CHAPTER III
III.1. Introduction

Including gravity-induced contributions, although the SU(5) model can be consistent with a very stable proton and the accepted values of $\sin^2 \theta_W \approx 0.22-0.24$, there exists a grand desert in between the electroweak unification scale $M_W \approx 10^2 \text{GeV}$ and the grand unification scale $M_U \geq 10^{15} \text{GeV}$. However, if some new physics are discovered between these energy gaps in foreseeable future, the SU(5) model can be ruled out. Also neutrino is massless in the SU(5) model discussed in Chapter II. Any experimental evidence for non vanishing neutrino mass would also question the validity of the model. A very attractive GUT, which can provide new physics populating the grand desert is the SO(10) model. Some of the attractive features of the SO(10) GUT compare to many other GUT's are described below. It is the minimal left-right-symmetric extension of SU(5), and contains all known fermions (plus the right-handed neutrino) of one generation in a single spinorial representation. It can explain the origins of parity-(P) and CP-violations which arise spontaneously as a result of symmetry breaking. It can provide massive neutrinos over a wide range of values in order to solve the solar neutrino puzzle via the so-called Mickhayev-Smirnov-Wolfenstein (MSW) mechanism or eV-keV-MeV energy spectrum of neutrinos, near their present experimental limits. Using the mechanism of decoupling P- and SU(2)
breakings\textsuperscript{21}, it is possible to have a natural solution to the domain-wall problem\textsuperscript{24}. The introduction of one or more intermediate symmetries in SO(10) promises experimental verification of interesting theoretical ideas such as the quark-lepton unification based upon SU(4)\textsubscript{C} and left-right symmetry\textsuperscript{22}, besides explaining the observed proton stability.

After the discovery of SO(10) model\textsuperscript{4}, several attempts have been made to obtain a low SU(4)\textsubscript{C}-breaking scale (M\textsubscript{C}) in the presence of the gauge groups SU(2)\textsubscript{L} \times SU(2)\textsubscript{R} \times SU(4)\textsubscript{C} (\equiv B\textsubscript{224}) or SU(2)\textsubscript{L} \times U(1) \times SU(4)\textsubscript{C} (\equiv B\textsubscript{214}). Such a low M\textsubscript{C} \approx 10^{-5}-10^{-6}\text{GeV}, is expected to provide the low-energy signature of quark-lepton unification through rare-kaon decays (K\textsubscript{L} \rightarrow \mu e) and small Majorana neutrino masses. In addition the existence of such scale for B\textsubscript{224}-breaking would predict the experimental observation of phenomenon such as n-\bar{n} oscillation\textsuperscript{38}. Due to the nonavailability of the free neutron sources, as the neutron oscillation experiments are very difficult, it might be useful to search a GUT where the consequences of SU(4)\textsubscript{C}-breaking can be testified by a relatively easier class of experiments, such as the rare-kaon decays\textsuperscript{39} only. Even with two intermediate symmetries\textsuperscript{65} in the SO(10) GUT,

\[
\begin{align*}
M\textsubscript{U} & \rightarrow B_{214} \rightarrow B_{2113} \rightarrow G_{st} \rightarrow B_{13},
\end{align*}
\]

where B\textsubscript{2113} = SU(2)\textsubscript{L} \times U(1) \times U(1) \times SU(3)\textsubscript{C}, it has not been possible to obtain M\textsubscript{C}=10^{-5}-10^{-6}\text{GeV}, for the presently accepted values\textsuperscript{2} of \sin^2\theta\textsubscript{W}=0.230\pm0.005. Although, two or more intermediate symmetries populating the grand desert provide possibilities of richer
physical structure, the prediction with single intermediate symmetry are very appealing because of the minimal nature of the GUT scenario.

In this Chapter\textsuperscript{40} we note that the conventional SO(10) GUT, with the single $G_{214}$ intermediate symmetry, is ruled out as it predicts a proton lifetime lower than the Irvine–Michigan–Brookhaven (IMB) limit\textsuperscript{19} for the $p \rightarrow e^+\pi^0$ mode. On the other hand for the first time we find that, when the effects of a five-dimensional operator\textsuperscript{27,28} is included in the GUT Lagrangian corresponding to the single intermediate symmetry, in the chain

\[ \text{SO}(10) \rightarrow G_{214} \rightarrow \mathbf{126} \rightarrow \mathbf{10} \]

it is possible to have $M_C \sim 10^{5}-10^{11}\text{GeV}$, $M_U \sim 10^{15}-10^{17}\text{GeV}$, for $\sin^2 \theta_W \approx 0.22-0.24$. For $M_C \sim 10^{5}-10^{6}\text{GeV}$, corresponding to observable rare-kaon decays\textsuperscript{39}, some of the Majorana neutrino masses could be measured in the laboratory. In this chain the proton lifetime is found to be significantly larger than the IMB limit\textsuperscript{19} ($\tau_p(p \rightarrow e^+\pi^0) \geq 3 \times 10^{32}\text{yr}$) depending upon the values of $\sin^2 \theta_W$ and $M_C$. For still larger values of $M_C \sim 10^{7}-10^{11}\text{GeV}$, corresponding to undetectable rare-kaon decays, Majorana neutrino masses decrease further and the proton lifetime also decreases, saturating the IMB limit for $M_C \sim 10^{11}\text{GeV}$.

This Chapter is organized in the following manner. In Sec.III.2 we derive modified GUT-boundary conditions, formulas for unification mass, electroweak mixing angle, and the GUT coupling constant including the effects of the five-dimensional operator.
In Sec. III.3 we discuss the solutions with single intermediate symmetry without gravity-induced effect. In Sec. III.4 we report our new results with single intermediate symmetry by including the effects of the five-dimensional operator. Majorana neutrino masses of various ranges have been predicted in Sec. III.5. A brief summary and conclusion of the Chapter are stated in Sec. III.6.

III.2. Derivation of the formulas for unification mass, electroweak mixing angle, and GUT coupling

In this section we derive the formulas for the unification mass \( M_U \), electroweak mixing angle \( \sin^2 \theta_W \), and grand unification coupling \( \alpha_G \). For chain (3.2), it is usually stated that the vacuum expectation value (VEV) of the Higgs field \( \chi(1,0,1) \subset 45 \subset SO(10) \), where the transformation property of \( \chi \) is under \( G_{214} \), might achieve the spontaneous symmetry breaking (SSB) at the first stage. But, according to the observations made by Yasue several years ago, both \( 54 \) and \( 45 \) are needed to break \( SO(10) \rightarrow G_{214} \). In this case, as \( 45 \) is antisymmetric, it does not contribute to the gravity-induced corrections through the five-dimensional operator; but the necessary presence of \( 54 \) is sufficient to induce significant modifications to the GUT predictions through the five-dimensional operators. Following the similar techniques of Refs. 27 and 28, the nonrenormalizable five-dimensional operator can be written as

\[
\eta
\alpha_{NR} = - \frac{1}{2M_G} \text{Tr} [F_{\mu\nu} (54) F^{\mu\nu}],
\]  

(3.3)
where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu], \]

\[ (A_i^a)_{\mu}^b = A^i_\mu (\lambda_i^a)^b. \]

\[ \text{Tr}(\lambda_i^a \lambda_j^b) = (1/2) \delta_{ij}, \] \quad (3.4)

In Eqs. (3.3)-(3.4) \( A_\mu \)'s are the gauge field matrices, \( \lambda_i \)'s are the \text{SO}(10) generators, \( \eta \) is an unknown parameter, \( M_B \) is the compactification scale, and \( \phi \) is the scalar field \( 54 \subset \text{SO}(10) \).

When \( \phi \) acquires a nonzero VEV
\[ \langle \phi \rangle = \frac{1}{\sqrt{2}} \text{diag}(1,1,1,1,1,-3/2,-3/2,-3/2,-3/2), \] \quad (3.5)

the presence of the nonrenormalizable term (3.3) modifies the usual gauge kinetic energy terms,
\[ \alpha_R = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F^{\mu\nu}), \] \quad (3.6)

of the \text{SU}(2)_L, \text{U}(1)_R, and \text{SU}(4)_C gauge fields which can be written as
\[ \alpha = -\frac{1}{2} \left[ 1 - \frac{3}{2} \epsilon \right] \text{Tr}(F^{(2L)} F^{(2L)}\mu\nu) - \frac{1}{4} \left[ 1 - \frac{3}{2} \epsilon \right] F^{(1R)} F^{(1R)}\mu\nu \\
- \frac{1}{2} (1+\epsilon) \text{Tr}(F^{(4C)} F^{(4C)}\mu\nu), \] \quad (3.7)

where
\[ \epsilon = \frac{\eta \phi}{M_B^{1/2}}. \] \quad (3.8)

The superscripts (2L), (1R), and (4C) stand for the \text{SU}(2)_L, \text{U}(1)_R, and \text{SU}(4)_C, respectively. Now rescaling of the gauge fields changes their coupling constants as
\[ g_{2L}^2(M_U) \rightarrow g_{2L}^2(M_U) \left[ 1 - \frac{3}{2} \epsilon \right], \quad g_{1R}^2(M_U) \rightarrow g_{1R}^2(M_U) \left[ 1 - \frac{3}{2} \epsilon \right], \]
\[ g_{4C}^2(M_U) \rightarrow g_{4C}^2(M_U) (1+\epsilon), \] \quad (3.9)

where \( g_{2L}^2(M_U), g_{1R}^2(M_U), \) and \( g_{4C}^2(M_U) \) denote the coupling constants of \text{SU}(2)_L, \text{U}(1)_R, and \text{SU}(4)_C, respectively, without gravity-
induced corrections. As the three coupling constants are equal at
the scale $\mu \geq M_U$ for achieving unification, the GUT condition is
imposed through the equations

$$g_0^2(M_U) = 4\pi \left( 1 - \frac{\alpha_i}{2} \right)$$

where $g_0$ is the bare-GUT coupling constant. The one-loop RGE's for
the chain (3.2) are given by

$$\frac{1}{\alpha_i(M_W)} = \frac{1}{\alpha_i(\mu)} + \frac{a_i}{2\pi \ln \frac{\mu}{M_W}}, \quad i = Y, 2L, 3C, \quad (3.11)$$

$$\frac{1}{\alpha_j(M_U)} = \frac{1}{\alpha_j(\mu)} + \frac{a'_j}{2\pi \ln \frac{\mu}{M_U}}, \quad j = 2L, 1R, 4C, \quad (3.12)$$

where $a_i$ ($a'_j$) is the one-loop coefficient in the lower (higher)
scale. Confining to the minimal fine-tuning conditions the Higgs
scalars, needed for SSB in the two different mass ranges are $M_W \leq$
$\mu \leq M_C$, $\Phi(1,2,1)$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$, $M_C \leq \mu \leq M_U$, $\Phi(1,1/2,1)$ and $\Delta_R(1,1,10)$ under $SU(2)_L \times U(1)_R \times SU(4)_C$. With the
minimal number of Higgs scalars and the three fermion generations,
the values of the one-loop coefficients are $a_Y = 41/10$, $a_{2L} = 19/6$, $a_{3C} = -7$, $a'_{2L} = 19/6$, $a'_{1R} = 15/2$, and $a'_{4C} = -29/3$.

Using Eqs. (3.10)-(3.12) and the combinations $\alpha^{-1}(M_W) - (8/3)$
$\alpha^{-1}_{3C}(M_W)$, $\alpha^{-1}(M_W) - (8/3)$ $\alpha^{-1}_{2L}(M_W)$, $\alpha^{-1}(M_W) = (5/3)\alpha^{-1}_Y(M_W) + \alpha^{-1}_{2L}(M_W)$, we
obtain the following equations for the unification mass, $\sin^2 \theta_W$,
and the GUT coupling constant ($\alpha_B = g_0^2/4\pi$):

$$\ln \frac{M_W}{M_U} = \frac{6\pi}{71-74\epsilon} \left[ \frac{1}{\alpha} - \frac{8}{3\alpha_S} + \left\{ \frac{7}{3\alpha_S} + \frac{1}{\alpha} \right\} \epsilon \right] + \left( \frac{4-36\epsilon}{71-74\epsilon} \right) \ln \frac{M_C}{M_W},$$

(3.13)
$$\sin^2 \theta_W = \frac{1}{71-74\varepsilon} \left[ \left\{ \frac{39}{2} + (19-\frac{38}{3}) \alpha \varepsilon - 53\varepsilon \right\} - \frac{\alpha}{\pi} \left\{ \frac{245}{3} - 170\varepsilon \right\} \ln \frac{M_C}{M_W} \right] \right\},$$

Equation (3.14)

$$\frac{1}{\alpha_g} = \frac{1}{71-74\varepsilon} \left[ \frac{29}{\alpha} - \frac{19}{3\alpha_g} - \frac{226}{3\pi} \ln \frac{M_C}{M_W} \right],$$

Equation (3.15)

where $\alpha_s = \alpha_{3C}(M_W) = g^2_{3C}(M_W)/4\pi$ and $\alpha(M_W) = e^2(M_W)/4\pi$.

III.3. Solutions with $G_{214}$ intermediate symmetry without five-dimensional operator

From Eq. (3.10), it is clear that $\varepsilon = 0$ corresponds to the absence of gravity-induced effects. Substituting $\varepsilon = 0$ in Eqs. (3.13)-(3.15), we obtain

$$\ln \frac{M_U}{M_W} = \frac{6\pi}{71} \left\{ \frac{1}{\alpha} - \frac{8}{3\alpha_g} \right\} + \frac{4}{71} \ln \frac{M_C}{M_W},$$

Equation (3.16)

$$\sin^2 \theta_W = \frac{1}{71} \left[ \left\{ \frac{39}{2} + 19 - \frac{\alpha}{\alpha_g} \right\} - \frac{245\alpha}{3\pi} \ln \frac{M_C}{M_W} \right],$$

Equation (3.17)

$$\frac{1}{\alpha_g} = \frac{1}{71} \left[ \frac{29}{\alpha} - \frac{19}{3\alpha_g} - \frac{226}{3\pi} \ln \frac{M_C}{M_W} \right].$$

Equation (3.18)

Now using $\alpha_s = 0.1088$ ($\Lambda_{\text{MS}} = 160$ MeV), $\alpha^{-1}(M_W) = 127.54$, we compute the numerical solutions for $M_C$, $M_U$, and $\sin^2 \theta_W$. Some of our solutions are presented in Table 5. It is clear from Table 5 that with a purely renormalizable Lagrangian (3.6), the chain (3.2) yields a maximum $M_U \propto 3 \times 10^{14}$ GeV for the allowed value of $\sin^2 \theta_W = 0.22-0.24$. For the maximum $M_U$, the corresponding proton lifetime $\tau_p \propto 10^{29.7} \text{ yr}$, which is significantly less than the IMB limit. Here the uncertainty in $\tau_p$ arises due to the uncertainties in the proton decay matrix element and the QCD parameter. Thus, purely renormalizable SO(10) model with single
Table 5. One-loop solutions for SO(10) with single intermediate symmetry, SU(2) \_L \times U(1) \_R \times SU(4) \_C, in the absence of gravity-induced corrections.

<table>
<thead>
<tr>
<th>$M_C$ (GeV)</th>
<th>$M_U$ (GeV)</th>
<th>$\sin^2 \theta_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>$9.2 \times 10^{13}$</td>
<td>0.273</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$1.2 \times 10^{14}$</td>
<td>0.260</td>
</tr>
<tr>
<td>$10^9$</td>
<td>$1.5 \times 10^{14}$</td>
<td>0.247</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>$2.0 \times 10^{14}$</td>
<td>0.233</td>
</tr>
<tr>
<td>$10^{13}$</td>
<td>$2.57 \times 10^{14}$</td>
<td>0.220</td>
</tr>
<tr>
<td>$10^{14}$</td>
<td>$2.95 \times 10^{14}$</td>
<td>0.214</td>
</tr>
</tbody>
</table>
G_{214} intermediate symmetry is ruled out.

III.4. Solutions with $G_{214}$ intermediate symmetry with five-dimensional operator

To see whether the effects of the five-dimensional operator improves the predictions, we compute the solutions using Eqs. (3.13)-(3.15) with the same input parameters as in Sec.II.3. In this case, interesting solutions are obtained for allowed regions for $M_C < 10^{13}$ and $\epsilon$ within the available experimental constraint (Refs. 2 and 19) on $M_U$ and $\sin^2 \theta_W$. Some of our allowed solutions obtained with $\epsilon > 0$ are presented in Tables 6 and 7 and Figs. 1-3. At first, Fig.1 is plotted using Eq. (3.13), and Fig.2 using Eq. (3.14). In Fig.1 the horizontal lines are the IMB and the Planck limits on the unification mass. The projection of the line PQ onto Fig.2 has been denoted as the IMB limit in the latter. The horizontal lines in Fig.2 represent the 2\sigma limits of the world average, $\sin^2 \theta_W = 0.230 \pm 0.005$. The projection of the Planck limit from Fig.1 onto Fig.2 does not provide any useful boundary for the allowed region. But, a much better limit exists from the experimentally observed bounds on the rare-kaon decay mode, $K_L \rightarrow \bar{\mu}e$, corresponding to $M_C > 3 \times 10^5$ GeV. Specifying the four sides of the quadrilateral in Fig.2 in this fashion, the allowed solutions are shown by the shaded area.

The numerical value of $M_C$, $\epsilon$, $M_U$, $\sin^2 \theta_W$, and $\alpha_B^{-1}$ are shown in Table 6 for $M_C = 10^5-10^6$GeV and, in Table 7 for $M_C = 10^7-10^{11}$GeV. For chain (3.2), we find that the modification due to five-dimensional operator permit $10^5 \leq M_C \leq 10^{11}$GeV for allowed values of $M_U$ and $\sin^2 \theta_W$. For every $M_C$, the parameter $\epsilon$ and the
Table 6. Predictions for $M_U$, $\sin^2 \theta_W$, and values of $M_C$, corresponding to observable rare-kaon decay, as a function of $\epsilon$, in the presence of gravity-induced corrections, for the same chain as Table 5. The proton lifetime is for the $P \rightarrow e^+\pi^0$ mode excluding uncertainties.

<table>
<thead>
<tr>
<th>$M_C$ (GeV)</th>
<th>$\epsilon$</th>
<th>$M_U$ (GeV)</th>
<th>$\sin^2 \theta_W$</th>
<th>$a_B^{-1}$</th>
<th>$\tau_P$ (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-5}$</td>
<td>0.10</td>
<td>$1.3 \times 10^{17}$</td>
<td>0.225</td>
<td>54.56</td>
<td>$4.8 \times 10^{40}$</td>
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<tr>
<td></td>
<td>0.09</td>
<td>$5.9 \times 10^{16}$</td>
<td>0.230</td>
<td>53.93</td>
<td>$2.0 \times 10^{39}$</td>
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<tr>
<td></td>
<td>0.08</td>
<td>$2.7 \times 10^{16}$</td>
<td>0.235</td>
<td>53.32</td>
<td>$8.6 \times 10^{37}$</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.09</td>
<td>$6.0 \times 10^{16}$</td>
<td>0.224</td>
<td>53.08</td>
<td>$2.1 \times 10^{39}$</td>
</tr>
<tr>
<td></td>
<td>0.08</td>
<td>$2.8 \times 10^{16}$</td>
<td>0.229</td>
<td>52.47</td>
<td>$9.6 \times 10^{37}$</td>
</tr>
<tr>
<td></td>
<td>0.07</td>
<td>$1.3 \times 10^{16}$</td>
<td>0.234</td>
<td>51.88</td>
<td>$4.4 \times 10^{36}$</td>
</tr>
<tr>
<td></td>
<td>0.06</td>
<td>$6.3 \times 10^{15}$</td>
<td>0.239</td>
<td>51.30</td>
<td>$2.4 \times 10^{35}$</td>
</tr>
</tbody>
</table>
Table 7. Same as Table 6, but for larger values of $M_C$.

<table>
<thead>
<tr>
<th>$M_C$ (GeV)</th>
<th>$\varepsilon$</th>
<th>$M_U$ (GeV)</th>
<th>$\sin^2 \theta_W$</th>
<th>$a^{-1}_B$</th>
<th>$\tau_P$ (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^7$</td>
<td>0.07</td>
<td>1.3x10^{16}</td>
<td>0.228</td>
<td>51.04</td>
<td>4.2x10^{36}</td>
</tr>
<tr>
<td></td>
<td>0.06</td>
<td>6.7x10^{15}</td>
<td>0.233</td>
<td>50.48</td>
<td>2.9x10^{35}</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>3.3x10^{15}</td>
<td>0.238</td>
<td>49.92</td>
<td>1.7x10^{34}</td>
</tr>
<tr>
<td>$10^8$</td>
<td>0.06</td>
<td>7.2x10^{15}</td>
<td>0.227</td>
<td>49.65</td>
<td>3.8x10^{35}</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>3.6x10^{15}</td>
<td>0.232</td>
<td>49.10</td>
<td>2.3x10^{34}</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>1.8x10^{15}</td>
<td>0.236</td>
<td>48.57</td>
<td>1.4x10^{33}</td>
</tr>
<tr>
<td>$10^9$</td>
<td>0.05</td>
<td>3.8x10^{15}</td>
<td>0.225</td>
<td>48.28</td>
<td>2.8x10^{34}</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>1.9x10^{15}</td>
<td>0.230</td>
<td>47.75</td>
<td>1.7x10^{33}</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>0.04</td>
<td>2.1x10^{15}</td>
<td>0.223</td>
<td>46.94</td>
<td>2.4x10^{33}</td>
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<tr>
<td>$10^{11}$</td>
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<td>1.2x10^{15}</td>
<td>0.221</td>
<td>45.63</td>
<td>2.0x10^{32}</td>
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Fig. 1. Solutions of one-loop renormalization-group equations for $M_U$ as a function of $\epsilon$, and for $M_C=10^n_c$, $n_c=5-12$. The horizontal lines are the IMB (lower) and the Planck (upper) limits. The allowed upper limit, for $n_c=5$ shown as point R is obtained as the projection of the corresponding point in Fig. 2.
Fig. 2. Solutions of one-loop renormalization-group equations for $\sin^2\theta_W$ as a function of $M_C$ and $\epsilon$. 
Fig. 3. Predictions on proton lifetime for the $p \rightarrow e^+\pi^0$ mode with $\Lambda_{\text{MS}} = 160$ MeV, as a function of $M_c$ and $\sin^2\theta_W$. The solid lines are for values of $\sin^2\theta_W$ corresponding to 1σ and 2σ limits. The dot-dashed line is for $\sin^2\theta_W = 0.230$. The IMB limit is shown by the dashed line.
unification mass $M_U$ are allowed over a wider range depending upon the $2\sigma$ or $1\sigma$ limit of $\sin^2\theta_W$. The solutions with smaller (larger) values of $\sin^2\theta_W$ are associated with larger (smaller) values of $M_U$ and $\tau_p$. In Table 6, we have presented the solutions for the values of $M_C \sim 10^5-10^6 \text{GeV}$ which predict rare-kaon decays to be observable for any value of $\sin^2\theta_W$ in the range of $0.22-0.24$. The highest value of $M_U \sim 3 \times 10^{17} \text{GeV}$ is possible for $M_C = 10^5 \text{GeV}$ and $\sin^2\theta_W = 0.22$. This has been shown by the point R in Fig.1 which has been obtained by the projection of the corresponding point in Fig.2.

It is clear from Table 6 and 7 that for the increase in the value of $M_C$, the value of $M_U$ for a fixed value of $\sin^2\theta_W$, and the proton lifetime for the $p \rightarrow e^+\pi^0$ mode decreases. This has been shown in Fig.3 for the $1\sigma$ and $2\sigma$ boundaries and the central value of $\sin^2\theta_W = 0.230$. For $M_C > 10^8 \text{GeV}$, the allowed range of $\tau_p$ also decreases being restricted by the IMB limit\textsuperscript{19} from below. The IMB limit\textsuperscript{19} is found to be saturated nearly at $M_C \sim 10^{10} (10^{11}) \text{GeV}$ if the values of $\sin^2\theta_W$ is allowed to be $0.225 (0.220)$.

The order of magnitude of the compactification scale $M_G$ that makes these gravity-induced corrections important can be calculated by using $\eta = (40\pi\alpha_B)^{1/2} \epsilon M_G/M_U$. Our estimation depends, crucially, on the assumption that $|\eta| \sim 1$ as in the Shafi and Wetterich\textsuperscript{27} case. Solutions having $\epsilon \approx 0.03-0.05$ are found to be associated with lower values of the unification mass, $M_U \sim 10^{15} \text{GeV}$ which require $M_G$ to be nearly 2 orders of magnitude smaller than $M_p$. The other class of solutions found in this model are associated with $\epsilon \approx 0.07-0.10$, and $M_U \sim 10^{16}-10^{17} \text{GeV}$ which require $M_G \sim 10^{17}-10^{18} \text{GeV}$. In particular the observable predictions for
rare-kaon decay corresponding to $M_C \approx 10^5 - 10^6$ GeV are found to be possible with $\sin^2 \theta_W \approx 0.22 - 0.23$, $\varepsilon \approx 0.1$, and $M_U \sim 10^{17}$ GeV, requiring $M_B \sim 10^{18}$ GeV. This scale is generally expected from the Kaluza-Klein type compactification, where $M_B = M_{pl}/2\pi \approx 1.6 \times 10^{18}$ GeV. If, on the other hand $\eta$ is allowed to be $\sim 0.1(10)$, our estimation would require $M_B$ 1 order less (more) for every value of $\varepsilon$. For example, with $M_C \sim 10^5$ GeV, and $M_U \sim 10^{17}$ GeV, consistency of the solutions with $\varepsilon \approx 0.1$ requires $M_B \sim 10^{17}(10^{19})$ GeV, if $|\eta| \sim 0.1(10)$, instead of $|\eta| \sim 1$.

III.5. Predictions on Majorana neutrino masses

At the second stage of the chain (3.2), the scalar representation $126 \circ SO(10)$ is used to break the intermediate gauge symmetry spontaneously to the standard group. The scalar representation $126$ contains right-handed SU(2)$_R$ triplet $\Delta_R(1,3,10)$ under SU(2)$_L \times$ SU(2)$_R \times$ SU(4)$_C$, which carries 2 units of lepton number. Therefore, when VEV is given to $\Delta_R$, lepton number is broken by 2 units, as a result of which Majorana neutrino masses are generated at this stage fulfilling the formula\(^{16,17}\)

$$m_{\nu_i} \approx \frac{D^2}{M_C}, \quad i=1, 2, 3,$$

(3.19)

where $m_{\nu_i}$ is the neutrino mass of the $i$th generation. Bell-Mann, Ramond, and Slansky\(^ {16}\) have identified the Dirac mass $m_i^D$ with the up quark mass ($m_1^D = m_u$, $m_2^D = m_c$, $m_3^D = m_t$), where as Mohapatra and Senjanovic\(^ {17}\) and others have taken the Dirac mass as the charged lepton mass ($m_1^D = m_e$, $m_2^D = m_\mu$, $m_3^D = m_\tau$) of the corresponding generation. Using quark masses, $m_u = 5$ MeV, $m_c = 1.25$ GeV, and $m_t \approx 100$ GeV in formula (3.19), the Majorana neutrino masses for different generations corresponding to the allowed range of $M_C \approx 10^5 - 10^{11}$ GeV
are given by
\[ m_{\nu_e} \sim (2.5 \times 10^{-7} - 0.25) \text{eV}, \quad m_{\nu_\mu} \sim (1.5 \times 10^{-2} \text{eV} - 15.6 \text{keV}), \quad \text{and} \]
\[ m_{\nu_\tau} \sim (100 \text{eV} - 100 \text{MeV}), \quad (3.20) \]
where the lower (upper) limit corresponds to \( M = 10^5 (10^7) \text{GeV} \).
But the same formula with the charged lepton masses, \( m_e = 0.51 \text{ MeV}, \)
\( m_\mu = 106 \text{MeV}, \quad m_\tau = 1.78 \text{ GeV} \), and the same values of \( M \approx 10^5 - 10^{11} \text{GeV} \)
provides
\[ m_{\nu_e} \sim (2.6 \times 10^{-9} - 2.6 \times 10^{-3}) \text{eV}, \quad m_{\nu_\mu} \sim (1.1 \times 10^{-4} - 112) \text{eV}, \quad \text{and} \]
\[ m_{\nu_\tau} \sim (3.2 \times 10^{-2} \text{eV} - 31.7 \text{keV}). \quad (3.21) \]
Out of these, the masses for \( \nu_\mu \) and \( \nu_\tau \) obtained in Eqs. (3.20) and
(3.21) are measurable by laboratory experiments. However, the mass
of \( \nu_\mu \) and \( \nu_\tau \) corresponding to \( M \approx 10^5 \text{GeV} \) clearly violate the
cosmological bound\(^{42}\) which states that \( \sum_{i=e,\mu,\tau} m_\nu \leq 65 \text{eV} \), where the
sum is over the stable and light neutrino species. One possible
way to evade the cosmological bound is to make these \( \nu_\mu \) and \( \nu_\tau \)
unstable with respect to Majoron emission\(^{43}\) which is a massless scalar carrying 2 units of lepton number, and is created by
introducing an additional \( U(1)_L \)-global symmetry (\( l=\text{lepton number}) \), and breaking it spontaneously at a scale \( M \gg M_W \).

III.6. Summary and Conclusion

In the absence of gravity-induced corrections, the \( SO(10) \) BUT
with single \( G_2 \) intermediate symmetry is ruled out as it predicts
proton lifetime significantly below the IMB limit\(^{19}\). However, we
found that such a model can provide a stable proton \( (\tau_P \approx 3 \times 10^{32} \text{yr} \)
for \( p \rightarrow e^+ \pi^0 \) mode) with the allowed values of \( \sin^2 \theta_W \approx 0.22 - 0.24 \),
when five-dimensional operator, induced by gravity, scaled by the
compaction mass, is included. Such higher-dimensional nonrenormalizable operators are usually present in theories with Kaluza-Klein type unification with gravity. With five-dimensional operator, the SU(4) -breaking is found to be permitted over a wide range $M_C \sim 10^5-10^{11}$ GeV. Such allowed value of $M_C \sim 10^5-10^{11}$ GeV provide two different values of Majorana neutrino masses (i) $m_{\nu_e} \sim (2.5\times10^{-7}-0.25)$ eV, $m_{\nu_\mu} \sim (1.5\times10^{-2}$ eV-15.6 keV), $m_{\nu_\tau} \sim (100$ eV-100 MeV) and (ii) $m_{\nu_e} \sim (2.6\times10^{-9}-2.6\times10^{-3})$ eV, $m_{\nu_\mu} \sim (1.1\times10^{-4}-112)$ eV, $m_{\nu_\tau} \sim (3.2\times10^{-2}$ eV-31.7 keV), depending upon whether the Dirac masses are taken as quarks or charged lepton masses respectively. Out of them, although $\nu_e$ mass is too small compared to $\nu_\mu$ and $\nu_\tau$ masses, these masses could be measured by laboratory experiments depending upon $M_C$ and the choice of the Dirac mass.

For the first time, with single $G_{214}$ intermediate symmetry, we have obtained interesting SU(4) predictions, for the observable SU(4) -breaking by rare-kaon decay modes at low energies with $M_C \sim 10^5-10^6$ GeV, and any value of $\sin^2 \theta_W$ in the range 0.22-0.24. For such lower values of $M_C$, the proton lifetime is larger depending upon the value of $\sin^2 \theta_W$. For larger values of $M_C > 10^8$ GeV, the allowed range of $\tau_P$ decrease with the increasing $M_C$. For a fixed $\sin^2 \theta_W$, $\tau_P$ decreases with $M_C$ and the IMB limit$^{19}$ is saturated when $M_C \sim 10^{11}$ GeV. The order of magnitude of the compactification scale, estimated in this model, is found to be in the range $10^{17}-10^{18}$ GeV, unless the parameter in the nonrenormalizable term has the value $|\eta| \sim 10$, or larger.

In this model, as the mass of the right-handed neutral gauge
boson $M_{Z_R} \geq 10^5$ GeV, and that of the charged gauge bosons $M_{W_R} = M_{U_R} \sim 10^{15} - 10^{17}$ GeV, there is negligible contribution to the V+A structure of charged and neutral currents. Similarly the $K_L - K_S$ mass difference and other CP-violating parameters have, essentially, the same prediction as the standard model. At low energies, this model does not seem to predict any other detectable signatures, except rare-kaon decays and neutrino masses.