General Remarks and Conclusions

6.1. General Remarks: Other Lattices and Related Models

We have studied the lattice-anisotropy crossover for the s.q. to s.c. and s.q. to f.c.c. lattices only. However, following modern ideas about universality, we expect the scaling function $X(x)$ to be universal for all other crossovers from two to three dimensions. Only the values of the non-universal parameters will depend on the lattice type.

Since all the two-dimensional lattices with nearest neighbour interactions have been solved exactly, the values of $A$ and $K_0(0)$ are known. The value of $B$ follows from Liu-Stanley relations

$$
\left( \frac{\partial X}{\partial q} \right)_0 = \lambda K \left[ X(0) \right]^2,
$$

where $\lambda$ is the number of extra interactions in the off-plane directions. Then from the universal values of $X$ and $X$ one can get the values of $\omega$ and $A$. For illustration, we display the results for s.q. to b.c.c. and triangular to f.c.c. crossovers in Table 13. (For completeness, we also collect the results for
the s.q. to s.c. and s.q. to f.c.c. ). Other lattices can be dealt with similarly.

As examples of related models, consider our Hamiltonian (equation 1.8), in the s.c. case. We have considered the case (i) J > 0, g > 0. For the other choices of signs, we have the following three possibilities:

(ii) J > 0 , g < 0 ,
(iii) J < 0 , g > 0 ,
(iv) J < 0 , g < 0 .

All these cases can be studied by making appropriate scaling hypotheses in a manner similar to our case. The only difference is that the usual susceptibility will not be the ordering susceptibility in these cases. In case (ii), \( \chi \) will diverge strongly for \( g = 0 \) but only weakly for \( g \neq 0 \). In cases (iii) and (iv), even the \( g = 0 \) susceptibility will diverge only weakly. All these cases can be handled by using the techniques of Gerber and Fisher\(^9\) who discussed a similar problem for the case of spin-space anisotropy.

6.2. Comparison of our Results with others

Even though our results verify the scaling and universality theories and are internally consistent, it would be very useful to study the model by using other methods. We have already compared our results of \( \hat{\omega} \) and \( A \) with those of Harbus and Stanley\(^7\) in Section 4.3.

The problem of obtaining the scaling function can be attacked by renormalization group (RG) technique, newer extrapolation methods and experiments.
There has been considerable interest recently in the RG approach to the theory of critical phenomena. For our model, Grover and Chang and Stanley proved that \( \phi = \gamma \) using these methods. Bruce studied the problem of obtaining the scaling function. But so far, the scaling function has not been obtained by these methods. Therefore, it would be highly desirable if the scaling function can be obtained using the RG method, which will be an indirect check on our calculations.

Recently, a more powerful technique, the partial differential approximants technique for the series analysis has been developed. An application of this method to several test functions and dimensional crossover in the Ising model was made by Stilck and Salinas. Again, there have not been a calculation of the crossover scaling function by this method. It would be of great interest to apply this method to the present problem.

Regarding experimental studies, there are very few of them but in most cases, either one or both of the interactions J and J' are antiferromagnetic. The only known example of the present model is perhaps FeCl₂. In this case, the value of \( g \) is estimated to be \( 7.5 \times 10^{-2} \), but from our study of the effective exponent, we can easily see that this value of \( g \) is outside the crossover scaling region.

Thus, further work, both experimental and theoretical, would be most welcome.
6.3. Concluding Remarks

We have studied the crossover behaviour of the susceptibility of quasi-two-dimensional Ising models, occurring when small anisotropic exchange interactions are introduced into an otherwise isotropically coupled system. The two lattices studied are a.c. and f.c.c. lattices. It has been shown, with good numerical precision, that a scaling formulation describes the crossover of the susceptibility. We have verified the detailed predictions of the extended crossover scaling theory both for \( g = 0 \) and \( g \neq 0 \) and have obtained accurate estimates for the non-universal and universal parameters.

By studying derivatives of the susceptibility with respect to the anisotropy in the isotropic limit, we have obtained the expansion for the scaling function \( X(x) \), in powers of \( x \). Furthermore, the universality of the expansion with respect to lattice structure has been convincingly demonstrated for the two lattices studied.

In addition, we have done careful analyses for finite, non-vanishing values of anisotropy to reveal the universality of the scaling function \( X(x) \), in particular, to evaluate the singular behaviour as \( x \) approaches a characteristic critical value \( x \) related directly to the anisotropy-induced shift in critical temperature. In this way, we have been able to construct accurate approximants to a crossover scaling function valid in the whole critical region.

The crossover of the effective susceptibility exponent, \( \gamma_{\text{eff}} \), has been studied over a wide range of temperatures for a range of anisotropies. The results indicate that the full
crossover behaviour may be experimentally unobservable for physical anisotropies of a few percent.