Uhlenbeck and Goudsmit, in 1925, put forward the hypothesis that the electron possesses a magnetic moment, and that, in a magnetic field its direction is quantized so that it orients itself either parallel or anti-parallel to the field. This extra degree of freedom is called spin. In the same year, Lenz suggested to his student Ising\(^2\) that if an interaction was introduced between the spins so that parallel spins in a lattice attract one another, and anti-parallel spins repel one another, then at sufficiently low temperatures the spins would all be aligned and the model might provide a microscopic description of ferromagnetism. The corresponding Hamiltonian is of the form

\[ \mathcal{H} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - mH \sum_i \mathbf{S}_i, \]  

(1.1)

Here \( J > 0 \) represents the interaction between spins, \( H \) is an external magnetic field, \( m \) the magnetic moment of a single spin and \( \mathbf{S}_i \) is a dummy variable which can take the values \( \pm 1 \). The suffix \( i \) runs over all sites of the lattice and \( \langle ij \rangle \) over all pairs of sites \( i \) and \( j \) which are nearest neighbours (n.n.). This is
called the Ising model. (For an interesting historical review of this model, see the article by Brush).

Ising solved the model in one dimension and found that the solution is analytic without any singularities and the spontaneous magnetization vanishes for all \( T > 0 \). Eleven years later Peierls\(^4\) showed that the two-dimensional model does have a non-zero spontaneous magnetization and therefore can be regarded as a valid model of a ferromagnet. (For a summary of rigorous results on the Ising model, see the article by Griffiths in the Domb-Green Series\(^5\)).

In 1941, Kramers and Wannier\(^6\) discovered a transformation which enabled them to calculate the exact value of the Curie temperature of a simple quadratic (s.q.) lattice. They also showed how to develop exact series expansions for the partition function at high and low temperatures. Their paper was followed by Onsager's famous calculation in 1944 of the partition function\(^7\) of the s.q. lattice in zero field, which has served as a landmark in the theory of critical behaviour. During subsequent years exact information was obtained on the Ising model as calculations were extended to the spontaneous magnetization,\(^8-9\) correlations\(^10-11\) and susceptibility;\(^11-13\) solutions also became available for a variety of additional two-dimensional lattices. (See the reviews by Domb\(^14\) and Syoji\(^15\)). However, it has so far proved impossible to solve the model exactly in non-zero field or in three dimensions.

In the absence of exact solutions the two alternative approaches available were closed form approximations and series expansions. The former had been developed in the 1930s by Bragg and Williams,\(^16\) Bethe,\(^17\) Guggenheim\(^18\) and others. But, a comparison by Kramers and Wannier\(^6\) with exact series expansions showed that even the best approximations available gave only a
few terms correctly. Also such approximations were suggested by the authors to be unreliable in the critical region and this suggestion was substantiated by Onsager.\textsuperscript{7}

Series expansions had been introduced by Kramers and Wannier\textsuperscript{6} with the aim of testing the validity of closed form approximations as mentioned above. However, Domb\textsuperscript{19} suggested that if expansions of sufficient length could be derived they might provide a direct assessment of critical behaviour. Such expansions were derived in two dimensions for the s.q. lattice by Domb\textsuperscript{19} and in three dimensions for the simple cubic (s.c.) lattice by Wakefield.\textsuperscript{20} These calculations were extended to a variety of two- and three-dimensional lattices\textsuperscript{14} and methods of extracting information regarding critical behaviour were steadily improved.

Series for the initial susceptibility at high temperatures provided the smoothest and most regular pattern of behaviour of coefficients, they were all found to be positive in sign, and the ratio method\textsuperscript{21} was used to estimate the Curie temperatures and critical exponents. For the s.q. and plane triangular (p.t.) lattices in two dimensions, the Curie temperatures were known exactly, and hence a more accurate estimate could be made of the critical exponents. Domb and Sykes\textsuperscript{22} suggested the value $\gamma = 1.75$ for the susceptibility exponent of these lattices and this was later justified rigorously.\textsuperscript{11-13,23} For three-dimensional lattices the corresponding estimate\textsuperscript{24} was 1.25.

A dramatic step forward was taken by Baker\textsuperscript{25} in 1961 who applied Padé approximant\textsuperscript{21} to these Ising series. For series with positive terms, the results were in excellent accord with those of the ratio method. But for irregular series, e.g., the low temperature series for spontaneous magnetiza-
tion, it was possible to obtain estimates for the critical exponent of the spontaneous magnetization.

Due to the existence of the exact solutions and the possibility of deriving extensive series expansions both at low and high temperatures, the Ising model has served as a pioneer in the exploration of critical behavior and many important results in the theory of critical phenomena started with application to the Ising model. These include accurate estimates of the critical exponents, stable accurate estimates of critical values of thermodynamic functions, observation that dimension rather than lattice structure determines critical behaviour of antiferromagnets, critical amplitudes, critical equation of state, critical correlations, surface and finite size effects, lattice-lattice scaling, correction terms to the equation of state, the crossover exponent.

From all these and many more studies, it has now been established that, in general, the critical exponents depend on very few details of the system, e.g., the dimensionality, the symmetry properties of the Hamiltonian, etc. A systematic method of getting expansions for the exponents in terms of these parameters was introduced by Wilson and Fisher. This so-called renormalization group method has been a valuable tool in getting numerical values for critical parameters, sometimes comparable in accuracy with the other methods. (Recent reviews are by Wilson and Kogut and also by Fisher).

1.2. Brief History of Scaling and Universality

The scaling hypothesis was introduced by Widom, Kadanoff, Domb and Hunter, Patashinskii and Pokrovskii.
Fig. 1

$T > T_0$

$T = T_c$

$T < T_c$

$T = 0$

$H$

$M$
Fig. 2
more or less independently. (For recent reviews of these and other ideas see Fisher, Stanley. The basic idea is to look at the isotherms in Figure 1 and notice that they are very similar to one another. The scaling hypothesis exploits this similarity to bring all the isotherms on one curve by introducing a change in scale which is temperature-dependent. The hypothesis states that

$$\frac{M(H,T)}{a|t|^\lambda} \approx f_{\pm} \left( \frac{bH}{|t|^\Delta} \right), \quad t = \frac{(T-T_c)}{T_c}, \quad (1.2)$$

where and are two exponents and 'a' and 'b' are scale factors which have been introduced to make the scaling functions universal. The functions \( f_{\pm} \) are valid for \( t > 0 \) and \( t < 0 \), respectively. A typical sketch of functions \( f_{\pm} \) is shown in Figure 2. Following comments are to be noted regarding the hypothesis.

The symbol "\( \approx \)" denotes that it is valid in the critical region, i.e., for both \( H \) and \( t \) very small, typically \( \lesssim 10^{-2} \). The variable \( \chi = bH/|t|^{\Delta} \) has the range \(-\infty \leq \chi \leq \infty \). The functions \( f_{+} \) and \( f_{-} \) are different from each other owing to the fact that the isotherms for \( T > T_c \) are essentially different from those for \( T < T_c \). No particular value of \( \chi \) corresponds to the critical point (\( H = 0, T = T_c \)) because \( \chi \) is indeterminate for \( H = 0, T = T_c \). However, there exists one value of \( \chi \) which corresponds to each path taken by an approach to the critical point. Of course, the hypothesis does what it was designed to do. If we change \( H \) and \( t \) by factors of \( l \) and \( l^{1/\Delta} \) respectively, \( M \) is simply multiplied by \( l^{-\Delta/\lambda} \). In fact, this can be explicitly noted by writing (1.2) in the following form

$$M(H,t) \approx a l^{-\lambda} M\left( l^{\Delta} bH, lt \right), \quad (1.3)$$
<table>
<thead>
<tr>
<th>Exponent</th>
<th>Definition</th>
<th>Condition</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$C_H \sim (-t)^{-\lambda}$</td>
<td>0 0 0 0</td>
<td>Specific heat at constant magnetic field.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$C_H \sim t^{-\lambda}$</td>
<td>0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>$M \sim (-t)^{\beta}$</td>
<td>0 0 0 0</td>
<td>Zero-field magnetization.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\chi_T \sim (-t)^{-\gamma'}$</td>
<td>0 0 $\neq 0$</td>
<td>Zero-field isothermal susceptibility.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\chi_T \sim t^{-\gamma}$</td>
<td>0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>$H \sim</td>
<td>M</td>
<td>^{\delta}_{\text{sgn}(M)}$</td>
</tr>
</tbody>
</table>
where 1 is arbitrary. In this form, it is called the generalized homogeneity hypothesis.\(^{46}\)

Now, we will study some of the implications of the scaling hypothesis.

\(\beta\)

(a) For \(H = 0\), we know that \(C/M = 0\) for \(t > 0\) and \(H = B(t)\) for \(t \leq 0\). (For the sake of completeness, in Table 1, we give the standard definitions of the exponents\(^{45,46}\).) In order to reproduce this behavior, equation (1.2) should satisfy the following

\[
\lambda = \beta, \quad \alpha = B.
\]

(b) For \(T = T_c\), \(H \to 0\) a new exponent is defined for the shape of the critical isotherm.

\[
M = D H^{1/6}, \quad T = T_c, \quad H \to 0.
\]

The functions \(f(x)\) should satisfy the following conditions:

\[
f_+ (x) = x^{4/6}, \quad x \to \infty,
\]

\[
\lambda/\Delta = 1/6, \quad D = a b^{4/6}.
\]

(c) Differentiating equation (1.2) with respect to \(H\), we get a scaling hypothesis for \(\lambda\)

\[
\chi(H, t) \approx a b \left| t \right|^{\lambda} f_+ \left( \frac{b H}{\left| t \right|^\Delta} \right).
\]

Using this, we can show that, e.g.,

\[
\gamma' = \gamma = \Delta - \lambda.
\]

Similarly, all other thermodynamic functions can be shown to obey the scaling hypothesis. So, all the thermodynamic exponents can be expressed in terms of just two exponents, \(\lambda\) and \(\Delta\). By eliminating these one can get relations between exponents. Obviously, the number of independent rela-
tions will be two less than the number of exponents. An example of these relations\textsuperscript{26} is
\[ \gamma = \beta (6 - 1). \]

Similarly, all the amplitudes can be expressed in terms of just two amplitudes and the values of \( f \) and their derivatives at special points.\textsuperscript{47,48} The thermodynamic scaling has been verified experimentally in both fluids and magnets. The detailed predictions like the exponent relations have also been verified both theoretically and experimentally.\textsuperscript{49,50}

In order to understand the universality hypothesis, we should first note that all the quantities that characterize the critical behaviour of the system can be divided into two categories: Universal and Non-universal. Quantities which depend on very few properties of the system are called universal quantities, e.g., critical exponents and scaling functions. The remaining quantities which depend on details of the system are called non-universal quantities, e.g., critical temperature, critical amplitude and scale factors.

Most studies of universality\textsuperscript{41,50} have been devoted to the so-called "n-vector model" or the "general exchange Hamiltonian". The basic variables in the model are n-component classical unit vectors sitting on a d-dimensional lattice, interacting through a rotationally symmetric "exchange". The Hamiltonian is written as
\[
\mathcal{H} = - J \sum_{i,j} \vec{\beta}_i \cdot \vec{\beta}_j , \quad J > 0 , \quad (1.5)
\]
where \( \vec{\beta}_l = \{ \beta_l^\alpha, \alpha = 1, 2, 3, \ldots n \} \) satisfy the constraint
\[
\sum_{\alpha=1}^{n} (\beta_l^\alpha)^2 = 1 .
\]
The Hamiltonian has been studied in all its generalizations.\textsuperscript{41,50}
Fig. 3

- $n=\infty$ (Spherical)
- $n=8$
- $n=4$ (Structural)
- $n=3$ (Ferromagnets)
- $n=2$ (Helium)
- $n=1$ (Ising)

- $d=1$
- $d=2$
- $d=3$
- $d=4$
- $d>4$

- Heisenberg
- Onsager
- Helium
Some examples are, differing exchanges in different lattice directions, in different spin directions; interactions becoming long ranged, etc.

Before discussing the results it is useful to point out that the Hamiltonian (1.5) provides a model for many interesting physical systems, as indicated in Figure 3. The values \( n = 1, 2, 3 \) are realized in magnetic materials. Superfluids are described by \( n = 2 \) and ordinary fluids, binary mixtures and alloys by \( n = 1 \). Recently, it has been shown that certain structural transitions can be described by \( n = 4, 6, 8 \). The limit \( n \to \infty \) corresponds to the spherical model. \(^{51, 52-54}\)

The universal quantities depend on the following:

(i) Dimensionality \( d \) of the system, which is defined as the number of physical dimensions in which the system has infinite extent.

(ii) Symmetry number \( n \), which is the number of independent components of the order parameter needed to describe the ordered state.

(iii) Range of interactions, short-range interactions are those satisfying \( \sum_{|\mathbf{R}_i|} R_i^2 |J(\mathbf{R}_i)| < \infty \). All others are classified as long-range. When interactions are long-range, they are usually taken to be \( J(R) \sim R^{-(d+\sigma)} \) for \( R \to \infty \). For \( 0 < \sigma < 2 \), the interactions are long-range and the exponents are \( \sigma \)-dependent in general. \(^{41}\)

The universal quantities do not depend on the details of lattice structure, the length of the system in the direction in which it is finite, and the strength of the interactions in different directions. They are also independent of the magnitude of the spin quantum number \( S = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \infty \), number of spin components, amount of anisotropy in the spin space as long as it does not change the symmetry properties of the Hamiltonian.
Lastly, they do not depend on the strength of the next nearest (or more) neighbours as compared with that of nearest neighbour interactions.

The basic reason for universality is the existence of long-range correlations near the critical point. Due to their range, they do not "see" the details of the system, but only the gross features like the dimensionality, symmetry properties, etc.

Summing up, to each set of \((d,n,\sigma)\) corresponds a unique set of two independent thermodynamic exponents and a unique scaling function. Systems having the same \((d,n,\sigma)\) are said to belong to the same universality class. A natural question is as to how one extrapolates between different universality classes. This leads naturally to crossover phenomena which is the subject of the next section.

### 1.3. Brief History of Crossover Scaling and Universality

Crossover occurs when the exponents of the system change discontinuously at a special point when a parameter, say \(g\), is changed continuously. Let us call the special value of \(g\) zero. Then, for \(g = 0\), we have one set of exponents \(\alpha\), \(\beta\), \(\gamma\), etc. And, by the universality hypothesis, for \(g \neq 0\), we have another set of exponents, say, \(\alpha\), \(\beta\), \(\gamma\), etc. For convenience, let us call the \(g = 0\) and \(g \neq 0\) systems as the isotropic and anisotropic systems, respectively.

According to the crossover scaling theory,\(^{55,56}\) which is developed systematically in Section 2.1, one expects that for weak anisotropy, the system behaves first as if fully iso-
tropic when the critical point is approached. However, on going closer to the critical temperature $T_c(g)$, the system starts to respond to the anisotropy until, eventually, its behaviour becomes fully characteristic of $g \neq 0$ systems. The change to anisotropic form occurs in the vicinity of a crossover temperature $T^x = T_c(g) + \Delta T^x$ whose variation is determined by a crossover exponent $\phi$ according to $\Delta T^x \sim g^{1/\phi}$. The value of $\phi$ is characteristic of the isotropic system and the type of crossover involved but, as usual, should not depend on the details of lattice structure, etc.

To describe this behaviour, a scaling hypothesis for the case of spin-space anisotropy (e.g., Heisenberg to Ising crossover) in terms of the variables $g/t^\phi$, where

$$ t = \left[\frac{T - T_c(g)}{T_c(0)}\right] / T_c(0) $$

was made by Riedel and Wegner. They verified the predictions of the theory in the mean-field approximation and in the spherical model. This, however, left open the value of the shift exponent $\Psi$, defined as

$$ T_c(g) - T_c(0) \approx g^{1/\Psi}, \quad g \rightarrow 0. $$

This question was discussed in detail by Fisher and Jasnow (unpublished). They showed that, in general, $\Psi$ may or may not be equal to $\phi$. If $\Psi = \phi$, then one can make an extended scaling hypothesis in terms of the variables $g/t^\phi$ where

$$ t = \left[\frac{T - T_c(0)}{T_c(0)}\right] / T_c(0). $$

Conversely, if the extended form is assumed, then the conclusion $\Psi = \phi$ is forced. If $\Psi \neq \phi$, then the extended scaling can still be made but in terms of more complicated variables. (For many examples of this last case, see Singh (unpublished)).
To see the kind of crossovers that are possible in physical systems, let us turn to Figure 3. Let us denote each point by its co-ordinates \((d,n)\). (We restrict ourselves to the case of short-range forces, for simplicity). In principle, each point is characterized by a different set of exponents, so we can have crossover from any one point to another. Most thoroughly studied, theoretically as well as experimentally, are the cases \((3,3)\) to \((3,2)\) and \((3,1)\). All the predictions were verified and scaling functions were obtained theoretically by Pfeuty, Jasnow, Fisher and Singh.\(^{58-60}\) These results had a direct application in Heisenberg systems exhibiting a bicritical point.\(^{61}\) Such points were studied experimentally by Rohrer\(^{62}\) who found good agreement between theory and experiment. Various crossovers in the spherical model were studied by Riedel and Wegner\(^{55}\) and Singh and Jasnow\(^{63}\) (unpublished). The only other case which has been studied in some detail is the \((2,1)\) to \((3,1)\) case by Stanley and coworkers.\(^{50}\) We discuss their work in Section 2.2 in detail. Many other interesting crossovers have been studied mainly by the renormalization group techniques. (For a recent review see the article by Aharony\(^{64}\)). In the coming section we introduce the present model.

1.4. Present Model

Onsager's paper\(^{7}\) on the two-dimensional Ising model also contained a discussion of the anisotropic rectangular lattices with different interactions \(J\) and \(J'\) in the two principal lattice directions. This work has been extended to the spontaneous magnetization in the anisotropic lattice (see Domb\(^{14}\)). For these exact solutions, critical exponents remain unchanged as long as \(J' \neq 0\).
The smoothness postulate\(^{65}\) generalizes this result to any isotropic system in \(d\) dimensions. For example, if we consider a set of parallel s.q. lattices with internal interactions \(J\) coupled to one another with interaction \(J'\), we should expect the system to display three-dimensional critical exponents as long as \(J' > 0\), and to revert to two-dimensional exponents only when \(J' = 0\).

In this work, we shall be dealing with quasi-two-dimensional systems, i.e., three-dimensional systems in which the interactions in the off-plane directions are weaker than those within the planes. The most important parameter determining the behaviour of such a system is the lattice-anisotropy, \(g\), i.e., the ratio of the interplanar to intraplanar couplings. For \(g\) large, say more than \(10^{-2}\) or so, the properties of the system are essentially three-dimensional in nature. So one can use the standard Padé approximant method to construct the thermodynamic functions for all temperatures.\(^{66}\) On the other hand, for smaller values of \(g\), the crossover behaviour makes itself felt, so more refined techniques must be used.

Our model is the ferromagnetic quasi-two-dimensional Ising model without magnetic field. The interactions which are purely in the \(xy\)-plane are denoted by \(J\) and all others by \(J'\). The anisotropy parameter is \(g = J/J = J_z/J_{xy}\). The two lattices studied are the s.c. and face-centered cubic (f.c.c.) lattices. We are considering a crossover from 2-d to 3-d. The Hamiltonian of our model is

\[
\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - g J \sum_{\langle ij \rangle} \delta_{ij}, s_i \neq 1, J > 0. \tag{1.8}
\]

The first summation runs over all the n.n. pairs of spins in the \(xy\)-plane, and the second summation is over all other n.n. pairs. For \(g = 0\), the Hamiltonian describes a set of mutually non-interacting two-dimensional Ising models. For \(g = 1\), it
describes either the s.c. or f.c.c. model.

In Chapters 3 to 5, we will study the crossover scaling behaviour of this model in detail. Such models have also been studied experimentally. (For a recent review, see de Jongh and Miedema\textsuperscript{67}). The only known example of the present model is perhaps FeCl\textsubscript{2}. But in this case, the value of $g$ is estimated to be $3 \times 10^{-2}$, so it is outside the crossover scaling region. Further experimental studies on such systems would be most welcome.

The coming section gives the summary of all the remaining chapters.

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1.5. Summary of Remaining Chapters

The outline of the remaining chapters is as follows. In Chapter 2, the crossover scaling theory is presented. Also given is a summary of previous work on the same model. In Chapter 3, the low-$g$ expansion of the scaling function is obtained through the analysis of the isotropic critical behaviour of the susceptibility. The study of the critical behaviour in the presence of small but finite anisotropy is the subject of Chapter 4. In Chapter 5, we construct the closed form approximants for the scaling function and we also examine the crossover of the susceptibility exponent based on this function. Chapter 6 which is the last chapter contains a summary of the work done and also our concluding remarks.