CHAPTER 1

Notation and Terminology

1.1. BASIC DEFINITIONS

As the notation and terminology used in Graph Theory vary widely, we give below some basic definitions and notation often used in this thesis. Some of the terms not defined here are defined either in the respective chapters where they are studied or taken as in Harary [17].

1.1 Definitions A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements called vertices (points), and $E(G)$ is a finite set of unordered pairs of distinct elements of $V(G)$ called edges (lines). $V(G)$ and $E(G)$ are called the vertex set and the edge set of $G$ respectively. The value $|V(G)|$ is called the order of $G$. The value $|E(G)|$, denoted by $e(G)$ is called the size of $G$. A graph with order $n$ and size $m$ will be called an $(n,m)$-graph. A $(1,0)$-graph is called the trivial graph.

1.2 Definitions Let $G$ be a graph. The edge $\{v, w\}$ where $v, w \in V(G)$ will be usually denoted by $vw$ or $wv$. If $e = vw$ is an edge of $G$, then $e$ is said to join the vertices $v$ and $w$, and these vertices are then said to be adjacent. In this case, we also say that $e$ is incident with $v$ and $w$, and that $w$ is a neighbour of $v$ and vice versa. The set of all neighbours of a vertex $u$ of a graph $G$ is denoted by $N(u)$ and its cardinality by $n(u)$. When $W \subseteq V(G)$, $W \cap N(u)$ is denoted by $N_W(u)$ and its cardinality by $n_W(u)$.

1.3 Definitions For each vertex $v$ in a graph $G$, the number of edges incident with $v$ is called the degree of $v$, denoted by $\deg_G v$ (or simply $\deg v$). The maximum and minimum degree in $G$ will be denoted by $\Delta(G)$ and $\delta(G)$ or simply by $\Delta$ and $\delta$. 
respectively. A vertex with degree $m$ is referred to as a $m$-vertex. A 0-vertex and a 1-vertex are respectively called an isolated vertex and an endvertex. The unique neighbour of a 1-vertex is called its base. A graph is called regular if all vertices have the same degree.

1.4 Definitions The degree sequence of a graph $G$, denoted by $DS(G)$ and represented by $[d_1^{m(1)}, d_2^{m(2)}, \ldots, d_k^{m(k)}]$ means that the vertices of $G$ can be labeled $v_1, v_2, \ldots, v_{m(1)}$, $v_{m(1)+1}, v_{m(1)+2}, \ldots, v_{m(1)+m(2)}$ each have degree $d_1$, $v_{m(1)+m(2)+1}, v_{m(1)+m(2)+2}, \ldots, v_{m(1)+m(2)+m(3)}$ each have degree $d_2$, and so on. The neighbourhood degree sequence of a vertex $v$ in a graph $G$, denoted by $NDS_G(v)$ or simply $NDS(v)$ and represented by $[d_1^{m(1)}, d_2^{m(2)}, \ldots, d_p^{m(p)}]$ means that the vertices of $G$ adjacent to $v$ can be labeled $v_1, v_2, \ldots, v_{m(1)}$ each have degree $d_1$, $v_{m(1)+1}, v_{m(1)+2}, \ldots, v_{m(1)+m(2)}$ each have degree $d_2$, $v_{m(1)+m(2)+1}, v_{m(1)+m(2)+2}, \ldots, v_{m(1)+m(2)+m(3)}$ each have degree $d_3$, and the last $m(p)$ vertices each have degree $d_p$.

1.5 Definitions A graph $H$ is said to be a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $W \subseteq V(G)$, then the subgraph induced by $W$ (denoted by $<W>$) is the subgraph of $G$ whose vertex set is $W$ and edge set is $\{vw \mid vw \in E(G) \text{ and } v, w \in W\}$. An induced subgraph of $G$ is a subgraph which is induced by some subset $W$ of $V(G)$. If $W$ is a proper subset of $V(G)$, then $G-W$ denotes the induced subgraph $<V(G) - W>$. If $v$ is a vertex of $G$, then the vertex-deleted subgraph $G-v$ is the subgraph of $G$ induced by $V(G) - \{v\}$. If $H$ is a subgraph of $G$ and $v$ is a vertex of $G$ not in $H$, then the vertex-added subgraph $H+v$ is the subgraph of $G$ induced by $V(H) \cup \{v\}$. If $uv$ is an edge of $G$, then the edge-deleted subgraph $G-uv$ is the graph $(V(G), E(G) - \{uv\})$. If $H$ is a
subgraph of $G$ and $uv$ where $u, v \in V(H)$ is an edge of $G$ not in $H$, then the **edge-added subgraph** $H + uv$ of $G$ is the graph $(V(H), E(H) \cup \{uv\})$.

1.6 Definitions A sequence $v_1, v_2, \ldots, v_n$ of distinct vertices in a graph $G$ where $v_i$ is adjacent with $v_{i+1}$, $1 \leq i \leq n-1$ is called a **path of length $n-1$** from $v_1$ to $v_n$. A path of length $k-1$ (i.e., with $k$ vertices) is denoted by $P_k$. A **cycle of length $n$**, $n \geq 3$ (denoted by $C_n$), in a graph $G$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ where $v_i$ is adjacent with $v_{i+1}$, $1 \leq i \leq n-1$ and $v_n$ is adjacent to $v_1$. A **chord** of a cycle $C_m$ is an edge not on $C_m$, which has its endvertices in $C_m$.

1.7 Definitions A graph $G$ is **connected** if for every $u, w \in V(G)$, there exists a path from $u$ to $w$; a graph which is not connected is called **disconnected**. The maximal connected subgraphs of a disconnected graph $G$ are called the **components** of $G$. A connected graph having no cycles is called a **tree**.

1.8 Definitions A vertex $v$ of a graph $G$ is called a **cut vertex** of $G$ if $G - v$ has more components than $G$. A connected graph is called **separable** if it has cut vertices and is called **non-separable** otherwise.

1.9 Definitions A **block** of a graph $G$ is a maximal non-separable subgraph. If $G$ itself is connected and non-separable, then $G$ is called a block. A block of $G$ is called an **end-block** of $G$ if it has only one cut vertex of $G$ and is called a **non-endblock** otherwise.

1.10 Definitions The **connectivity** $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected graph or in the trivial graph. A graph $G$ is called **$k$-connected** if $\kappa(G) \geq k$. Thus a **2-connected graph** is a block with more than two vertices.
1.11 Definitions A graph in which every pair of distinct vertices is adjacent is called a complete graph; the complete graph of order \( p \) is denoted by \( K_p \). A graph of the form \( (V \cup W, \{\{v, w\} : v \in V \text{ and } w \in W\}) \) where \( V \) and \( W \) are disjoint nonempty sets is called a complete bipartite graph and is denoted by \( K_{r,s} \) where \( |V| = r \) and \( |W| = s \).

1.12 Definitions The distance between two vertices \( u \) and \( v \) in a graph \( G \), denoted by \( d(u, v) \), is the length of a shortest path joining them if any; otherwise \( d(u, v) = \infty \). The eccentricity of a vertex \( u \) in a connected graph \( G \) is the maximum of its distances to other vertices. The radius and diameter of a graph \( G \), denoted by \( rad(G) \) and \( diam(G) \) respectively, are the minimum and maximum of the vertex eccentricities respectively.

1.13 Definition The complement of a graph \( G \), denoted by \( \overline{G} \) is the graph having the same vertex set as \( G \) and \( uw \) is an edge of \( \overline{G} \) if and only if it is not an edge of \( G \).

1.14 Definitions Two graphs \( G \) and \( H \) are said to be isomorphic (written \( G \cong H \)) if there is a bijection \( f \) from \( V(G) \) on to \( V(H) \) such that for every \( v, w \in V(G) \), \( \{v, w\} \in E(G) \Leftrightarrow \{f(v), f(w)\} \in E(H) \). Such a bijection \( f \) is called an isomorphism of \( G \) to \( H \). An isomorphism of a graph with itself is called an automorphism. Two vertices \( u \) and \( v \) of a graph \( G \) are said to be similar if there is an automorphism of \( G \) taking \( u \) to \( v \); they are said to be bisimilar if there is an automorphism of \( G \) interchanging \( u \) and \( v \).

1.2 ULAM'S CONJECTURE

Probably the foremost unsolved problem in Graph Theory is Ulam's Conjecture. This problem is due to P.J. Kelly and S.M.Ulam. Kelly's Ph.D thesis [19] written under S.M.Ulam in 1942 dealt with this. Ulam proposed it as a set theory problem in his famous book "A Collection of Mathematical Problems" [49].
1.15 Ulam's Problem

This is how Ulam’s problem was originally stated [49]:

"Suppose that in two sets A, B each of n elements, there is defined a distance function \( \rho \) for every pair of distinct points, with values either 1 or 2 and \( \rho(P, P) = 0 \). Assume that for every subset of \((n-1)\) points of A, there exists an isometric system of \((n-1)\) points of B, and that the number of distinct subsets isometric to any given subset of \((n-1)\) points is the same in A as in B. Are A and B isometric?"

Kelly [20] has given the graph theoretic version of this problem as below and solved it for trees and disconnected graphs, and verified it for graphs on up to six vertices. Mc Kay has extended this verification to eleven vertices [34].

1.16 Ulam's Conjecture. Let G and H be graphs on \( n (\geq 3) \) vertices \( v_1, v_2, \ldots, v_n \) and \( w_1, w_2, \ldots, w_n \) respectively. If \( G - v_i \cong H - w_i \) for \( i = 1 \) to \( n \), then \( G \cong H \).

The current version of this problem, popularly known as the Reconstruction Conjecture is the one formulated by Harary [15].

1.17 Reconstruction Conjecture. Every graph G on at least three vertices is uniquely determined up to isomorphism by the collection of its vertex-deleted subgraphs.

Bondy and Hemminger [5] have reformulated this precisely as follows:

1.18 Definitions A vertex-deleted unlabeled subgraph \( G - v \) of a graph G is called a card of G. The deck of G, denoted \( \mathcal{D}(G) \), is the collection of all its cards. A graph H with \( \mathcal{D}(H) = \mathcal{D}(G) \) is called a reconstruction of G. If every reconstruction of G is isomorphic to G, then G is said to be reconstructible. A property (parameter) Q defined on a class S
of graphs is called a **recognizable property (reconstructible parameter)** if $Q(G) = Q(H)$ whenever $G \in S$ and $H$ is a reconstruction of $G$.

1.19 **Definitions** A family $F$ of graphs is called **recognizable** if, for each graph $G \in F$, every reconstruction of $G$ is also in $F$, and called **weakly reconstructible** if, for each graph $G \in F$, all reconstructions of $G$ that are in $F$ are isomorphic to $G$. A family $F$ of graphs is called **reconstructible** if " $G \in F \Rightarrow G$ is reconstructible " (i.e. if $F$ is both recognizable and weakly reconstructible).

1.20 **Reconstruction Conjecture (RC).** All graphs on at least three vertices are reconstructible.

1.3 **SET RECONSTRUCTION CONJECTURE**

The Set Reconstruction Conjecture was proposed by Harary [15]. We state it after a few definitions.

1.21 **Definitions** For a graph $G$, $S(G)$ denotes the set of its (non-isomorphic) cards. A graph $H$ with $S(H) = S(G)$ is called a **set reconstruction** of $G$. If every set reconstruction of $G$ is isomorphic to $G$, then $G$ is said to be **set reconstructible**. Definitions of **set reconstructible classes of graphs, set reconstructible parameters** and **set recognizable properties** are analogous to those for graphs in Definitions 1.18 and 1.19.

1.22 **Set Reconstruction Conjecture (SRC) [15].** All graphs with at least four vertices are set reconstructible.