CHAPTER 4

Strongly Prime Gamma - Near Rings

4.1 Introduction

Strongly prime rings were introduced by Handelmann and Lawrence [25] and in [24] Groenewald and Heyman investigated the upper radical determined by the class of all strongly prime rings. In [22], Groenewald introduced the concept of strongly prime to near-rings and in [18], G.L. Booth, N.J. Groenewald and S. Veldsman introduced the concept of equiprime near-rings.

In this chapter we extend the concepts of strongly prime and equiprime to $\Gamma$ - near rings. In the second section we give some
characterizations of strongly prime $\Gamma-$ near rings. In the third section we show that the strongly prime radical $\mathcal{P}_s(N)$ of $N$ coincides with $\mathcal{P}_s(L)^+$ where $\mathcal{P}_s(L)$ is strongly prime radical of the left operator near-ring $L$ of $N$. Finally in the last section we shall prove that the equiprime radical $\mathcal{P}_e(N)$ of $N$ coincides with $\mathcal{P}_e(L)^+$ where $\mathcal{P}_e(L)$ is the equiprime radical of the left operator near - ring $L$ of $N$.

4.2 Strongly prime $\Gamma-$ near rings

In this section we shall prove some equivalent conditions for strongly prime $\Gamma-$ near rings.

**Definition 4.2.1.** Let $N$ be a $\Gamma-$ near ring, then the right $\alpha-$ annihilator of a subset $A$ of $N$ is $r_\alpha(A) = \{ x \in N / A\alpha x = 0 \}$.

**Definition 4.2.2.** A $\Gamma-$ near ring $N$ is said to be strongly prime if for each $a \neq 0 \in N$, there exists a finite subset $F$ of $N$ such that $r_\alpha(aF) = 0 \ \forall \alpha \in \Gamma$. $F$ is called an insulator for $a$ in $N$.

**Lemma 4.2.3.** If a $\Gamma-$ near ring $N$ is strongly prime, then $N$ is prime.

**Proof.** Let $0 \neq A, B \triangleleft N$. We shall show that $A\Gamma B \neq 0$. Since $A \neq 0$ there exists a finite subset $F$ of $A$ such that $r_\alpha(F) = 0$, for each $\alpha \in \Gamma$. Hence for each $0 \neq b \in B$ we have $F\Gamma b \neq 0$. Therefore $A\Gamma B \neq 0$. 


Definition 4.2.4. A $\Gamma-$ near ring $N$ is said to be **left(right) weakly semiprime** if $[x, \Gamma] \neq 0 ([\Gamma, x] \neq 0)$ $\forall x \neq 0 \in N$.

$N$ is said to be **weakly semiprime** if it is both left and right weakly semiprime.

**Proposition 4.2.5.** If $N$ is strongly prime $\Gamma-$ near ring, then $N$ is weakly semiprime $\Gamma-$near ring.

**Proof.** Suppose that $N$ is a strongly prime $\Gamma-$near ring. We shall prove that $N$ is a weakly semiprime $\Gamma-$ near ring. Let $x \neq 0 \in N$. It is enough to prove that $[x, \Gamma] \neq 0$ and $[\Gamma, x] \neq 0$. Suppose that $[x, \Gamma] = 0$. Since $N$ is a strongly prime $\Gamma-$ near ring, for every $\beta \in \Gamma$ there exists a finite subset $S_\beta(x)$ such that for $b \in N$, $\{x_\beta c \alpha b/c \in S_\beta(x)\} = 0$, $\forall \alpha \in \Gamma$ implies that $b = 0$. Now $x_\beta c \alpha x = [x, \beta] c \alpha x = 0c \alpha x = 0$, $\forall \beta, \alpha \in \Gamma, c \in S_\beta(x)$. Hence $x = 0$, a contradiction. Thus $N$ is a weakly semiprime $\Gamma-$near ring.

**Proposition 4.2.6.** If a $\Gamma-$ near ring $N$ is strongly prime then, the left operator near-ring $L$ is strongly prime.

**Proof.** Let $\sum_i [x_i, \alpha_i] \neq 0 \in L$, then there exists $x \in N$ such that $\sum_i [x_i, \alpha_i] x \neq 0$, i.e., $\sum_i x_i \alpha_i x \neq 0$. Since $N$ is strongly prime, there exists a finite subset $F = \{a_1, a_2, \ldots, a_n\}$ (say) such that for any
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\[ \sum_i x_i \alpha_i x \Gamma F \Gamma b = 0 \text{ implies } b = 0 \] \hspace{1cm} (4.2.1)

Fix \( \alpha, \beta \in \Gamma \). Consider \( G = \{[x \alpha a_1, \beta], \ldots [x \alpha a_n, \beta]\} \). Our claim is that \( G \) is an insulator for \( \sum_i [x_i, \alpha_i] \). Let \( \sum_j [y_j, \beta_j] \in L \) such that
\[ \sum_i [x_i, \alpha_i] G \sum_j [y_j, \beta_j] = 0. \]
We shall prove that \( \sum_j [y_j, \beta_j] = 0 \).

Now
\[ \sum_i [x_i, \alpha_i] G \sum_j [y_j, \beta_j] = 0 \]

implies
\[ \sum_i [x_i, \alpha_i] [x \alpha a_k, \beta] \sum_j [y_j, \beta_j] = 0 \quad \forall k = 1, 2, \ldots n. \]

Hence
\[
\left( \sum_i [x_i, \alpha_i] [x \alpha a_k, \beta] \sum_j [y_j, \beta_j] \right) z = 0 \quad \forall z \in N; k = 1, 2, \ldots n.
\]

This implies that
\[ \sum_i [x_i, \alpha_i] [x \alpha a_k, \beta] \sum_j [y_j, \beta_j] z = 0 \quad \forall z \in N; k = 1, 2, \ldots n. \]

Hence
\[ \sum_i x_i \alpha_i x \alpha a_k \beta \sum_j y_j \beta_j z = 0 \quad \forall z \in N; k = 1, 2, \ldots n. \]

By (4.2.1), \( \sum_j y_j \beta_j z = 0 \quad \forall z \in N. \) Therefore \( \sum_j [y_j, \beta_j] = 0 \). Thus \( L \) is strongly prime.
Theorem 4.2.7. Let $N$ be a left weakly semiprime $\Gamma$—near ring having no zero divisor, then $N$ is strongly prime if and only if $L$ is strongly prime.

Proof. Suppose that $L$ is strongly prime. To prove $N$ is strongly prime, let $x \neq 0 \in N$. Since $N$ is left weakly semiprime, $[x, \Gamma] \neq 0$ and since $L$ is strongly prime, there exists a finite subset $F = \left\{ \sum_{j=1}^{n} [y_{jk}, \beta_{jk}] / k = 1, 2, \ldots m \right\}$ (say) such that for any $\sum_{\ell} [z_{\ell}, \delta_{\ell}] \in L$.

$$[x, \Gamma] F \sum_{\ell} [z_{\ell}, \delta_{\ell}] = 0 \implies \sum_{\ell} [z_{\ell}, \delta_{\ell}] = 0 \quad (4.2.2)$$

Consider $F' = \{ y_{jk} \beta_{jk} x / j = 1, 2, \ldots, n; k = 1, 2, \ldots m \}$. Our claim is that $F'$ is an insulator for $x$. Let $y \in N$ such that $x \Gamma F' \Gamma y = 0$. We shall prove that $y = 0$. Now $x \Gamma F' \Gamma y = 0$ implies $x \alpha y_{jk} \beta_{jk} x \beta y = 0$ for all $\alpha, \beta \in \Gamma$. Therefore

$$[x \alpha y_{jk} \beta_{jk} x \beta y, \Gamma] = 0 \quad \forall j = 1, 2, \ldots n; k = 1, 2, \ldots m.$$

Hence

$$[x, \alpha] [y_{jk}, \beta_{jk}] [x \beta y, \Gamma] = 0 \quad \forall k = 1, 2, \ldots m.$$

By (4.2.2), $[x \beta y, \Gamma] = 0$. Therefore $x \beta y = 0$. Since $N$ is weakly semiprime and $N$ has no zero divisor, $y = 0$ and consequently $F'$ is an insulator for $x$. Therefore $N$ is strongly prime.

Converse part follows from Proposition 4.2.6.
We recall that for \( X \subseteq \mathbb{N} \), \( \langle X \rangle \) is constructed by the following recursive rules

(i) \( a \in \langle X \rangle \) \( \forall a \in X \).

(ii) If \( b, c \in \langle X \rangle \), then \( b + c \in \langle X \rangle \).

(iii) If \( b \in \langle X \rangle \) and \( x, y \in \mathbb{N}, \alpha \in \Gamma \), then \( x\alpha (b + y) - x\alpha y \in \langle X \rangle \).

(iv) If \( b \in \langle X \rangle \) and \( x \in \mathbb{N}, \alpha \in \Gamma \), then \( b\alpha x \in \langle X \rangle \).

(v) If \( b \in \langle X \rangle \) and \( x \in \mathbb{N} \), then \( x - b \in \langle X \rangle \).

(vi) Nothing else is in \( \langle X \rangle \).

**Definition 4.2.8.** Suppose \( X \subseteq \mathbb{N} \) and \( d \in \langle X \rangle \). We call a sequence \( s_1, s_2, \ldots, s_n \) of elements of \( \mathbb{N} \), a generating sequence of length \( m \) for \( d \) with respect to \( X \). If \( s_1 \in X, s_m = d, \alpha \in \Gamma \) and for each \( i = 2,3,\ldots m \), one of the following applies

\[
\begin{align*}
  s_i & \in X \\
  s_i & = s_j + s_\ell, \ 1 \leq j, \ell < i \\
  s_i & = s_j\alpha x, \ 1 \leq j < i \text{ and } x \in \mathbb{N} \\
  s_i & = x\alpha (s_j + y) - x\alpha y, \ 1 \leq j < i \text{ and } x, y \in \mathbb{N} \\
  s_i & = x + s_j - x, \ 1 \leq j < i \text{ and } x \in \mathbb{N}
\end{align*}
\]

The complexity of \( d \) with respect to \( X \) denoted by \( C_X (d) \), is the
length of a generating sequence of least length for $d$ with respect to $X$.

**Lemma 4.2.9.** Let $N$ be a $\Gamma-$ near ring. If $X \neq 0$ and $X\Gamma N = 0$, then $\langle X \rangle \Gamma N = 0$.

**Proof.** Let $X\Gamma N = 0$ and suppose $x \in \langle X \rangle$ arbitrary. We use induction on $C_X(x)$. If $C_X(x) = 1$, then $x \in X$ and from our assumption we have $X\Gamma N = 0$. Suppose $y\Gamma N = 0 \ \forall y \in \langle X \rangle$ such that $C_X(y) < n$ and let $C_X(x) = n$. We have the following possibilities:

(i) $x = a + b$ where $a, b \in \langle X \rangle$ and $C_X(a), C_X(b) < n$. Hence

\[
x\Gamma N = (a + b)\Gamma N = a\Gamma N + b\Gamma N = 0
\]

(ii) $x = a\alpha n$ where $a \in \langle X \rangle, n \in N, \alpha \in \Gamma$ and $C_X(a) < n$. Hence

\[
x\Gamma N = (a\alpha n)\Gamma N \subseteq a\Gamma N = 0
\]

(iii) $x = a\alpha (d + b) - a\alpha b$ where $d \in \langle X \rangle$ and $a, b \in N, \alpha \beta \in \Gamma$ with $C_X(d) < n$. If $m$ is arbitrary element of $N$, then

\[
x\beta m = (a\alpha (d + b) - a\alpha b)\beta m
\]
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$$= a\alpha (d\beta m + b\beta m) - (a\alpha b) \beta m$$

$$= a\alpha b\beta m - a\alpha b\beta m$$

$$= 0.$$

Hence $x\Gamma N = 0$.

(iv) If $x = a + b - a$ where $b \in \langle X \rangle$, $a \in N$, $\alpha \in \Gamma$ and $C_X(b) < n$.

Let $m \in N$, then

$$x\alpha m = (a + b - a) \alpha m$$

$$= a\alpha m + b\alpha m - a\alpha m$$

$$= 0.$$

This completes the proof.

**Corollary 4.2.10.** If every non zero ideal of a $\Gamma$-near ring $N$ contains a subset $F$ with $r_\alpha(F) = 0$, $\forall \alpha \in \Gamma$, then for each $a \in N$, $a \neq 0$, $\beta \in \Gamma$, there is a $y \in N$ with $a\beta y \neq 0$.

**Proof.** Let $a \neq 0 \in N$ and suppose $F$ is a subset of $\langle a \rangle$ such that $r_\alpha(F) = 0$ $\forall \alpha \in \Gamma$. For every $n \neq 0 \in N$, we have $F\Gamma n \neq 0$ and therefore $\langle a \rangle \Gamma N \neq 0$. From Lemma 4.2.9, there exists $y \neq 0 \in N$ such that $a\beta y \neq 0$, for all $\beta \in \Gamma$.

**Theorem 4.2.11.** Let $N$ be a $\Gamma$-near ring, then $N$ is strongly prime if and only if every non zero ideal of $N$ contains a finite subset $F$ with $r_\alpha(F) = 0$, $\forall \alpha \in \Gamma$. 
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**Proof.** Let \( I \neq 0 \) be an ideal in \( N \) and \( a \neq 0 \in I \). Since \( N \) is strongly prime, there exists a finite subset \( F \subseteq N \) such that \( r_{\alpha} (a\Gamma F) = 0 \), \( \forall \alpha \in \Gamma \). Put \( F_1 = a\Gamma F \). Hence \( F_1 \) is a finite subset subset of \( I \) with \( r_{\alpha} (F_1) = 0 \), \( \forall \alpha \in \Gamma \).

Conversely, let \( a \neq 0 \in N \), then \( \langle a \rangle \neq 0 \). From our assumption, there exists a finite subset \( F \) of \( \langle a \rangle \) such that \( r_{\alpha} (F) = 0 \), \( \forall \alpha \in \Gamma \). It follows from the Corollary 4.2.10 that there exists \( y \in N \) with \( a\beta y \neq 0 \) for all \( \beta \in \Gamma \). Again we use our assumption, we can find a finite subset \( G_1 = \{g_1, g_2, \ldots, g_n\} \subseteq \langle a\beta y \rangle \) with \( r_{\alpha} (G) = 0 \), \( \forall \alpha, \beta \in \Gamma \). For each \( i \), let \( s_{i_1}, s_{i_2}, \ldots, s_{i_m} \) be the corresponding generating sequence of \( g_i \). Each of these sequence involve a finite number of terms of the form \( a\beta y \) or \( (a\beta y) \gamma t_k, t_k \in N, \forall \alpha, \beta, \gamma \in \Gamma \). Let \( G_1 = \{a\beta y, (a\beta y) \gamma t_k/ \text{ these occur in the generating sequence of an element of } G\} \). Clearly \( G_1 \) is finite and \( r_{\alpha} (G_1) \subseteq r_{\alpha} (G) = 0 \), \( \forall \alpha \in \Gamma \). Take \( H = \{x/a\beta x \in G_1, \forall \beta \in \Gamma\} \). Our claim is that \( H \) is an insulator for \( a \). Now \( r_{\alpha} (G_1) = 0 \) implies that for any \( n \in N, G_1 \alpha n = 0 \), \( \forall \alpha \in \Gamma \) implies \( n = 0 \). Since \( a\Gamma H \subseteq G_1 \), we have \( H \) is an insulator for \( a \) and consequently \( N \) is strongly prime.

**Proposition 4.2.12.** Let \( N \) be zero symmetric \( \Gamma \)-near ring then the following are equivalent.

(i) \( N \) is strongly prime \( \Gamma \)-near ring.
(ii) Every non zero right $\Gamma-$ subgroup of $N$ contains a finite subset $F$ such that $r_\alpha(F) = 0$, $\forall \alpha \in \Gamma$.

(iii) Every non zero right ideal of $N$ contains a finite subset $F$ such that $r_\alpha(F) = 0$, $\forall \alpha \in \Gamma$.

(iv) Every non zero ideal of $N$ contains a finite subset $F$ such that $r_\alpha(F) = 0$, $\forall \alpha \in \Gamma$.

**Proof.** $(i) \Rightarrow (ii)$: Let $I \neq 0$ be a right $\Gamma-$ subgroup of $N$ and let $a \neq 0 \in I$. Since $N$ is strongly prime, $a$ has an insulator $F$ such that $r_\alpha(a \Gamma F) = 0$, $\forall \alpha \in \Gamma$. Let $G = a \Gamma F$. Then $G \subseteq I$ and $r_\alpha(G) = 0$, $\forall \alpha \in \Gamma$.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (i)$ follows from Theorem 4.2.11.

**Proposition 4.2.13.** Let $N$ be a zero symmetric $\Gamma-$ near ring with d.c.c. on right annihilators, then $N$ is 3-prime if and only if $N$ is strongly prime.

**Proof.** Suppose $N$ is strongly prime. To prove $N$ is 3-prime, let $a, b \in N$ such that $a \neq 0$ and $b \neq 0$. Since $N$ is strongly prime, there exists a finite subset $F$ of $N$ such that $a \Gamma F \Gamma b \neq 0$. Hence $a \Gamma N \Gamma b \neq 0$. Conversely, let $I \neq 0$ be an ideal in $N$ and for each $\alpha \in \Gamma$, consider
the collection of right $\alpha-$ annihilators $\{r_\alpha (F)\}$ where $F$ runs over all finite subset of $I$. From our hypothesis, there exists a minimal element $M = r_\alpha (F_0)$. If $M \neq 0$, let $m \neq 0 \in M$ and $a \neq 0 \in I$. Since $N$ is 3-prime, there exists $n \neq 0 \in N$ such that $a\beta n \gamma m \neq 0$ for all $\beta, \gamma \in \Gamma$. Hence $a \gamma n \neq 0$. Let $S_\alpha = r_\alpha (F_0 \cup \{a \gamma n\}) \ \forall \alpha \in F$. Now $m \in M$ but $m \not\in S_\alpha$ implies that $S_\alpha$ is smaller than $M$, a contraction. This forces that $M = (0)$. Hence for every non zero ideal $I$ of $N$, there exists a finite subset $F$ such that $r_\alpha (F) = 0 \ \forall \alpha \in \Gamma$ and consequently $N$ is strongly prime.

4.3 Radicals of strongly prime $\Gamma-$ near rings.

In this section we shall prove that the strongly prime radical $P_s (N)$ of $N$ coincides with $P_s (L)^+$ where $P_s (L)$ is the strongly prime radical of the left operator near - ring $L$ of $N$.

Definition 4.3.1. An ideal $I$ of a $\Gamma-$ near ring $N$ is said to be strongly prime if for each $a \not\in I$, there exists a finite subset $F$ such that for any $b \in N$, $a \Gamma FTb \subseteq I$ implies that $b \in I$. $F$ is called an insulator for $a$.

Proposition 4.3.2. Let $N$ be a $\Gamma-$ near ring. If $P$ is a strongly prime ideal of $N$, then $P^+ = \{\ell \in L/\ell x \in P \ \forall x \in N\}$ is a strongly prime ideal of $L$. 
Proof. Suppose that $P$ is a strongly prime ideal of $N$. We shall prove that $P'$ is a strongly prime ideal of $L$. Let $\sum_i [x_i, \alpha_i] \notin P'$, then there exists $x \in N$ such that $\sum_i [x_i, \alpha_i] x \notin P$, that is $\sum_i x_i \alpha_i x \notin P$. Since $P$ is strongly prime in $N$, there exists a finite subset $F = \{f_1, f_2, \cdots, f_n\}$ of $N$ such that for any $b \in N$, 

$$\sum_i x_i \alpha_i x \Gamma FT \subseteq P \implies b \in P. \quad (4.3.1)$$

Fix $\alpha, \beta \in \Gamma$.

Consider the collection $F' = \{[x \alpha f_1, \beta], \cdots, [x \alpha f_n, \beta]\}$. Our claim is that $F'$ is an insulator for $\sum_i [x_i, \alpha_i]$. Let $\sum_j [y_j, \beta_j] \in L$ such that 

$$\sum_i [x_i, \alpha_i] F' \sum_j [y_j, \beta_j] \subseteq P'.$$

To prove $\sum_j [y_j, \beta_j] \in P'$, now 

$$\sum_i [x_i, \alpha_i] F' \sum_j [y_j, \beta_j] \subseteq P'$$

implies 

$$\sum_i [x_i, \alpha_i] [x \alpha f_k, \beta] \sum_j [y_j, \beta_j] \in P' \quad \forall k = 1, 2, \cdots, n,$$

i.e., 

$$\left(\sum_i [x_i, \alpha_i] [x \alpha f_k, \beta] \sum_j [y_j, \beta_j]\right) z \in P$$

$\forall z \in N; k = 1, 2, \cdots, n$. Hence 

$$\sum_i x_i \alpha_i x \Gamma FT \sum_j y_j \beta_j z \subseteq P \quad \forall z \in N.$$ 

By (4.3.1), $\sum_j y_j \beta_j z \in P \quad \forall z \in N$. i.e., $\sum_j [y_j, \beta_j] z \in P \quad \forall z \in N$. Hence 

$$\sum_j [y_j, \beta_j] \in P'$$ 

and therefore $F'$ is an insulator for $\sum_i [x_i, \alpha_i]$ and consequently $P'$ is a strongly prime ideal of $L$. 


Proposition 4.3.3. Let $N$ be a distributive strongly 2-primal $\Gamma$-near
ring with strong left unity. If $Q$ is a strongly prime ideal of $L$, then
$Q^+ = \{ x \in N / [x, \alpha] \in Q \ \forall \alpha \in \Gamma \}$ is a strongly prime ideal of $N$.

Proof. Suppose $Q$ is a strongly prime ideal of $L$. We shall prove that $Q^+$ is a strongly prime ideal of $N$. Let $x \notin Q^+$, then there exists $\alpha \in \Gamma$ such that $[x, \alpha] \notin Q$. Since $Q$ is a strongly prime ideal of $L$, there exists a finite subset $F = \left\{ \sum_{j=1}^{n} [y_{j}, \beta_{jk}] / jk = 1, 2, \ldots, m \right\}$ (say) such that for any $\sum_{\ell} [z_{\ell}, \delta_{\ell}] \in L$,

$$[x, \alpha] F \sum_{\ell} [z_{\ell}, \delta_{\ell}] \subseteq Q \text{ implies that } \sum_{\ell} [z_{\ell}, \delta_{\ell}] \in Q. \quad (4.3.2)$$

Consider $F' = \{ y_{jk} \beta_{jk} x / j = 1, 2, \ldots, n; k = 1, 2, \ldots, m \}$. Our claim is that $F'$ is an insulator for $x$. Let $a \in N$ such that $x \Gamma F' \Gamma a \subseteq Q^+$. To prove $a \in Q^+$. Now $x \Gamma F' \Gamma a \subseteq Q^+$ implies

$$[x \Gamma F' \Gamma a, \Gamma] \subseteq Q, \quad \text{i.e., } [x \alpha y_{jk} \beta_{jk} x \beta a, \gamma] \in Q, \quad \forall j = 1, 2, \ldots, n; k = 1, 2, \ldots, m \text{ and } \forall \alpha, \beta, \gamma \in \Gamma.$$

This implies that

$$[x, \alpha] F [x \beta a, \gamma] \subseteq Q. \quad (4.3.3)$$

By (4.3.2) $[x \beta a, \gamma] \in Q$. Now since $Q$ is strongly prime in $L$, $Q$ is prime in $L$. By Proposition 1.1.26, $Q^+$ is prime ideal of $N$. Since $N$
is strongly 2-primal, \(Q^+\) is completely prime in \(N\). Hence \(x\gamma a \in Q^+\) and \(x \notin Q^+\) implies \(a \in Q^+\). Thus \(Q^+\) is strongly prime in \(N\).

**Proposition 4.3.4.** Let \(N\) be a distributive strongly 2-primal \(\Gamma\)-near ring with strong left unity and \(L\), a left operator near-ring of \(N\). Then \(\mathcal{P}_s(N) = \mathcal{P}_s(L)^+\).

**Proof.** Let \(P\) be a strongly prime ideal of \(L\). Then by Proposition 4.3.3, \(P^+\) is a strongly prime ideal of \(N\). Moreover \((P^+)^{+'} = P\) by Theorem 1.1.27. Suppose \(Q\) is a strongly prime ideal in \(N\), then by Proposition 4.3.2, \(Q^{+'}\) is strongly prime in \(L\) and \((Q^{+'})^+ = Q\) by Theorem 1.1.27. Thus the mapping \(P \rightarrow P^+\) defines a 1-1 correspondence between the set of strongly prime ideals of \(L\) and \(N\).

Hence \(\mathcal{P}_s(L)^+ = (\cap P)^+ = \cap P^+ = \mathcal{P}_s(N)\).

### 4.4 Equiprime radicals of \(\Gamma\)-near rings

In this section we shall prove that the equiprime radical \(\mathcal{P}_e(N)\) of \(N\) coincides with \(\mathcal{P}_e(L)^+\) where \(\mathcal{P}_e(L)\) is the equiprime radical of left operator near-ring \(L\) of \(N\).

**Definition 4.4.1.** Let \(N\) be a \(\Gamma\)-near ring, and \(P\) be an ideal in \(N\). Then \(P\) is said to be **equiprime** if \(a, x, y \in N, a \notin P, a\alpha n\beta x - a\alpha n\beta y \in P\) for all \(n \in N, \alpha, \beta \in \Gamma\) implies \(x - y \in P\).
Proposition 4.4.2. Let $N$ be a $\Gamma$-near ring. If $P$ is an equiprime ideal of $N$, then $P^{+'} = \{\ell \in L/\ell x \in P \ \forall x \in N\}$ is an equiprime ideal of $L$.

Proof. Let $\ell \notin P^{+'}$ and $\ell', \ell'' \in N$ such that $\ell' - \ell'' \notin P^{+'}$. From definition of $P^{+'}$, there exist $a, b \in N$ such that $\ell a \notin P$ and $(\ell' - \ell'')b \notin P$, that is $\ell a \notin P$ and $\ell'b - \ell''b \notin P$. From the hypothesis, there exists $c \in N$ such that

\[(\ell a) \alpha c \beta (\ell'b) - (\ell a) \alpha c \beta (\ell''b) \notin P, \ \forall \alpha, \beta \in \Gamma\]

i.e., $[\ell a, \alpha] [c, \beta] \ell'b - [\ell a, \alpha] [c, \beta] \ell''b \notin P, \ \forall \alpha, \beta \in \Gamma$

i.e., $\ell [a, \alpha] [c, \beta] \ell'b - \ell [a, \alpha] [c, \beta] \ell''b \notin P, \ \forall \alpha, \beta \in \Gamma$.

Hence

\[(\ell [a \alpha c, \beta] \ell' - \ell [a \alpha c, \beta] \ell'') b \notin P, \ \forall \alpha, \beta \in \Gamma\]

This proves that

\[\ell [a \alpha c, \beta] \ell' - \ell [a \alpha c, \beta] \ell'' \notin P^{+'}, \ \forall \alpha, \beta \in \Gamma\]

and consequently $P^{+'}$ is an equiprime ideal of $L$.

Proposition 4.4.3. Let $N$ be a $\Gamma$-near ring. If $Q$ is an equiprime ideal of $L$, then $Q^+ = \{x \in N/ [x, \alpha] \in Q \ \forall \alpha \in \Gamma\}$ is an equiprime ideal of $N$. 
Proof. Let \( x \notin Q^+ \) and \( a, b \in N \) such that \( a - b \notin Q^+ \). We claim that \( x\Gamma N \Gamma a - x\Gamma N \Gamma b \notin Q^+ \). Since \( x \notin Q^+ \) and \( a - b \notin Q^+ \), then there exist \( \alpha, \beta \in \Gamma \) such that \( [x, \alpha] \notin Q \) and \( [a - b, \beta] \notin Q \) implies that \( [x, \alpha] \notin Q \) and \( [a, \beta] - [b, \beta] \notin Q \). Since \( Q \) is a equiprime ideal in \( L \), there exists \( \ell = \sum [y_i, \beta_i] \in L \) such that \( [x, \alpha] \ell [a, \beta] - [x, \alpha] \ell [b, \beta] \notin Q \). Hence \( [x\alpha a - x\alpha b, \beta] \notin Q \). This implies that \( x\alpha a - x\alpha b \notin Q^+ \).

i.e., \( x\alpha \sum [y_i, \beta_i] a - x\alpha \sum [y_i, \beta_i] b \notin Q^+ \)

i.e., \( x\alpha \sum y_i \beta_i a - x\alpha \sum y_i \beta_i b \notin Q^+ \).

But clearly \( x\alpha \sum y_i \beta_i a - x\alpha \sum y_i \beta_i b \in x\Gamma N \Gamma a - x\Gamma N \Gamma b \). Thus \( x\Gamma N \Gamma a - x\Gamma N \Gamma b \notin Q^+ \) and consequently \( Q^+ \) is an equiprime ideal of \( N \).

Theorem 4.4.4. Let \( N \) be a \( \Gamma \)-near ring with left operator near-ring \( L \), then \( \mathcal{P}_e(L)^+ = \mathcal{P}_e(N) \).

Proof. Let \( P \) be an equiprime ideal of \( L \). Then by Proposition 4.4.3, \( P^+ \) is an equiprime ideal of \( N \). Moreover \( (P^+)^{+'} = P \) by Theorem 1.1.27. Suppose \( Q \) is an equiprime ideal in \( N \), then by Proposition 4.4.2, \( Q^{+'} \) is an equiprime ideal in \( L \) and \( (Q^{+'})^+ = Q \) by Theorem 1.1.27. Thus the mapping \( P \to P^+ \) defines a 1-1 correspondence between the set of equiprime ideals of \( L \) and \( N \).

Hence \( \mathcal{P}_e(L)^+ = (\cap P)^+ = \cap P^+ = \mathcal{P}_e(N) \).