \quad \text{in } \Omega, \quad \text{and} \quad (3.4.6) \]

iii) \( u(x, y) \) vanishes on the boundary \( \partial \Omega \).

Then

\[ \int_{\Omega} |\nabla u(x)|^2 dxdy \leq \frac{1}{2} A |\nabla u(x)|^2_M \]

where \( |\nabla u|^2_M = \max |\nabla u|^2 \) and \( A \) is the area of \( \Omega \).
NONEXISTENCE THEOREM:

Theorem 3.10. If (3.3.2) is satisfied in a convex domain $\Omega$ then no non-trivial solution of

$$
\Delta^2 u + a(x, y)g(\Delta u) + b(x, y)f(u) = 0, \quad \text{in} \quad \Omega \quad (3.4.7)
$$

$$
u(x, y) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega. \quad (3.4.8)
$$

exists.

Proof: We prove this theorem by contradiction method. Suppose that there exist a nontrivial solution $u$ of the given BVP (3.4.7) - (3.4.8). From equation (3.3.3), we have

$$
P = |\nabla u(x)|^2 - u\Delta u
$$

Using Theorem 3.2 and boundary condition (3.4.8), we get

$$
u_{,i}u_{,i} - u\Delta u \leq 0. \quad (3.4.9)
$$

Integrate equation (3.4.9) over $\Omega$,

$$
\int_{\Omega} \left[u_{,i}u_{,i} - u\Delta u\right] dxdy \leq 0. \quad (3.4.10)
$$
Using Green’s first identity

\[ \int_{\Omega} \left[ v\Delta u + \nabla v \cdot \nabla u \right] dxdy = \int_{\partial \Omega} v \frac{\partial u}{\partial n} d\sigma, \quad \text{with} \quad v = u \quad (3.4.11) \]

and boundary condition (3.4.8) in equation (3.4.12), we obtain

\[ \int_{\Omega} |\nabla u|^2 dxdy \leq 0. \quad (3.4.12) \]

Consequently $|\nabla u| = 0$ in $\Omega$ and by continuity $u \equiv 0$ in $\Omega \cup \partial \Omega$. Thus it is a contradiction. Therefore there is no nontrivial solution of BVP (3.4.7) - (3.4.8) exists.

**Theorem 3.11.** If equation (3.3.2) is satisfied in a convex domain $\Omega$ then no nontrivial solution of

\[ \Delta^2 u + a(x,y)g(\Delta u) + b(x,y)f(u) = 0, \quad \text{in} \quad \Omega \quad (3.4.13) \]

\[ u(x,y) = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial \Omega. \quad (3.4.14) \]

exists.

**Proof:** We prove this theorem by contradiction method. Suppose that a nontrivial solution $u$ of the given BVP (3.4.13) - (3.4.14) exists.

From equation (3.3.3),

\[ P = |\nabla u(x)|^2 - u\Delta u \]
Then by Theorem 3.2, $P$ takes its maximum on the boundary $\partial \Omega$ at a point, say $Q$. Using Hopf’s second maximum principle, we see that either $\frac{\partial P}{\partial n}(Q) > 0$ or $P$ is constant in $\Omega \cup \partial \Omega$.

**Case I.** Assume that $\frac{\partial P}{\partial n}(Q) > 0$. First we differentiate $P$ partially in the normal direction and use boundary condition (3.4.14) to get

$$\frac{\partial P}{\partial n}(Q) = 2 \frac{\partial u}{\partial n} \frac{\partial^2 u}{\partial n^2}. \quad (3.4.15)$$

We have well known identity in the differential geometry [83] for curvature $k$,

$$\frac{\partial^2 u}{\partial n^2} + k \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = u_{,ii} = \Delta u \quad (3.4.16)$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ are normal and tangential derivatives respectively.

The tangential component $\frac{\partial^2 u}{\partial s^2}$ is zero. Therefore equation (3.4.16) becomes

$$\frac{\partial^2 u}{\partial n^2} = \Delta u - k \frac{\partial u}{\partial n}. \quad (3.4.17)$$

By using equation (3.4.17) and boundary condition (3.4.14) in equation (3.4.15), we get

$$\frac{\partial P}{\partial n}(Q) = -2k \left( \frac{\partial u}{\partial n} \right)^2. \quad (3.4.18)$$

Since $\Omega$ is convex, $k > 0$. Thus $\frac{\partial P}{\partial n}(Q) < 0$ is not possible. Thus we see that no nontrivial solution exists.
**Case II.** Assume that $P$ is constant say $c$ in $\Omega \cup \partial \Omega$. Therefore

$$|\nabla u|^2 = \left( \frac{\partial u}{\partial n} \right)^2 = c \quad \text{on} \quad \partial \Omega. \quad (3.4.19)$$

Now as $P = c$ in $\Omega \cup \partial \Omega$, we have $\frac{\partial P}{\partial n} = 0$ on $\partial \Omega$. But from equation (3.4.18), we have

$$\frac{\partial P}{\partial n}(Q) = -2kc.$$

For a bounded convex domain with a continuously turning tangent on the boundary, $k \neq 0$. Moreover $c \neq 0$, if $c = 0$, then $|\nabla u|_M = 0$ and by Lemma 3.9 and Theorem 3.2 we can conclude that $u \equiv 0$ in $\Omega$. Therefore $P = c$ is not possible. This is contradiction to $P$ is constant. Thus there is no nontrivial solution exists.