Chapter- 3

GENERALIZATIONS OF BASIC AND LARGE SUBMODULES
CHAPTER-III

GENERALIZATIONS OF BASIC AND LARGE SUBMODULES

3.1. Introduction

In the last chapter we studied large submodules with the help of basic submodules. A basic submodule is $h$-pure in the containing module but may not be an isotype and totally projective.

Here we define generalized basic submodules for limit ordinals $\alpha$ which are totally projective. Totally projective modules are defined[32,36] in different ways and their equivalence was established.

In the second section we study the family $\alpha$ of all $QTAG$-modules such that $M/H_\beta(M)$ is totally projective for all ordinals $\beta < \alpha$. This is equivalent to saying that $M/H_\beta(M)$ is a direct sum of countably generated modules. We name these modules as $\alpha$-modules. The $\alpha$-basic submodules are the generalization of basic submodules and every $\alpha$-module contains an $\alpha$-basic submodule. This helps us examine the structure of $\alpha$-pure submodules of $\alpha$-modules for limit ordinals $\alpha$ such that $\alpha \neq \beta + \omega$ for any ordinal $\beta$.

In section three we study $\alpha$-large submodule $L$ of an $\alpha$-module such that $M = L + B$ where $B$ is an $\alpha$-basic submodule of $M$. We establish that $L$ is determined by an increasing sequence of ordinals less than $\alpha$. These $\alpha$-large submodules are also studied in the light of totally projective modules.

Since totally projective modules are also defined with the help of nice systems, emphasizing their importance section four is devoted to nice submodules and nice bases of $QTAG$-modules.
3.2. α- Modules and α- Basic Submodules

We recall that for \( k \in \mathbb{Z}^+ \), a module \( M \) is \( k \)-projective if \( H_k(M) = 0 \) and for a limit ordinal \( \sigma \), it is \( \sigma \)-projective if there exists a submodules \( N \) bounded by \( \sigma \) such that \( M/N \) is a direct sum of uniserial modules. Moreover it is totally projective if it is \( \sigma \)-projective for all limit ordinals \( \sigma \).

Also a reduced \( QTAG \)-module \( M \) is totally projective if it has a nice system and direct sums and direct summands of totally projective modules are also totally projective[36].

we start with the following.

**Definition 3.2.1.** Let \( \alpha \) denote the class of all \( QTAG \)-modules \( M \) such that \( M/H_\beta(M) \) is totally projective for all ordinals \( \beta < \alpha \). These module are called \( \alpha \)-modules.

**Definition 3.2.2.** A submodule \( B \subseteq M \) is an \( \alpha \)- basic submodule of an \( \alpha \)- module \( M \) if:

(i) \( B \) is totally projective of length at most \( \alpha \);
(ii) \( B \) is \( \alpha \)- pure submodule of \( M \); and
(iii) \( M/B \) is h-divisible .

**Remark 3.2.3.** For an \( \alpha \)- basic submodule \( B \) of \( M \) and \( \beta < \alpha \), the \( \beta \)-th Ulm invariant of \( B \) coincides with \( \beta \)-th Ulm invariant of \( M \). Therefore just like basic submodules [22], any two \( \alpha \)- basic submodules are isomorphic.

**Definition 3.2.4.** An ordinal \( \alpha \) is said to be confinal with \( \omega \) if \( \alpha \) is the limit of a countable ascending sequence of ordinals.

**Definition 3.2.5.** Let \( \alpha \) be an ordinal confinal with \( \omega \) and \( M \) a
QTAG- module. An $\alpha$- high confinal tower of $M$ is an ascending chain $\{M_k\}$ of submodules of $M$ such that:

(i) for each $k \in \mathbb{Z}^+$, $M_k$ is a $H_{\sigma(k)}(M)$- high submodule of $M$;

(ii) $\alpha = \sup\{\sigma(k)\}$, $\sigma(k) < \sigma(k + 1)$;

(iii) if $\alpha = \beta + \omega$ where $\beta$ is a limit ordinal then $\sigma(k) = \beta + l$ for some $l \in \mathbb{Z}^+$ and;

(iv) if $\alpha \neq \beta - \omega$, for any ordinal $\beta$, then $\sigma(k) = \beta(k) + \omega$ for some limit ordinal $\beta(k)$.

Now we prove the following result:

**Lemma 3.2.6.** Let $M/H_\beta(M)$ be a totally projective module and $B$ a basic submodule of $H_\beta(M)$. If $N$ is a submodule of $M$ such that $M/B = (N/B) \oplus (H_\beta(M)/B)$ then $N$ is totally projective.

**Proof.** Let $N$ be a submodule of $M$ such that $M/B = (N/B) \oplus (H_\beta(M)/B)$. Now $M = N + H_\beta(M)$ and $N$ is maximal in $M$ such that $N \cap H_\beta(M) = B$. Therefore $N \cap H_{\beta+1}(M) = H_{\beta+1}(N)$ and $H_\alpha(M) \cap N = H_\alpha(N)$ for all $\alpha \leq \beta + 1$ implying that $H_\beta(N) = H_\beta(M) \cap N = B$. Now

$$N/H_\beta(N) = N/(H_\beta(M) \cap N) \cong (H_\beta(M) + N)/H_\beta(M) = M/H_\beta(M)$$

and $H_\beta(N) = B$ is a direct sum of uniserial modules. This implies that $N$ is also totally projective.

**Lemma 3.2.7.** Let $\alpha$ be a limit ordinal confinal with $\omega$ such that $\alpha = \beta + \omega$ for some ordinal $\beta$ and $M = \bigcup M_k$ with $\{M_k\}$ an $\alpha$- high confinal tower of $M$. If $N \subseteq M$ such that:

(i) $N = \bigcup N_k$ where $N_1 \subseteq N_2 \subseteq \cdots$ and $N_k$ is nice in $M_k$ for each $k$, and

(ii) $N \subseteq H_\sigma(M) + N_k$ for all $\sigma < \sigma(k)$; then $N$ is nice in $M$. 39
Proof. In order to prove that $N$ is nice in $M$, we have to show that each coset $x+N$ contains an element $x+y$ which is proper with respect to $N$. Consider $x \in M$, $x \notin N$ and choose $k$ such that $x \in M_k$. Let $eta = H_M(x) < \sigma(k)$. Now for $j \geq k$, there exists $y_j \in N_j$ such that

$$H_M(x+y_j) = H_{M_j}(x+y_j) \geq H_{M_k}(x+y') = H_M(x+y') \quad \text{for any } y' \in N_j.$$ 

Now suppose $H_M(x+y_n) > \beta = H_M(x)$ for some $n \geq k$. Then

$$H_M(y_{n+i}) = H_M(x) \quad \text{for } i = 1, 2, 3, \ldots .$$

Put $\lambda = H_M(x+y_n)$, then $\lambda < \sigma(n)$ because $x+y_n \in M_n$.

Again $\lambda + 1 < \sigma(n)$ as $\sigma(n)$ is a limit ordinal. Suppose $x+y_n$ is not proper with respect to $N$. Then for some $l$, $H_M(x+y_{n+l}) > H_M(x+y_n) = \lambda$ and $x+y_{n+l} \in H_{\lambda+1}(M)$. As $H_{\lambda+1}(M) + N_n \supseteq N$, we have $y_{n+l} = z_l + y_{n,l}$ where $z_l \in H_{\lambda+1}(M)$ and $y_{n,l} \in N_n$. Therefore $x+y_{n,l} = x + z_l + y_{n,l} \in H_{\lambda+1}(M)$ and $x+y_{n,l} \in H_{\lambda+1}(M)$, which is not possible because $H_M(x+y_{n,l}) \leq H_M(x+y_n) = \lambda$. This implies that $x+y_n$ is proper with respect to $N$ and $N$ is nice in $M$.

Remark 3.2.8. Let $\alpha$ be a limit ordinal confinal with $\omega$ such that $\alpha = \beta + \omega$ for some ordinal $\beta$ and $M = \bigcup M_k$ with $\{M_k\}$ an $\alpha$-high confinal tower of $M$. If $A_k$ denotes a collection of nice submodules of $M_k$, forming a nice system [38] and $A$ contains all submodules $N$ of $M$ such that:

(i) $N = \bigcup N_k$ with $N_1 \subseteq N_2 \subseteq \ldots$ and $N_k \in A_k$ for every $k$ and

(ii) $N \subseteq H_\sigma(M) + N_k$ for all $\sigma < \sigma(k)$, then the members of $A$ are nice by Lemma 3.2.7.

Lemma 3.2.9. With $A$ as in Remark 3.2.8, if $N \in A$ and $K \subseteq M$ such that $(K+N)/N$ is countably generated, there exists $P \in A$ such that $P \supseteq N + K$ and $P/N$ is countably generated.

Proof. For each $k$ we have $\sigma(k) = \beta(k) + \omega$, where $\beta(k)$ is a limit
ordinal. Therefore $\alpha = \sup\{\sigma(k)\} = \sup\{\beta(k)\}$. In order to show that $P \subseteq H_\sigma(M) + P_k$ for each ordinal $\sigma < \sigma(k)$, we have to prove that $P \subseteq H_{\beta(k)+1}(M) + P_k$ for every $l < \omega$.

Let $N \in \mathcal{A}$ and $K$ a submodule of $M$ such that $K/N$ is countably generated. Now $K = N + (\sum x_i R), \ i < \omega$. Put $T = \sum x_i R$ and $T_k = T \cap M_k$. Now we have to construct submodules $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_l$ such that:

(a) $N_i \subseteq P_i$ for $i \leq k$,
(b) $P_j \subseteq P_i$ for $l \leq k$,
(c) $P_i^k \in \mathcal{A}_i$,
(d) $P_{i+1}^k \subseteq H_{\beta(i)+1}(M) + P_i^k$ for $l < \omega$,
(e) $T_i \subseteq P_i^k$ and
(f) $P_i^k / N_i$ is countably generated for $i \leq k$.

For each $i < \omega$, put $P_i = \cup_{k \geq i} P_i^k$. Now $P_i \in \mathcal{A}_i$ and $P_i \subseteq P_{i+1}$. Again $P_{i+1} = \cup_{k \geq i+1} P_i^{k+1} \subseteq \cup_{k \geq i+1} (H_{\beta(i)+1}(M) + P_i^k) = H_{\beta(i)+1}(M) + P_i$ for $l < \omega$, thus $P_i = P_{i+n} \subseteq H_{\beta(i)+1}(M) + P_i$ for all $n < \omega$, $l < \omega$.

If $P = \cup_{i < \omega} P_i$ then $K \subseteq P$ and $P/N$ is countably generated. Moreover $P \in \mathcal{A}$ because $P \subseteq H_{\beta(i)+1}(M) + P_i$ for every $i$ and $l$.

Now we shall construct $P_i^{k+1}$ for $1 \leq i \leq k+1$. For $1 \leq i \leq k$, put $P_{i,0} = N_i$, $P_{i,1} = P_i^k$, $P_{k+1,0} = N_{k+1}$. Let $P_{k+1,1} \in \mathcal{A}_{k+1}$ such that $P_{k+1,1} \supseteq P_k^k + N_{k+1} + T_{k+1}$ and $P_{k+1,1}/N_{k+1}$ is countably generated. Inductively we may define a family of submodules $P_{i,j}$ with $1 \leq i \leq k+1$ and $j < \omega$, satisfying the following conditions.

(i) $P_{i,j} \subseteq P_i$ for $j \leq l$,
(ii) $P_{i,j} \in \mathcal{A}_i$,
(iii) $P_{i,j+1}/P_{i,j}$ is countably generated,
(iv) $P_{i+1,2j} \subseteq H_{\beta(i)+l}(M) + P_{i,2j}$ for $1 \leq i \leq k$ and $j, l < \omega$ and $P_i,2j+1 \subseteq P_{i+1,2j+1}$ for all $1 \leq i \leq k$ and $j < \omega$.

Now if we put $P_i^{k+1} = \bigcup_{j<\omega} P_{i,j}$, then $\bigcup_{j<\omega} P_{i,2j} = P_i^{k+1} = \bigcup_{j<\omega} P_{i,2j+1}$, and by (iv)

$$P_i^{k+1} = \bigcup_{j<\omega} P_{i+1,2j} \subseteq \bigcup_{j<\omega} (H_{\beta(i)+l}(M) + P_{i,2j}) = H_{\beta(i)+l}(M) + P_i^{k+1},$$

for all $l < \omega$, $1 \leq i \leq k$, and by (v)

$$P_i^{k+1} = \bigcup_{j<\omega} P_{i,2j+1} \subseteq \bigcup_{j<\omega} P_{i+1,2j} = P_i^{k+1}$$

for all $1 \leq i \leq k$. The above five conditions are satisfied by the sub-modules $P_i^j$, $1 \leq i \leq j \leq k + 1$.

Since $P_{i,j+1}/P_{i,j}$ is countably generated for each

$$j < \omega, \quad P_i^{k+1}/N_i = P_i^{k+1}/P_{i,0} = \bigcup_{j<\omega} (P_{i,j+1}/P_{i,0})$$

is also countably generated for all $1 \leq i \leq k + 1$ and the condition (f) is also satisfied.

To apply induction we assume that $P_{i,j}$ satisfy conditions (i) $\rightarrow$ (v) for all $1 \leq i \leq k + 1$ and $j \leq 2m + 1$.

To define $P_{i,2m+2}$ for $1 \leq i \leq k + 1$, we put $P_{k+1,2m+2} = P_{k+1,2m+1}$ and assume that for some positive integer $t \leq k$, $P_{t+1,2m+2}$ is defined for each $t + 1 \leq i \leq k - 1$.

Let $P_{t+1,2m+2} = P_{t+1,2m} + (\Sigma_{j<\omega} y_j R)$. Since $M \subseteq H_{\beta(t)+l}(M) + M_t$, $\Sigma y_i R \subseteq P_{t+1,2m+2}$, we have $y_j = x_{j,l} + x'_{j,l}$ where $x_{j,l} \in H_{\beta(t)+l}(M)$ and $x'_{j,l} \in M_t$ for each $j, l < \omega$. Let

$$Q_{t,2m+2} = \Sigma x_{j,l}' R \subseteq M_t.$$

Now $P_{t,2m+2} \in \mathcal{A}_t$ such that

$$P_{t,2m+2} \supseteq P_{t,2m+1} + Q_{t,2m+2} \text{ and } P_{t,2m+2}/P_{t,2m+1} \text{ is countably generated.}$$
Since $P_{t+1,2m} \subseteq H_{\beta(t)+1}(M) + P_{t,2m} \subseteq H_{\beta(t)+1}(M) + P_{t,2m+1}$
\[ \subseteq H_{\beta(t)+1}(M) + P_{t,2m+2} \]
and $\sum y_i R \subseteq H_{\beta(t)+1}(M) + Q_{k,2m+2} \subseteq H_{\beta(t)+1}(M) + P_{t,2m+2}$.

We have
\[ P_{t+1,2m+2} \subseteq H_{\beta(t)+1}(M) + P_{t,2m+2} \text{ for each } l < \omega. \]

Now we have submodules $P_{i,j}$ satisfying the above five conditions for all $1 \leq i \leq k + 1$ and $j \leq 2m + 2$.

We may define $P_{1,2m+3} = P_{1,2m+2}$ and assume that for some integer $t \leq k$, $P_{i,2m+3}$ has been defined for each $1 \leq i \leq t$. Since $P_{t,2m+3}/P_{t,2m+1}$ is countably generated and
\[ P_{t,2m+1} \subseteq P_{t+1,2m+1} \subseteq P_{t+1,2m+2}, \quad (P_{t,2m+3} + P_{t+1,2m+2})/P_{t+1,2m+2} \]
is countably generated.

Therefore there exists $P_{t+1,2m+3} \in A_{t+1}$ such that
\[ P_{t+1,2m+3} \supseteq P_{t+1,2m+3} + P_{t+1,2m+2} \]
and $P_{t+1,2m+3}/P_{t+1,2m+2}$ is countably generated and the submodules $P_{i,j}$ for $1 \leq i \leq k + 1$ and $0 \leq j \leq 2m + 3$ satisfy the five conditions.

**Lemma 3.2.10.** For an ordinal $\sigma$, confinal with $\omega$, if $M/H_\sigma(M)$ is totally projective, then every $H_\sigma(M)$- high submodule of $M$ is totally projective.

**Proof.** Let $N$ be a $H_\sigma(M)$- high submodule of $M$. Since
\[ N \cong (N + H_\sigma(M))/H_\sigma(M) \]
and $(N + H_\sigma(M))/H_\sigma(M)$ is $\sigma$- pure in the $\sigma$- projective module $M/H_\sigma(M)$, $N$ is $\sigma$- projective. Since $\sigma$ is a limit ordinal, $N/H_{\rho}(N) \cong M/H_{\rho}(M)$ is $\rho$- projective for all ordinals $\rho < \sigma$. 

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Therefore $N$ is totally projective.

**Theorem 3.2.11.** Let $\alpha$ be a limit ordinal confinal with $\omega$, and $\{M_k\}$ an $\alpha$-high confinal tower of $M$. If $M$ is an $\alpha$-module, then $\bigcup M_k$ is totally projective of length at most $\alpha$.

**Proof.** Since $\bigcup M_k$ is an isotype submodule of $M$, its length can not be greater than $\alpha$. If $\alpha = \beta + \omega$ for some ordinal $\beta$, then consider the submodule $\{\bigcup M_k\} \cap H_\beta(M)$ of $H_\beta(M)$. Now $M_k \cap H_{\beta+\omega}(M) = 0$ for every $k$, therefore

$$H_\omega((\bigcup M_k) \cap H_\beta(M)) = (\bigcup M_k \cap H_{\beta+\omega}(M)) = 0,$$

and $\bigcup M_k \cap H_\beta(M)$ is a QTAG-module free from the elements of infinite height. Since

$$(\bigcup M_k) \cap H_\beta(M) = (\bigcup (M_k \cap H_\beta(M))) = \bigcup H_\beta(M_k)$$

is the union of an ascending chain of bounded modules. Therefore by [40] $(\bigcup M_k) \cap H_\beta(M)$ is a direct sum of uniserial modules. Now $(\bigcup M_k) \cap H_\beta(M)$ is a $h$-pure submodule of $H_\beta(M)$ and $\bigcup M_k + H_\beta(M) = M$. Therefore $(\bigcup M_k) \cap H_\beta(M)$ is a basic submodule of $H_\beta(M)$. Since $M$ is an $\alpha$-module, $M/H_\beta(M)$ is totally projective, by Lemma 3.2.6, $\bigcup M_k$ is totally projective.

On the other hand if $\alpha \neq \beta + \omega$ for any ordinal $\beta$, then by Lemma 3.2.10, $M_k$ is totally projective for every $k$. In order to show that $\bigcup M_k$ is totally projective, we have to show that it has a nice system of modules. Let $\mathcal{A}$ denote the family of nice submodules $N$ of $\bigcup M_k$ such that

(a) $N = \bigcup N_k$ with $N_1 \subseteq N_2 \subseteq \ldots$ and $N_k$ is a nice submodule of $M_k$ for every $k$,

(b) $N \subseteq H_\sigma(M) + N_k$ for all $\sigma < \sigma(k)$.

Trivially $\{0\} \in \mathcal{A}$. Then for $N \in \mathcal{A}$ and a submodule $K$ of $M$ such that $N + K/N$ is countably generated, there exists a submodule $P \in \mathcal{A}$
such that $N + K \subseteq P$ and $P/N$ is countably generated. We have to show that sum of any two members of $A$ is in $A$.

Suppose $\{N_i\}_{i \in I} \subseteq A$ with $N_i = \cup_{k<\omega} N_{k,i}$, where

(i) $N_{k,i} \subseteq N_{l,i}$ for $k \leq l$,

(ii) $N_{k,i} \in A_k$ for each $k$ and

(iii) for each $k$ and $\sigma < \sigma(k)$, $A_i \subseteq H_\sigma(M) + N_{i,k}$.

Now

$$\Sigma_{i \in I} N_i = \Sigma_{i \in I} (\cup_{k<\omega} N_{k,i}) = \cup_k (\Sigma_{i \in I} N_{k,i})$$

and

$$\Sigma_{i \in I} N_{k,i} \subseteq \Sigma_{i \in I} N_{l,i}, \text{ and } \Sigma_{i \in I} N_{k,i} \in A_k$$

Since $\sigma < \sigma(k)$ for every $k$, we have

$$\Sigma_{i \in I} N_i \subseteq \Sigma_{i \in I} (H_\sigma(M) + N_{k,i}) = H_\sigma(M) + \Sigma_{i \in I} N_{k,i}.$$ 

Therefore $\Sigma_{i \in I} N_i \in A$.

**Lemma 3.2.12.** Let $\{M_k\}$ be an $\alpha$-high confinal tower of $M$. If $N = \cup \{M_k\}$, then $N$ is $\alpha$-pure in $M$.

**Proof.** Since $N$ is an isotype it is $h$-neat submodule of $M$. For $\sigma < \alpha$, there exists a positive integer $k$ such that $\sigma < \sigma(k)$ and

$$\text{Soc}(M) = \text{Soc}(M_k) + \text{Soc}(H_\sigma(M)) = \text{Soc}(N) + \text{Soc}(H_\sigma(M)).$$

Thus $N$ is $\alpha$-pure in $M$. With the help of above discussion we are able to infer the following:

**Theorem 3.2.13.**

(i) If $M$ is an $\alpha$-module with $\alpha$ confinal with $\omega$. Then $M$ contains an $\alpha$-basic submodule.
(ii) If $M$ is a $h$-reduced $QTAG$-module which contains a proper $\alpha$-basic submodule then $M$ is an $\alpha$-module where $\alpha$ is confinal with $\omega$.

**Proof.** The preceding two results imply (i). For (ii) consider a proper $\alpha$-basic submodule $B$ of the $h$-reduced $QTAG$-module $M$. For $\sigma < \alpha$,

$$M/H_\sigma(M) = (B + H_\sigma(M))/H_\sigma(M) \cong B/(B \cap H_\sigma(M)) = B/H_\sigma(B)$$

is totally projective. Therefore $M$ is an $\alpha$-module and $\alpha$ must be confinal with $\omega$.

For the limit ordinals $\alpha$ such that $\alpha \neq \beta + \omega$ for any ordinal $\beta$, we shall investigate the $\alpha$-pure submodules of $\alpha$-modules.

**Proposition 3.2.14.** Let $M$ be a direct sum of countably generated modules such that $M = \cup_{\beta < \alpha} M_\beta$, where $\{M_\beta\}$ is an $\alpha$-high confinal tower. If $N$ is an $\alpha$-pure submodule of $M$ such that for each $\beta$, $N \cap M_\beta$ is a $\beta$-high submodule of $N$, then $N$ is a direct summand of $M$.

**Proof.** Since $N \cap M_\beta$ is $(\beta + 1)$-pure in $N$ and $N$ is $\alpha$-pure in $M$, $N \cap M_\beta$ is $(\beta + 1)$-pure in $M$ and hence $(\beta + 1)$-pure in $M_\beta$. As $M_\beta$ is a direct sum of countably generated modules, $M_\beta$ is $\beta$-projective. Therefore there is a direct decomposition $M_\beta = (N \cap M_\beta) \oplus K_\beta$ for every $\beta < \alpha$. Now

$$M/N = \cup_{\beta < \alpha} ((M_\beta + N)/N) \quad \text{and} \quad (M_\beta + N)/N \cong M_\beta/(M_\beta \cap N) \cong K_\beta$$

is a direct sum of countably generated modules for every $\beta$. Since $N$ is $\alpha$-pure in $M$, we have

$$\text{Soc}(H_\beta(M/N)) = (\text{Soc}(H_\beta(M) + N)/N \text{ for } \beta < \alpha,$$

therefore

$$\text{Soc}(M/N) = \text{Soc} \left( (M_\beta + N)/N \right) \oplus \text{Soc}(H_\beta(M/N)).$$
Due to this direct decomposition it is sufficient to show that 
\((M_\beta + N)/N\) is \(h\)-pure in \(M/N\). for \(\beta \geq \omega\).

Now \(\text{Soc}(M_\beta + N) = \text{Soc}(K_\beta \oplus N) = \text{Soc}(K_\beta) \oplus \text{Soc}(N)\)

\[= \text{Soc}(K_\beta) \oplus \text{Soc}(N \cap M_\beta) \oplus \text{Soc}(H_\beta(N))\]

\[= \text{Soc}(M_\beta) \oplus \text{Soc}(H_\beta(N)).\]

If \(\beta \geq \omega\) and \(x \in \text{Soc}(M_\beta + N)\) then \(x = y + z\) where \(y \in \text{Soc}(M_\beta)\), \(z \in \text{Soc}(H_\beta(N)) \subseteq H_\omega(N)\). If \(H_M(x)\) is finite then \(H_M(x) = H_{M_\beta}(y) = H_M(y) = H_{(M_\beta + N)}(x)\) and if \(H(x)\) is infinite then \(H_{M_\beta}(y)\) is infinite in \(M_\beta\) and \(x\) has infinite height in \(M_\beta + N\). Therefore \(M_\beta + N\) is \(h\)-pure in \(M\) and \((M_\beta + N)/N\) is \(h\)-pure in \(M/N\), implying that \(N\) is a direct summand of \(M\).

The following remarks are the immediate consequences of the proposition 3.2.14 and the above discussion.

**Remark 3.2.15.** Let \(\alpha\) be an ordinal confinal with \(\omega\) and \(N\) an \(\alpha\)-pure submodule of \(M\) such that \(\{N_k\}\) is an \(\alpha\)-high confinal tower of \(N\). Then there exists an \(\alpha\)-high confinal tower \(\{M_k\}\) of \(M\) such that \(N_k \subseteq M_k\) and \(N_k = N \cap M_k\) for each \(k\).

**Remark 3.2.16.** Let \(\alpha \neq \beta + \omega\) for any \(\beta\) and \(M\) a totally projective module and that \(M = \bigcup M_k\) where \(\{M_k\}\) is an \(\alpha\)-high confinal tower. If \(N\) is an \(\alpha\)-pure submodule of \(M\) such that for each \(k\), \(N \cap M_k\) is a \(H_{\sigma(k)}(N)\)-high submodule of \(N\), then \(N\) is a summand of \(M\).

**Theorem 3.2.17.** Let \(\alpha\) be any limit ordinal such that \(\alpha \neq \beta + \omega\) for any \(\beta\) and \(M\) an \(\alpha\)-module. If \(N\) is an \(\alpha\)-pure submodule of \(M\), then \(N\) is an \(\alpha\)-module.

**Proof.** For an ordinal \(\alpha \neq \beta + \omega\), let \(\{N_k\}\) be a \(\alpha\)-high confinal tower of \(N\). Now by Remark 3.2.15 there exists an \(\alpha\)-high confinal tower \(\{M_k\}\) of \(M\) such that \(N_k = N \cap M_k\) for each \(k\). Since \(M\) is an \(\alpha\)-module, \(\bigcup M_k\) is totally projective by Lemma 3.2.11.
Again by Remark 3.2.16, $\cup N_k$ is an $\alpha$-basic submodule of $N$ and by Theorem 3.2.13, $N$ is an $\alpha$-module.

**Section-3**

### 3.3. $\alpha$-Large Submodules

A fully invariant submodule $L \subseteq M$ is large in $M$ if $L + B = M$, for every basic submodule $B \subseteq M$. In the last section we generalized basic submodules as $\alpha$-basic submodules and now with these submodules we shall define $\alpha$-large submodules and characterize them.

We have shown in the last section that an $\alpha$-module $M$ contains a proper $\alpha$-basic submodule if and only if $\alpha$ is confinal with $\omega$.

If $M$ is an $\alpha$-module of length $< \alpha$, then $M$ is totally projective, therefore we shall discuss $\alpha$-modules of length at least $\alpha$.

A submodule $B$ of $M$ is an $\alpha$-basic submodule if and only if the following hold:

(i) $B$ is totally projective module of length $< \alpha$,

(ii) $\text{Soc}(M) \subseteq H_\beta(M) + \text{Soc}(B)$ for all $\beta < \alpha$,

(iii) No submodule of $M$ properly contains $B$, having the socle equal to $\text{Soc}(B)$.

We start with the following:

**Definition 3.3.1.** A fully invariant submodule $L$ of the $\alpha$-module $M$ is $\alpha$-large if $M = B + L$ for all $\alpha$-basic submodules $B$ of $M$.

**Definition 3.3.2.** Let $\nu = (\sigma(0), \sigma(1), \ldots, \sigma(k) \ldots)$ denote a sequence of ordinals and the symbol $\infty$ such that for any $k$ and $t$, $\sigma(k) < \sigma(k+1)$ if $\sigma(k)$ is an ordinal and $\sigma(t+1) = \infty$ if $\sigma(t) = \infty$. Now $\nu$ satisfies the gap condition for an $QTAG$-module $M$ if $\sigma(k) + 1 < \sigma(k+1)$ for some $k$, implies that Ulm invariant of $M$ corresponding to $\sigma(k)$ is non zero.
Remark 3.3.3. If \( v \) satisfies the gap condition for a \( h \)-reduced \( QTAG \)-module \( M \) such that each ordinal is less than the length of \( M \), then \( v \) is called the \( U \)-sequence for \( M \).

Definition 3.3.4. Let \( M \) be a \( QTAG \)-module and \( v = (\sigma(0), \sigma(1), \ldots, \sigma(k) \ldots) \) is a sequence of ordinals and \( \infty \). Then we define \( M^v = \{ x \in M | x \in H_{\sigma(k)-k}(M) \) for every \( k \}. \) Here \( \sigma(k) = \infty \) for all \( k > n \) where \( n \) is a fixed integer if and only if \( H_{n+1}(M^v) = 0 \). Initially we discuss the \( \alpha \)-modules of length \( \alpha \).

Proposition 3.3.5. Let \( N \) be a submodule of an \( \alpha \)-module \( M \) such that \( N \cap H_{\beta}(M) = 0 \) for some \( \beta < \alpha \). Then \( N \) is contained in some \( \alpha \)-basic submodule of \( M \).

Proof. Let \( \beta \) be the first ordinal such that \( N \cap H_{\beta}(M) = 0 \) and \( \{ \beta(k) \}_{k<\omega} \) denote an increasing sequence of ordinals greater than \( \beta \) with supremum \( \alpha \).

We may define a sequence of submodules \( \{ T_k \} \) such that \( T_k \) is maximal in \( \text{Soc}(M) \) with respect to the property \( T_k \cap H_{\beta(k)}(M) = 0 \).

If \( B \) is maximal in \( \text{Soc}(M) \) such that \( B \supseteq N \) and \( \text{Soc}(B) = \cup \{ T_k \}_{k<\omega} \), then \( B \) is \( \sigma \)-summable, \( \text{Soc}(M) \subseteq H_{\beta}(M) + \text{Soc}(B) \) and there is no submodule \( K \) of \( M \), which properly contains \( B \) such that \( \text{Soc}(K) = \text{Soc}(B) \).

Thus for each \( \beta < \alpha \), \( B/H_{\beta}(B) \cong M/H_{\beta}(M) \) and \( B \) is an \( \alpha \)-module. Since an \( \alpha \)-module of length \( \alpha \) is totally projective if and only if it is \( \sigma \)-summable \( \alpha \)-module, \( B \) is totally projective hence an \( \alpha \)-basic submodule.

An immediate consequence of Proposition 3.3.5, may be state as follows.

Remark 3.3.6. If \( B \) is an \( \alpha \)-basic submodule of an \( \alpha \)-module \( M \),
length of \( M \geq \alpha \), and \( N, K \) are fully invariant submodules of \( M \), then
\[
(N + B) \cap K = (N \cap K) + (B \cap K).
\]

**Proposition 3.3.7.** Let \( N \) be a finitely generated submodule of an \( \alpha \)-module \( M \). If \( N \cap H_\beta(M) = 0 \) for some \( \beta < \alpha \), then \( M = K \oplus T \) where \( N \subseteq K \) and \( K \) is a summand of some \( \alpha \)-basic submodule of \( M \).

**Proof.** By Proposition 3.3.5, \( N \) is contained in an \( \alpha \)-basic submodule \( B \) of \( M \).

Now the length of \( B \) is \( \alpha \) which is a limit ordinal, \( B = \bigoplus \{B_i\}_{i \in I} \) and for every \( i \), \( B_i \) is totally projective module of length less than \( \alpha \). Since \( N \) is finitely generated, there exists a finite subset \( J \subseteq I \) such that \( N \subseteq \bigoplus \{B_j\}_{j \in J} = K \).

We put \( T = \bigoplus \{B_i\}_{i \in I - J} + H_\beta(M) \), where \( \beta \) is the supremum of length of \( \{B_j\}_{j \in J} \), thus \( M = K \oplus T \).

**Definition 3.3.8.** For a \( QTAG \)-module \( M \), a \( U \)-sequence \( v = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots) \) is said to be a \( U_\beta \)-sequence for \( M \) if each \( \sigma(k) \) is an ordinal less than \( \beta \).

**Theorem 3.3.9.** Let \( M \) be an \( \alpha \)-module of length \( \alpha \). Then \( L \) is an \( \alpha \)-large submodule of \( M \) if and only if \( L = M^v \) where \( v \) is a \( U_\alpha \)-sequence for \( M \).

**Proof.** Let \( L \) be an \( \alpha \)-large submodule in \( M \). Since \( L \) is fully invariant it is determined by \( v \), where \( v = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots) \) is a \( U \)-sequence for \( M \). In other words \( L = M^v \).

Now if \( \sigma(k) \) is an ordinal for some \( k \) then \( \sigma(k) < \alpha \). Moreover all the symbols \( \sigma(k) \) are ordinals because \( \alpha \)-large submodules are unbounded and this is a \( U_\alpha \)-sequence.

For the converse suppose \( L \) is determined by a \( U_\alpha \)-sequence \( v \) for \( M \) that is \( L = M^v \). If \( x \in M \), \( e(x) = n \), then we may write \( x = y + z \).
where \( y \in B, \ z \in H_{\sigma(n)}(M) \) and \( e(y) \leq e(x) \). Here \( B \) is an \( \alpha \)-basic submodule of \( M \). Thus \( z \in L \), a module determined by a \( U_\alpha \)-sequence of \( M \).

**Remark 3.3.10.** A submodule \( L \) is \( \alpha \)-large in \( M \) whenever \( L \) is determined by a \( U_\alpha \)-sequence of \( M \). This is true even if the length of \( M \) exceeds \( \alpha \).

We may infer that if \( M \) is an \( \alpha \)-module of length \( \alpha \), then \( L \) is an \( \alpha \)-large submodule of \( M \) if and only if \( L \) is unbounded, fully invariant submodule of \( M \).

Now we focus on the \( \alpha \)-modules of length greater than \( \alpha \).

For two fully invariant submodules \( N, K \subseteq M \) and an \( \alpha \)-basic submodule \( B \) of \( M \), by Remark 3.3.6, we have

\[
(N + B) \cap K = (N \cap K) + (B \cap K).
\]

By replacing \( N \) by an \( \alpha \)-large submodule \( L \) and \( K \) by \( H_\alpha(M) \), we may say that \( L \supseteq H_\alpha(M) \).

If \( H_k(L/H_\alpha(M)) = 0 \), then \( H_\alpha(M) = H_k(L) \) and \( M = B \oplus H_\alpha(M) \) for any \( \alpha \)-basic submodule of \( M \) because \( H_k(L) \) is also an \( \alpha \)-large submodule of \( M \). As \( \alpha \)-modules are \( h \)-reduced \( L/H_\alpha(M) \) is an unbounded submodule of \( M/H_\alpha(M) \).

For any \( QTAG \)-module \( M \), the submodules \( \{H_k(M)\}_{k}, \ k = 0,1,2,\ldots, \infty \) form a neighborhood system of zero, giving rise to \( h \)-topology \([28]\).

If \( k \) is replaced by any arbitrary limit ordinal less than or equal to \( \alpha \), then \( h \)-topology may be extended to \( \alpha \)-topology.

All the definitions and results which hold for \( h \)-topology may be extended for \( \alpha \)-topology. In \( \alpha \)-topology, any submodule \( N \) of \( M \), we may define the closure of \( N \) as \( \cap_{\beta<\alpha}(N + H_\beta(M)) \) denoted by \( \overline{N} \).

**Proposition 3.3.11.** Let \( N \) be a fully invariant submodule of \( M \) and
B an $\alpha$-basic submodule of $M$, then $N \subseteq \overline{N \cap B}$. The equality holds if $M$ has length $\alpha$ and $N$ is unbounded.

**Proof.** If $\beta < \alpha$, then by Remark 3.3.6, we get $N \subseteq (N \cap B) + H_\beta(M)$ and therefore $N \subseteq \overline{N \cap B}$. If the length of $M$ is $\alpha$ and $N = M^v$, determined by the $U_\alpha$-sequence $v = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots)$ for $M$, then for $x \in \overline{N \cap B}$, $e(x) = m$, we may write $x = b + z$ for some $z \in H_{\sigma(m)}(M)$, $b \in B^v$ with $e(b) \leq e(x)$. This implies that $x \in N$ or $N = \overline{N \cap B}$.

**Proposition 3.3.12.** If $B$ is an $\alpha$-basic submodule of $M$ and $L$ is an $\alpha$-large submodule of $M$, then $L = L \cap \overline{B}$.

**Proof.** As $M = L + B$ and $L \subseteq \overline{L \cap B}$, we have

$$\overline{L \cap B} = (L + B) \cap \overline{L \cap B} = L + (B \cap \overline{L \cap B}).$$

We have to show that $B \cap (\overline{L \cap B}) \subseteq L \cap B$.

Since $B$ is $h$-pure, $B \cap (\overline{L \cap B})$ is equal to the closure of $L \cap B$ in $B$, and the result holds by Proposition 3.3.11, because $L \cap B$ is a fully invariant submodule of $B$. For $z \in L \cap B$ and an endomorphism $\phi$ of $B$, we may get a submodule $N \subseteq B$ such that $zR + \phi(z)R \subseteq N$ and $M = N \oplus K$ for some $K$.

Therefore there exists an endomorphism of $N$ which maps $z$ onto $\phi(z)$ which may be extended to an endomorphism of $M$. Again $L$ is a fully invariant submodule of $M$, $\phi(z) \in L$. Now $x = \phi(z) + z \in L$, thus $L \cap B \subseteq L$ and the result follows.

**Corollary 3.3.13.** If $N$ is an unbounded fully invariant submodule of an $\alpha$-basic submodule $B$ of $M$, then $\overline{N}$ is a fully invariant submodule of $M$.

**Proof.** If $v$ is a $U$-sequence for $B$, then we may write $N = B^v$. Since $v$ is also a $U$-sequence for $M$, $M^v = \overline{M^v \cap B} = \overline{B^v} = \overline{N}$.
**Proposition 3.3.14.** Let $L$ be a submodule of $M$ if and only if $L/H_\alpha(M)$ is an $\alpha$-large submodule of $M$. 

**Proof.** Consider $L$ an $\alpha$-large submodule of $M$. Since $M/H_\alpha(M)$ is an $\alpha$-module of length $\alpha$, $L/H_\alpha(M)$ is unbounded. Let $B$ be an $\alpha$-basic submodule of $M$, then

$$(L/H_\alpha(M)) \cap [B + H_\alpha(M)/H_\alpha(M)] = ((L \cap B) + H_\alpha(M))/H_\alpha(M)$$

which is isomorphic to $L \cap B$, hence unbounded by Proposition 3.3.12, and $(B + H_\alpha(M))/H_\alpha(M)$ is isomorphic to $B$. This implies that $(L/H_\alpha(M)) \cap [(B + H_\alpha(M))/H_\alpha(M)]$ is an unbounded fully invariant submodule of $(B + H_\alpha(M))/H_\alpha(M)$ an $\alpha$-basic submodule of $M/H_\alpha(M)$.

Since $L \cap B = L$, by Proposition 3.3.12, $((L \cap B) + H_\alpha(M))/H_\alpha(M)$, the closure in $M/H_\alpha(M)$ is $L/H_\alpha(M)$. Moreover $(B + H_\alpha(M))/H_\alpha(M)$ is an $\alpha$-basic submodule of $M/H_\alpha(M)$, by Corollary 3.3.13, $L/H_\alpha(M)$ is fully invariant submodule of $M/H_\alpha(M)$.

Conversely if $L/H_\alpha(M)$ is an $\alpha$-large submodule of $M$, then $L$ is a fully invariant submodule of $M$. Since for an $\alpha$-basic submodule $B$ of $M$,

$$M/H_\alpha(M) = ((B + H_\alpha(M))/H_\alpha(M)) + (L/H_\alpha(M)), \quad M = L + B$$

and $L$ is an $\alpha$-large submodule of $M$.

**Theorem 3.3.15.** Let $L$ be an $\alpha$-large submodule of $M$ if and only if $L = M^v$, where $v$ is a $U_\alpha$-sequence for $M$.

**Proof.** Since $L$ is $\alpha$-large in $M$ if and only if $L/H_\alpha(M)$ is $\alpha$-large in $M/H_\alpha(M)$ therefore $L$ is $\alpha$-large in $M$ if and only if $L/H_\alpha(M) = (M/H_\alpha(M))^v$, for some $U_\alpha$-sequence $v$ for $M/H_\alpha(M)$.

Now $v$ is a $U_\alpha$-sequence for $M$ and $(M/H_\alpha(M))^v = M^v/H_\alpha(M)$. Thus $L$ is $\alpha$-large in $M$ if and only if $L/H_\alpha(M) = M^v/H_\alpha(M)$ and the result follows.
Now we shall discuss some properties shared by the $\alpha$-modules and their $\alpha$-large submodules.

**Proposition 3.3.16.** If $L$ is an $\alpha$-large submodule of $M$ and $\beta < \text{length of } L/H_\alpha(M)$, then $H_\beta(L)$ is $\alpha$-large submodule in $M$.

**Proof.** Let $\beta = \omega + \rho$ and the length of $L/H_\alpha(M)$ is $\omega + \gamma$ where $\rho < \gamma$. If $L = M^v$, where $v = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots)$ and $\tau = \sup\{\sigma(k)\}_{k<\omega}$, then $H_\omega(L) = H_\tau(M)$, therefore $H_\beta(L) = H_\rho(H_\omega(L)) = H_{\rho + \tau}(M)$ and $\tau + \rho < \alpha$.

**Remark 3.3.17.** Let $N$ be a fully invariant submodule of the totally projective module $M$. Then $N$ and $M/N$ are totally projective and length of $M/N \leq \text{length of } M$.

**Remark 3.3.18.** If $B$ is an $\alpha$-basic submodule of $M$ then

$$M/L = (L + B)/L \cong B/(L \cap B),$$

where $L \cap B$ is a fully invariant submodule of the totally projective module $B$. In other words, $M/L$ is totally projective whenever $L$ is an $\alpha$-large submodule of $M$.

**Theorem 3.3.19.** An $\alpha$-large submodule $L \subseteq M$ is totally projective only if $M$ is totally projective.

**Proof.** If the length of $M$ is $\alpha = \omega$, then the result follows the last chapter. Suppose the result holds for all the limit ordinals $\beta < \alpha$ where $\beta$ is cofinal with $\omega$. If $v = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots)$ is a $U_\alpha$-sequence for $M$ and $L = M^v$, then we put $\tau = \sup\{\sigma(k)\}_{k<\omega}$.

Now $\tau < \alpha$ or $\tau = \alpha$ and we shall discuss the two cases separately. If $\tau < \alpha$, $(M/H_\tau(M))^v = L/H_\tau(M) = L/H_\omega(L)$ is a totally projective module and it is a $\tau$-large submodule of the $\tau$-module $M/H_\tau(M)$. Inductively $M/H_\tau(M)$ is a totally projective module as $H_\tau(M) = H_\omega(L)$, thus $M$ is totally projective.

If $\tau = \alpha$, then $H_\omega(L) = H_\tau(M) = 0$ and by [41] $L$ is $\sigma$-summable.
therefore \( \text{Soc}(L) = \bigcup \{ T_k \}_{k<\omega} \), \( T_k \subseteq T_{k+1} \) and \( T_k \cap H_k(L) = 0 \) for each \( k < \omega \). Since \( \text{Soc}(H_{\sigma(0)}(M)) = \text{Soc}(L) \), \( H_{\sigma(0)}(M) \) is a \( \sigma \)-summable \( \rho \)-module of length \( \rho \), here \( \rho \) is a limit ordinal cofinal with \( \omega \).

For each \( k \in \mathbb{Z}^+ \), there is an ordinal \( \rho(k) (< \rho) \) equal to the length of \( H_{\sigma(0)}(M) \), such that \( \sigma(k) = \sigma(0) + \rho(k) < \sigma(0) + \rho = \alpha \). Now \( \rho = \sup \{ \rho(k) \}_{k<\omega} \), thus \( \rho \) is cofinal with \( \omega \).

Again \( T_k \cap \text{Soc} (H_{\rho(k)}(H_{\sigma(0)}(M))) \subseteq T_k \cap \text{Soc}(H_{\sigma(k)}(M)) \subseteq T_k \cap H_k(L) = 0 \), thus \( H_{\sigma(0)}(M) \) is \( \sigma \)-summable. Since \( M \) is an \( \alpha \)-module, for any ordinal \( \beta < \rho \), \( M/H_{\beta}(H_{\sigma(0)}(M)) \) and \( H_{\sigma(0)}(M/H_{\beta}(H_{\sigma(0)}(M))) \) are totally projective. Therefore \( H_{\sigma(0)}(M)/H_{\beta}(H_{\sigma(0)}(M)) \) is a totally projective module and \( H_{\sigma(0)}(M) \) is an \( \alpha \)-module. Again by [41], \( H_{\sigma(0)}(M) \) is a totally projective module as \( M/H_{\sigma(0)}(M) \) is totally projective. Thus \( M \) is also totally projective.

If the length of \( M \) is greater than \( \alpha \), then \( H_{\tau}(M) = H_{\omega}(L) \neq 0 \) where \( \tau = \sup \{ \sigma(k) \}_{k<\omega} \). Therefore \( L/H_{\omega}(L) = L/H_{\tau}(M) \) is a totally projective module and \( \tau \)-large in \( M/H_{\tau}(M) \). Also \( M/H_{\tau}(M) \) is totally projective because \( H_{\tau}(M) = H_{\omega}(L) \).

**Theorem 3.3.20.** If \( L \) is an \( \alpha \)-large submodule of an \( \alpha \)-module \( M \) and \( B \) is an \( \alpha \)-basic submodule of \( M \), then \( L \cap B \) is a \( \tau \)-basic submodule of \( B \) where \( \tau \) is the length of \( L/H_{\alpha}(M) \).

**Proof.** Let \( v \) be a \( U_{\alpha} \)-sequence for \( M \) and \( L = M^v \). Then \( L \cap B = B^v \) is a fully invariant submodule of the totally projective submodule \( B \). By Remark 3.3.17, \( L \cap B \) is a totally projective module.

If \( \rho = \sup \{ \sigma(k) \}_{k<\omega} \) and \( \alpha = \rho + \beta \), \( \tau = \omega + \beta \),

then

\[
H_{\tau}(B^v) = H_{\beta}(H_{\omega}(B^v)) = H_{\beta}(H_{\rho}(B)) = H_{\alpha}(B) = 0.
\]

Therefore the length of \( B^v \leq \tau \). Since \( H_{\beta}(L) \) is \( \alpha \)-large in \( M \), by Remark 3.3.6,
\[
\text{Soc}(H_\beta(M)) = \text{Soc}((H_\beta(M)) \cap (B + H_\beta(L)) = \text{Soc}(H_\beta(B)) + \text{Soc}(H_\beta(L)).
\]

Again \( B \) is an \( \alpha \)-basic submodule of \( M \),

\[
\text{Soc}(M) = \text{Soc}(H_\beta(M)) + \text{Soc}(B) = \text{Soc}(B) + \text{Soc}(H_\beta(L)).
\]

Now

\[
\text{Soc}(L) = L \cap \text{Soc}(M) = L \cap (\text{Soc}(B) + \text{Soc}(H_\beta(L)))
\]

\[
= \text{Soc}(L \cap B) + \text{Soc}(H_\beta(L)).
\]

In order to show that \( L \) does not contain a proper submodule \( N \) such that \( \text{Soc}(N) = \text{Soc}(L \cap B) \), we have to show that

\[
H_1(L) \cap (L \cap B) \subseteq H_1(L \cap B) \text{ or } H_1(L) \cap B \subseteq H_1(B^v).
\]

Let \( y \in H_1(L) \cap B \) such that \( d(xR/yR) = 1 \) for some \( x \in L \).

Now \( y \in (H_{\sigma(1)}(M) \cap B) \subseteq H_{\sigma(0)+1}(B) \) and \( y = b' \) such that \( d(bR/b'R) = 1 \), for some \( b \in H_{\sigma(0)}(B) \). This implies that \( b \in B^v \) and \( H_1(L) \cap B \subseteq H_1(B^v) \).

We can immediately conclude that for an \( \alpha \)-large submodule \( L \) of an \( \alpha \)-module \( M \), \( L \) is a \( \beta \)-module if \( \beta \) is the length of \( L/H_\alpha(M) \).

**Section-4**

**3.4. Nice Bases of QTAG- Modules**

Totally projective modules are defined in terms of nice submodules [32,36] thus are very significant. Here we extend this study to the modules containing nice bases submodule.

We start with the following:

**Definition 3.4.1.** A QTAG- module \( M \) has a nice basis if it can be expressed as \( M = \bigcup_{k<\omega} M_k, \ M_k \subseteq M_{k+1} \subseteq M \) and each \( M_k \) is nice in \( M \) and a direct sum of uniserial modules.
Remark 3.4.2. If each $M_k$ is bounded then $M$ has a bounded nice basis.

Proposition 3.4.3. Let $N$ be a submodule of $M$ such that $H_\omega(N) = H_\omega(M)$. If $M$ has a (bounded) nice basis, then $N$ also has a (bounded) nice basis.

Proof. Let $\{M_k\}_{k<\omega}$ be the (bounded) nice basis of $M$. Now $M = \bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1}$ and every $M_k$ is a (bounded) nice submodule of $M$, which is a direct sum of uniserial modules. Now $N = \bigcup_{k<\omega} (M_k \cap N)$ and all the intersections are (bounded) and the direct sums of uniserial modules. For a limit ordinal $\sigma$,

$$\bigcap_{\rho<\sigma} ((M_k \cap N) + H_\rho(N)) \subseteq \bigcap_{\rho<\sigma} (M_k + H_\rho(M)) \cap N = (M_k + H_\sigma(M)) \cap N = (M_k + H_\sigma(N)) \cap N = H_\sigma(N) + (M_k \cap N)$$

and the result follows.

Corollary 3.4.4. A direct summand of a module $M$ with a (bounded) nice basis and a separable complement also has a (bounded) nice basis.

Proof. Let $N$ be a direct summand of $M$, with a separable complement $K$. Now $M = K \oplus N$. Since $H_\omega(K) = 0, H_\omega(M) = H_\omega(N)$ and by proposition 3.4.3, $N$ has a (bounded) nice basis.

Proposition 3.4.5. Let $N$ be a nice submodule of $M$ such that $M/N$ has a bounded nice basis. Then

(i) if $N$ is bounded, then $M$ has a bounded nice basis;

(ii) if $N$ is a direct sum of uniserial modules, then $M$ has a nice basis.

Proof. We may express $M/N = \bigcup_{k<\omega} (M_k/N)$ where $M_k \subseteq M_{k+1} \subseteq M$, $M_k/N$ is nice in $M/N$ and it is bounded. Now by [32, 36] $M_k$ is nice in $M$. Since $N$ is bounded $M_k$ must be bounded and (i) follows.
Again $M_k/N$ is bounded by [41] $M_k$ is a direct sum of uniserial modules. Now $M = \bigcup_{k<\omega} M_k$ and $(ii)$ follows.

**Remark 3.4.6.** Since $H_\sigma(M)$ is nice in $M$ for every ordinal $\sigma$, if $M/H_\sigma(M)$ has a bounded nice basis and $H_\sigma(M)$ is a direct sum of uniserial modules, then $M$ has a nice basis

**Remark 3.4.7.** If $M/H_\sigma(M)$ has a bounded nice basis and $H_\sigma(M)$ is bounded then $M$ has a bounded nice basis. Here $\sigma$ is any ordinal number. Also, if $M$ has a bounded nice basis, then $H_\sigma(M)$ and $M/H_\sigma(M)$ have bounded nice bases.

**Remark 3.4.8.** If the length of $M$ is $\alpha < \omega.2$, i.e. $\alpha = \beta + \omega$ for some ordinal $\beta$ and $M/H_\beta(M)$ has a bounded nice basis, then $M$ has a bounded nice basis.

**Proposition 3.4.9.** If $H_\omega(M)$ is a direct sum of uniserial modules, then $M$ has a nice basis. If $H_\omega(M)$ is bounded too then $M$ has a bounded nice basis.

**Proof.** Since $M/H_\omega(M)$ is separable, it has a bounded nice basis by Remark 3.4.8. We may express $M/H_\omega(M)$ as $\bigcup_{k<\omega} H^k(M/H_\omega(M))$. Again $H^k(M/H_\omega(M)) = M_k/H_\omega(M)$ for some modules $M_k$ such that $M_k \subseteq M_{k+1} \subseteq M$ and $H_k(M_k) \subseteq H_\omega(M)$. Thus $M = \bigcup_{k<\omega} M_k$ and $H_k(M_k)$ is a direct sum of uniserial modules, $M_k$ is also a direct sum of uniserial modules [41]. Since $M_k/H_\omega(M) = H^k(M/H_\omega(M))$ is nice in $M/H_\omega(M)$ and $H_\omega(M)$ is nice in $M$ by [32, 36] every $M_k$ is nice in $M$ and the result follows.

**Proposition 3.4.10.** Let $\alpha$ be an ordinal such that $M/H_\alpha(M)$ is countably generated and $H_\alpha(M)$ has a (bounded) nice basis. Then $M$ has a (bounded) nice basis.

**Proof.** Let $M/H_\alpha(M) = \bigcup_{k<\omega} (M_k/H_\alpha(M))$ where $M_k \subseteq M_{k+1} \subseteq M$.
and $M_k/H_\alpha(M)$ are finitely generated for every $k \in \mathbb{Z}^+$. Now $M = \bigcup_{k<\omega} M_k$ and for every $k$, $M_k = H_\alpha(M) + T_k$ where $T_k$ are finitely generated and $T_k \subseteq T_{k+1}$. Again

$$H_\alpha(M) = \bigcup_{k<\omega} N_k, \quad N_k \subseteq N_{k+1} \subseteq H_\alpha(M),$$

such that

$N_k$'s are nice in $H_\alpha(M)$ and $M$, and $N_k$'s are (bounded) direct sums of uniserial modules. Now $M = \bigcup_{n<\omega} (N_n + T_n)$ where $N_n + T_n \subseteq N_{n+1} + T_{n+1} \subseteq M$. Since $T_n$'s are finitely generated all $N_n + T_n$ are nice in $M$ and are (bounded) direct sums of uniserial modules[41] and the result follows.

The following result is an immediate consequence of Proposition 3.4.10.

**Corollary 3.4.11.** If the length of the module $M$ is less than $\omega.2$ and $M/H_\omega(M)$ is countably generated then $M$ has a bounded nice basis.

**Proposition 3.4.12.** If $M$ is a module such that $H_\omega(M)$ is countably generated, then $M$ is the union of a countable ascending chain tower of nice direct sums of countably generated modules.

**Proof.** Since the separable modules have a bounded nice basis

$$M/H_\omega(M) = \bigcup_{k<\omega} (M_k/H_\omega(M))$$

where $M_k/H_\omega(M) \subseteq M_{k+1}/H_\omega(M)$ are nice submodules of $M/H_\omega(M)$ and they are bounded such that $H_k(M_k) \subseteq H_\omega(M)$. Now $M_k$'s are nice in $M$ and they are the direct sums of countably generated modules and $M = \bigcup_{k<\omega} M_k$, the result follows.

Following is the immediate consequence of the above Proposition.

**Corollary 3.4.13.** If $M$ is a QTAG- module of length at most $\omega.2$ such that $H_\omega(M)$ is countably generated, then $M$ has a nice basis.