CHAPTER 1

Preliminaries

In this chapter, we recall certain definitions and results needed for our purpose.

1.0 Notations

Throughout this thesis we use the following notations.

$I < N$ - $I$ is an ideal of $N$

$< a >$ - ideal generated by $a$

$< a >_r$ - $\Gamma$- subgroup generated by $a$

$\mathcal{P}_0(N)$ - $0$- prime radical of $N$

$\mathcal{P}_c(N)$ - completely prime radical of $N$

$N_c$ - constant part of $N$
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\[ \mathcal{N}(N) \] - set of all nilpotent elements of \( N \)

\[ \mathfrak{p}(I) \] - intersection of all prime ideals of \( N \) which contains \( I \)

\[ \mathcal{P}_s(N) \] - strongly prime radical of \( N \)

\[ \mathcal{P}_e(N) \] - equiprime radical of \( N \)

\[ \mathcal{P}_{se}(N) \] - strongly equiprime radical of \( N \)

\[ \mathcal{P}_{rs}(M) \] - right strongly prime radical of \( M \)

\[ \mathcal{P}_{ls}(M) \] - left strongly prime radical of \( M \)

a.c.c. - ascending chain condition

d.c.c. - descending chain condition

\[ r_\alpha(A) \] - right \( \alpha \)-annihilator of \( A \)

1.1 Γ-near rings

In this section we collect all basic concepts and results in near - rings, mostly from Pilz [40] and also basic results in Γ- near rings from Satyanaryana [44] and Booth [12], which are required for our study in the subsequent chapters.

**Definition 1.1.1.** A non-empty set \( N \) with two binary operations \(+\) (addition) and \( \cdot \) (multiplication) is called a near - ring if it satisfies the following axioms:

(i) \( (N, +) \) is a group (not necessarily abelian);

(ii) \( (N, \cdot) \) is a semigroup;
(iii) \((a + b)c = ac + bc\) for all \(a, b, c \in N\).

Precisely speaking, it is a right near-ring. Moreover, a near-ring \(N\) is said to be zero-symmetric near-ring if \(n0 = 0\) for all \(n \in N\), where 0 is the additive identity in \(N\).

**Definition 1.1.2.** A \(\Gamma\)-near ring is a triple \((N, +, \Gamma)\), where

(i) \((N, +)\) is a (not necessarily abelian) group;

(ii) \(\Gamma\) is a non-empty set of binary operations on \(N\) such that for each \(\gamma \in \Gamma\), \((N, +, \gamma)\) is a right near-ring and;

(iii) \((x\gamma y)\mu z = x\gamma (y\mu z)\) for all \(x, y, z \in N\) and \(\gamma, \mu \in \Gamma\).

**Example 1.1.3.** \(\Gamma\)-near rings generalize near-rings in the sense that every near-ring \(N\) is a \(\Gamma\)-near ring with \(\Gamma = \{\cdot\}\), where \(\cdot\) is the multiplication defined on \(N\). Another example is the following: Let \(X\) and \(G\) be a non-empty set and an additive group respectively. Let \(N = M(X, G)\) and let \(\Gamma = M(G, X)\), where \(M(A, B)\) denotes the set of all mappings from \(A\) into \(B\). Then \(N\) is a \(\Gamma\)-near ring with the operations pointwise addition and composition of mappings.

**Example 1.1.4.** Let \(N = \mathbb{Z}_6\) with \(\Gamma = \{\gamma_1, \gamma_2\}\) where \(\gamma_1, \gamma_2\) are given by the schemes 1: \((0, 1, 0, 0, 0, 0)\) and 2: \((0, 0, 1, 0, 0, 0)\) (see p.409, Pilz [40]).
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Then $N$ is a $\Gamma$-near ring.

**Definition 1.1.5.** Let $N$ be a $\Gamma$-near ring, then a normal subgroup $I$ of $(N, +)$ is said to be

(i) a **left ideal** if $a\alpha (b + i) - aab \in I$ for all $a, b \in N, \alpha \in \Gamma$ and $i \in I$;

(ii) a **right ideal** if $i\alpha a \in I$ for all $a \in N, \alpha \in \Gamma, i \in I$;
(iii) an ideal, if it is both left and right ideal.

**Definition 1.1.6.** Let $N$ be a $\Gamma$- near ring. An element $e \in N$ is said to be **left unity** (respectively right unity) in $N$ if $e \alpha m = m$ (respectively $m \alpha e = m$) $\forall m \in N$ and $\alpha \in \Gamma$.

**Definition 1.1.7.** Let $N$ be a near - ring. A subgroup $I$ of $N$ is said to be $N$ - **subgroup** if $NI \subseteq I$.

**Definition 1.1.8.** A $\Gamma$- near ring $N$ is said to be **zero- symmetric** if $a \alpha 0 = 0 \forall a \in N$ and $\alpha \in \Gamma$, where 0 is the additive identity in $N$.

**Definition 1.1.9.** A $\Gamma$- near ring $N$ is said to be **simple** if $\Gamma \Gamma N \neq 0$ and $N$ has no nontrivial ideals.

**Definition 1.1.10.** A $\Gamma$- near ring $N$ is said to be **integral** if $a \alpha b = 0$ where $a, b \in N$ and $\alpha \in \Gamma$ implies that either $a = 0$ or $b = 0$.

**Definition 1.1.11.** A $\Gamma$- near ring $N$ is said to be **regular** if for all $a \in N$, there exists $x \in N$ such that $a = a \gamma_1 x \gamma_2 a$ for all $\gamma_1$ and $\gamma_2 \in \Gamma$.

**Definition 1.1.12.** A $\Gamma$- near ring $N$ is said to be **left strongly regular** if for all $a \in N$, there exists $x \in N$ such that $a = x \alpha a \beta a$ for all $\alpha, \beta$ in $\Gamma$.

**Lemma 1.1.13.** If $N$ is a left strongly regular $\Gamma$- near ring, then $a = a \gamma_1 x \gamma_2 a$ and $a \gamma x = x \gamma a$ for all $\gamma_1, \gamma_2, \gamma \in \Gamma$. 
Definition 1.1.14. A $\Gamma$-near ring $N$ is said to fulfill the **insertion of factors property** (IFP) provided that for any $a, b, r \in N, \gamma \in \Gamma, a\gamma b = 0$ implies $a\alpha r\beta b = 0$ for all $\alpha, \beta \in \Gamma$.

Definition 1.1.15. Let $N$ be a near - ring and $I$ be an ideal of $N$. Then $I$ is said to be $0- (1-, 2-) prime$ if $A, B$ ideals (left ideals, $N$ -subgroups) of $N$, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. If $a, b \in N$, $aNb \subseteq I$ implies $a \in I$ or $b \in I$, then $I$ is called **3-prime** ideal of $N$.

Definition 1.1.16. Let $N$ be a $\Gamma$ - near ring, a non-empty subset $I$ of $N$ is said to be a **left $\Gamma$-subgroup** of $N$ if $I$ is a subgroup of $(N, +)$ and $M\alpha I \subseteq I \ \forall \alpha \in I$. $I$ is said to be **right $\Gamma$-subgroup** of $N$ if $I$ is a subgroup of $(N, +)$ and $I\alpha M \subseteq I \ \forall \alpha \in I$. $I$ is said to be **$\Gamma$-subgroup** of $N$ if it is both left and right $\Gamma$- subgroup.

Definition 1.1.17. A $\Gamma$ - near ring $N$ is said to be **3-prime** if $a, b \in N, a\Gamma N\Gamma b = 0$ implies $a = 0$ or $b = 0$.

Definition 1.1.18. Let $N$ be a $\Gamma$ - near ring. Let $\mathcal{L}$ be the set of all mappings of $N$ on to itself which act on the left. Then $\mathcal{L}$ is a right near - ring with operations pointwise addition and composition of mappings. Let $x \in N, \alpha \in \Gamma$, define $[x, \alpha] : N \to N$ by $[x, \alpha] y = x\alpha y \ \forall y \in N$. The sub near - ring $L$ of $\mathcal{L}$ generated by the set $\{[x, \alpha] / x \in N, \alpha \in \Gamma\}$ is called the **left operator near - ring**.
of $N$. If $I \subseteq L$, then $I^+ = \{x \in N/ [x, \alpha] \in I \ \forall \alpha \in I\}$. If $J \subseteq N$, $J^+ = \{\ell \in L/ \ell x \in J \ \forall x \in N\}$. It is shown in [12] that $I$ is an ideal in $L$ implies $I^+$ is an ideal in $N$ and $J$ is an ideal in $N$ implies $J^+$ is an ideal in $L$.

A right operator near - ring $R$ of $N$ is defined analogously to the definition of $L$. Let $\mathcal{R}$ be the left near - ring of all mappings of $N$ in to itself which act on the right. If $\gamma \in \Gamma, y \in N$, we define $[\gamma, y] : N \to N$ by $x [\gamma, y] = x\gamma y$ for all $x \in N$. $R$ is the sub near - ring of $\mathcal{R}$ generated by the set $\{[\gamma, y] / \gamma \in \Gamma, y \in N\}$.

**Definition 1.1.19.** An element $x$ of a $\Gamma$-near ring $N$ is called distributive if $x\alpha (a + b) = x\alpha a + x\alpha b$ for all $a, b \in N$ and $\alpha \in \Gamma$. If all the elements of a $\Gamma$-near ring $N$ are distributive, then $N$ is said to be distributive $\Gamma$-near ring.

**Definition 1.1.20.** An element $m$ in a $\Gamma$ - near ring $N$ is said to be left zero divisor if $m\alpha n = 0 \ \forall \alpha \in \Gamma$ implies that $n \neq 0$. An element $n$ is said to be right zero divisor if $m\alpha n = 0 \ \forall \alpha \in \Gamma$ implies that $m \neq 0$. An element in a $\Gamma$- near ring is said to be zero divisor if it is both left and right zero divisors of $N$.

**Definition 1.1.21.** An ideal $I$ of a $\Gamma$- near ring $N$ is called completely prime (completely semiprime) if $a, b \in N, \gamma \in \Gamma, a\gamma b \in I$ implies $a \in I$ or $b \in I$ ($a\gamma a \in I$ implies $a \in I$). An ideal $I$ of $N$ is said to
be prime if for any two ideals \( A, B \) of \( N, A \Gamma B \subseteq I \) implies \( A \subseteq I \) or \( B \subseteq I \). An ideal \( I \) of \( N \) is called semiprime if for any ideal \( A \) of \( N, A \Gamma A \subseteq I \) implies \( A \subseteq I \).

**Definition 1.1.22.** An element \( 0 \neq a \in N \) is called nilpotent if there exists a positive integer \( n \geq 1 \) such that \((a \gamma)^n a = 0\) for each \( \gamma \in \Gamma \). \( N \) is said to be reduced if it has no nonzero nilpotent elements.

**Definition 1.1.23.** Let \( N \) be a \( \Gamma \)-near ring with left operator near ring \( L \). If \( \sum_i [d_i, \delta_i] \in L \) has the property that \( \sum_i d_i \delta_i x = x \ \forall x \in N \), then \( \sum_i [d_i, \delta_i] \) is called a left unity for \( N \). A strong left unity for \( N \) is an element \([d, \delta] \) of \( L \) such that \( d \delta x = x \ \forall x \in N \).

**Definition 1.1.24.** Let \( N \) and \( N' \) be \( \Gamma \)-near ring (for the same \( \Gamma \)) and let \( f : N \to N' \) be a group homomorphism. Then, if \( f(\gamma xy) = f(x) \gamma f(y) \) for all \( x, y \in N \) and \( \gamma \in \Gamma \), \( f \) is called a \( \Gamma \)-near ring homomorphism.

**Remark 1.1.25.** The concepts of \( \Gamma \)-near ring epimorphism, monomorphism and isomorphism etc are defined in the usual way.

**Proposition 1.1.26.** [13] Let \( N \) be a \( \Gamma \)-near ring with a strong left unity. If \( Q \) is a prime ideal of \( L \), then \( Q^+ \) is a prime ideal of \( N \).

**Theorem 1.1.27.** [12] Suppose that a \( \Gamma \)-near ring \( N \) has a right unity and a strong left unity. Then the mapping \( A \to A^+ \) defines an isomorphism between the lattices of two sided ideals of \( N \) and \( L \).
Proposition 1.1.28. [6, Proposition 3.2] $P_0(N)$ (prime radical of $N$) is equal to the set of all strongly nilpotent elements of a near-ring $N$.

Proposition 1.1.29. [21, Lemma 3.1] Let $N$ be a near-ring. If $M$ is an $m$-system and $I$ a completely semiprime ideal such that $M \cap I = \phi$, then there exists an ideal $P$ which is maximal in the set of completely semiprime ideals containing $I$ and do not intersect $M$. Moreover $P$ is completely prime.

1.2 $\Gamma$-rings

In this section, we collect some basic concepts and results in rings from Handelmann and Lawrence[25] and Anderson and Fuller [1] and also basic result in $\Gamma$-rings from Barnes[3], Kyono[32] and Parvathi and Ramakrishna Rao[39] which are required for our study in the sixth chapter.

Definition 1.2.1. Let $M$ and $\Gamma$ be additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied

(i) $a \alpha b \in M$,

(ii) $(a + b) \alpha c = a \alpha c + b \alpha c$ $a (\alpha + \beta) c = a \alpha c + a \beta c$ $a \alpha (b + c) = a \alpha b + a \alpha c$,

(iii) $(a \alpha b) \beta c = a \alpha (b \beta c)$,
then $M$ is called a $\Gamma$-ring. If these conditions are strengthened to

(i') $a \alpha b \in M, \alpha a \beta \in \Gamma,$

(ii') same as (ii),

(iii') $(a \alpha b) \beta c = a(\alpha b \beta)c = a\alpha(b\beta c),$

(iv') $a\gamma b = 0$ for all $a, b \in M$ implies $\gamma = 0,$

then $M$ is called a $\Gamma$-ring in the sense of Nobusawa [37].

**Definition 1.2.2.** A right(left) ideal of a $\Gamma$-ring $M$ is additive sub-group of a $\Gamma$-ring $M$ such that $I \Gamma M \subseteq I (M \Gamma I \subseteq I).$ If $I$ is both a right and a left ideal, then we say that $I$ is an ideal of $M.$ An ideal $I$ of a $\Gamma$-ring $M$ is said to be prime(semiprime) if for any ideals $U, V \subseteq M, U \Gamma V \subseteq I$ implies $U \subseteq I$ or $V \subseteq I(U \Gamma U \subseteq I \Rightarrow U \subseteq I).$

If $a \in M,$ then the principal ideal generated by $a,$ denoted by $< a >,$ is the intersection of all ideal containing $a$ and is the set of all finite sum of the form $na + x\alpha a + a\beta y + u\gamma a\delta v$ where $n$ is an integer, $a, x, y, u, v$ are elements of $M$ and $\alpha, \beta, \gamma, \delta$ are elements of $\Gamma.$

**Definition 1.2.3.** Let $M$ be a $\Gamma$-ring. Consider the maps $[\alpha, x] : y \mapsto y \alpha x$ and $[x, \alpha] : y \mapsto x \alpha y, x \in M, \alpha \in \Gamma$ and for all $y \in M.$ Clearly $[x, \alpha], [\alpha, x]$ belongs to $\text{End} M.$ The bilinearity map $\Gamma \times M \rightarrow \text{End} M (M \times \Gamma \rightarrow \text{End} M)$ given by $(\alpha, m) \mapsto [\alpha, m]((m, \alpha) \mapsto [m, \alpha])$ gives rise to a linear
map from $\Gamma \otimes \mathbb{Z} M \mapsto \text{End}M \left( M \otimes \mathbb{Z} \Gamma \mapsto \text{End}M \right)$ given by

$$\sum_i \alpha_i \otimes m_i \mapsto \sum_i [\alpha_i, m_i] \left( \sum_i m_i \otimes \alpha_i \mapsto \sum_i [m_i, \alpha_i] \right), \alpha_i \in \Gamma \text{ and } m_i \in M.$$

The image of $\Gamma \otimes \mathbb{Z} M \left( M \otimes \mathbb{Z} \Gamma \right)$ in $\text{End}M$ is an associative ring denoted by $R (L)$ and call it right(left) operator ring of $M$. The ring multiplication in $R$ and $L$ is given by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j]$$

and

$$\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j] = \sum_{i,j} [x_i \alpha_i y_j, \beta_j].$$

Moreover, $M$ is faithful $L - R$ bimodule, where if $x \in M$,

$$r = \sum_t [\delta_t, z_i] \in R \text{ and } \ell = \sum_i [x_i, \alpha_i] \in L$$

then $xr = \sum_t x \delta_t z_t$ and $\ell x = \sum_i x_i \alpha_i x$.

**Definition 1.2.4.** A $\Gamma-$ ring $M$ is said to be left(right) weakly semiprime $\Gamma-$ring if $[x, \Gamma] \neq 0 \left( [\Gamma, x] \neq 0 \right)$ for all $x \neq 0 \in M$.

$M$ is said to be weakly semiprime if it is both left and right weakly semiprime.

**Definition 1.2.5.** Let $M$ be a $\Gamma-$ring, an element $m \in M$ is said to be left zero divisor if $m \alpha n = 0$ for some $\alpha \in \Gamma$ implies that $n \neq 0$. An element $n$ is said to be right zero divisor if $m \alpha n = 0$ implies that $m \neq 0$. An element in a $\Gamma-$ ring $M$ is said to be zero divisor if it is both left and right zero divisor.
Definition 1.2.6. [25] A ring $R$ is said to be prime if for given $x \neq 0, y \neq 0 \in R$, there exists $z \in R$ such that $xzy \neq 0$.

We recall that annihlator of a subset $A$ of a ring $X$ is $\text{Ann}(A) = \{r \in X/Ar = (0)\}$.

Definition 1.2.7. [25] A right insulator for $x \neq 0 \in R$ to be a finite subset of $R, S(x)$, such that $\text{Ann}\left(\{xy/y \in S(x)\}\right) = (0)$.

$R$ is said to be strongly prime if each non zero element of $R$ has a right insulator. That is, for each $x \neq 0$, there is a finite set $S(x)$ such that for $y \in R, \{xzy/z \in S(x)\} = (0)$ implies $y = 0$.

Definition 1.2.8. [1] Let $S$ and $T$ be arbitrary associative rings with unity. By $\text{Mod-}T(\text{Mod-}T)$ we denote the category of all right(left) $T$-modules. Then a module $M$ is said to be a generator (in $\text{Mod-}T$) if for every $T$-module $K$ there is a set $I$ such that the sequence $M^I \rightarrow K \rightarrow 0$ is exact. $M$ is said to be progenerator if it is finitely generated, projective and is a generator. The rings $S$ and $T$ are said to be Morita equivalent if $S\text{-Mod (Mod-}S)$ and $T\text{-Mod (Mod-}T)$ are equivalent categories. Equivalently $S$ and $T$ are Morita equivalent if there exists a progenerator $M_T$ with $S \cong \text{End}_T(M)$.

Theorem 1.2.9. [39] Let $M$ be a weakly semiprime $\Gamma$-ring, $L$ and $R$ be its operator rings. Then $L$ and $R$ are Morita Equivalent.
Definition 1.2.10. An element $x$ of a $\Gamma$-ring $M$ is called nilpotent if for any $\alpha \in \Gamma$ there exists a positive integer $n = n(\alpha)$ such that $(x\alpha)^n x = (x\alpha)(x\alpha) \cdots (x\alpha)x = 0$. The set of all nilpotent elements of $M$ is denoted by $\mathcal{N}(M)$. A $\Gamma$-ring $M$ is said to be reduced if $\mathcal{N}(M) = 0$. We use $\mathcal{P}(M)$ for the prime radical of $M$, i.e., the intersection of all prime ideals of $M$.

Definition 1.2.11. Let $I$ be an ideal of a $\Gamma$-ring $M$. If for each $a + I, b + I$ in the factor group $\frac{M}{I}$, and each $\alpha \in \Gamma$, we define $(a + I) \alpha (b + I) = a\alpha b + I$, then $\frac{M}{I}$ is a $\Gamma$-ring which we shall call the $\Gamma$-residue class ring of $M$ with respect to $I$.

Definition 1.2.12. Let $M$ be a $\Gamma$-ring with $L$ and $R$ its left and right operator rings. If there exists an element $\sum_i [e_i, \delta_i] \in L$ such that $\sum_i e_i \delta_i x = x$ for every $x \in M$, then $\sum_i [e_i, \delta_i]$ is called the left unity of $M$. Similarly we can define the right unity of $M$.

It is easily verified that $\sum_i [e_i, \delta_i]$ is the unity of $L$. Similarly the right unity of $M$ will be the unity of $R$.

Thus for a $\Gamma$-ring with left and right unities its left and right operator rings $L$ and $R$ are both rings with unities.

Definition 1.2.13. Let $M_i$ be $\Gamma_i$-rings ($i = 1, 2$). Then a pair of maps $(f, g)$ where $f : M_1 \to M_2$ and $g : \Gamma_1 \to \Gamma_2$ is said to be a $\Gamma$-ring
**homomorphism** from $M_1$ into $M_2$ if (i) $f$ and $g$ are group homomorphisms and (ii) $f(x\alpha y) = f(x)g(\alpha)f(y)$ for all $x, y \in M_1$ and $\alpha \in \Gamma_1$.

$(f, g)$ is said to be a $\Gamma$-ring **monomorphism** (epimorphism, isomorphism) if both $f$ and $g$ are monomorphisms (epimorphisms, isomorphisms).

**Definition 1.2.14.** For a subset $I \subseteq R$, we define $I^* = \{x \in M/ [\alpha, x] \in I \forall \alpha \in \Gamma\}$. It follows that if $I$ is an ideal of $R$, then $I^*$ is an ideal of $M$. For a subset $J \subseteq M$, we define $J^{*'} = \{r \in R/ xr \in J \forall x \in M\}$. It follows that if $J$ is an ideal of $M$, then $J^{*'}$ is an ideal of $R$.

When $M$ has unities on both sides we have the following proposition [32].

**Proposition 1.2.15.** (i) The mapping $I \rightarrow I^* = [I, \Gamma]$ defines an isomorphism from the lattice of right ideals of $M$ onto the lattice of right ideals of $L$, with inverse mapping $P \rightarrow P^{+'} = PM$.

(ii) The mapping $J \rightarrow J^* = [\Gamma, I]$ defines an isomorphism from the lattice of left ideals of $M$ onto the lattice of left ideals of $R$, with inverse mapping $Q \rightarrow Q^{*'} = MQ$.

Moreover the lattices of all two sided ideals of $M, L$ and $R$ are isomorphic.
Lemma 1.2.16. [31 Lemma 2.1] Let \( P, Q \) and \( S \) be a prime ideal of a \( \Gamma \)-ring \( M \), a prime ideal of the right operator ring \( R \) and a primal ideal of the left operator ring \( L \) respectively. Then \( P^* \) is a prime ideal of \( R \), \( P^+ \) is a prime ideal of \( L \), \( Q^* \) and \( S^+ \) are prime ideals of \( M \).

Theorem 1.2.17. [30, Theorem 1] If \( Q \) is an ideal in a \( \Gamma \)-ring \( M \), all the following conditions are equivalent:

(i) \( Q \) is a semiprime ideal;

(ii) If \( a \in Q \) such that \( a \Gamma M \Gamma a \subseteq Q \), then \( a \in Q \);

(iii) If \( < a > \) is a principal ideal in \( M \) such that \( < a > \Gamma < a > \subseteq Q \), then \( a \in Q \);

(iv) If \( U \) is a right ideal in \( M \) such that \( U \Gamma U \subseteq Q \), then \( U \subseteq Q \);

(v) If \( V \) is a left ideal in \( M \) such that \( V \Gamma V \subseteq Q \); then \( V \subseteq Q \).

Theorem 1.2.18. [30, Theorem 4] If \( M \) is \( \Gamma \)-ring, the following conditions are equivalent:

(i) \( M \) is a prime \( \Gamma \)-ring;

(ii) If \( a, b \in M \) and \( a \Gamma M \Gamma b = (0) \), then \( a = 0 \) or \( b = 0 \);

(iii) If \( < a > \) and \( < b > \) are principal ideals in \( M \) such that \( < a > \Gamma < b >= (0) \), then \( a = 0 \) or \( b = 0 \);
(iv) If $A$ and $B$ are right ideals in $M$ such that $\Lambda \Gamma B = (0)$, then
\[ A = (0) \text{ or } B = (0); \]

(v) If $A$ and $B$ are left ideals in $M$ such that $A \Gamma B = (0)$, then
\[ A = (0) \text{ or } B = (0). \]

**Definition 1.2.19.** [30] An ideal $Q$ in a $\Gamma$- ring $M$ is said to be **semiprime ideal** if for any ideal $U$ of $M$, $U \Gamma U \subseteq Q$ implies $U \subseteq Q$.

**Definition 1.2.20.** [30] A subset $S$ of a $\Gamma$- ring $M$ is said to be an **$m$- system** if $S = \phi$ or if $a, b \in S$ implies $\langle a \rangle \Gamma \langle b \rangle \cap S \neq \phi$.

**Definition 1.2.21.** [3] For any ideal $U$ of a $\Gamma$- ring, we define $m(U)$ to be the set of all elements $x$ of $M$ such that every $m$- system containing $x$ contains an element of $U$. 