CHAPTER 3

A PROPERTY OF RINGS OF FUNCTIONS

Chapter 3, section 0.

Introduction.

Consider the following question which was asked in [CDR:91]. (The relevant concepts have been defined in §3.1 below.)

Question A. If in a commutative anti-regular ring, every non-zero-divisor is a unit, is it necessarily regular? Equivalently, is the total quotient ring of a commutative anti-regular ring necessarily regular?

An example in the class of rings of continuous functions answers question A in the negative.

Basic results concerning anti-regular monoids and rings, as well as basic facts about rings of continuous functions are collected in §1.

The main result of next section characterises anti-regularity of the ring $C(X)$ of real-valued continuous functions on a topological space $X$ through a topological condition on $X$.

The example which answers Question A in the negative is furnished in the final section.
Chapter 3, section 1.

Preliminaries

As mentioned in the introduction, our interest in this chapter is in total quotient rings of anti-regular rings of continuous real-valued functions defined on topological spaces. However for the sake of completeness (and in the spirit of the approach adopted in Chapter 1) we shall record a few definitions and results in the most general setting possible.

3.1.1. Anti-regularity in monoids and rings.

3.1.1A. Basic definitions. Let $M$ be a monoid. Consider the following conditions for an element $a \in M$.

(i) $a \neq 0$.

(ii) There exists a non-zero element $b \in M$ such that $bab = b$. (i.e., in the terminology of 1.6.1. $b$ is a 2-inverse of $a$.)

Clearly (ii) $\Rightarrow$ (i) (always). If condition (ii) holds, then we call the element $a$ anti-regular. If (i) $\Rightarrow$ (ii) in $M$ then the monoid $M$ is called anti-regular. (Both these definitions extend to semigroups.)

Following the usual conventions, a ring is anti-regular if its multiplicative monoid is anti-regular.

3.1.1B Proposition. Let $a$ be a non-zero regular element of a semigroup $S$. Then $a$ is anti-regular.

Proof. Let $aba = a$. Then $b_1 = bab$ satisfies

$$b_1ab_1 = (bab)a(bab) = bab = b_1$$

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Further, \( ab_1a = a(bab)a = a \neq 0 \) shows that \( b_1 \neq 0 \).

3.1.1C. Corollary. Regular semigroups (monoids, rings) are anti-regular.

3.1.1D. (cf. Proposition 0.4.9.) Proposition. Let \( a \) be an element of a monoid \( M \). Then the following conditions are equivalent.

(i) The element \( a \) is anti-regular.

(ii) The (principal left) ideal \( Ma \) contains a non-zero idempotent.

Proof. (i) \( \Rightarrow \) (ii). Assume that \( b \in M \) satisfies \( bab = b \neq 0 \). Then \( ba \in I(M) \), \( ba \neq 0 \) and \( ba \in Ma \).

(ii) \( \Rightarrow \) (i). Let \( e \) be a non-zero idempotent element in \( Ma \). Therefore \( e = ba \) for some \( b \in M \). Write \( c := bab \). Then we have (since \( ba = (ba)^2 = (ba)^3 \))

\[
ca = bab \neq 0 \Rightarrow c \neq 0
\]

Next

\[
cac = bababa = bab = c \neq 0
\]

3.1.1E. (cf. Corollary 0.4.10.) Corollary. Anti-regular monoids are semiprime.

Proof. Similar to that of Corollary 0.4.10.

3.1.1F. Corollary. Semi-commutative, anti-regular monoids are reduced.

Proof. Apply Proposition 1.1.3.

For basic results on non-singularity in monoids we refer to §0.3

3.1.1G. Corollary. Let \( a \) be an anti-regular element of a monoid \( M \). Then \( a \) cannot belong to either the left singular ideal or the right singular ideal of \( M \).
Proof. Let \( a \) be an anti-regular element of \( M \). If possible, let \( a \in Z_i(M) \).

By 3.1.1D, the left ideal \( Ma \) contains a non-zero idempotent \( e \). Since \( e \in Z_i(M) \) (ideal!) \( l(e) \triangleleft M \) (as a left ideal). Therefore \( l(e) \cap Me \neq 0 \). So let \( ze \in l(e), ze \neq 0 \). Then \( ze = zee = 0 \), a contradiction!

3.1.1H. Corollary. Anti-regular monoids are left and right non-singular.

3.1.1I. Corollary. Regular monoids are left and right non-singular.

3.1.1J. Remarks. A systematic study of anti-regularity in monoids is not attempted here. However it is worth recording that natural analogues of some basic results on anti-regular rings fail in monoids. Indeed we have:

(i) Non-zero anti-regular domains are division rings, and

(ii) Noetherian anti-regular rings are regular (equivalently, semisimple).

(The first result above is immediate from the definitions, while (ii) follows as a corollary of Theorem 2.8 of [CDR : 90].)

However, analogues of both these results fail in monoids as can be seen from the following example:

Consider the element

\[ a = (\bar{2}, \bar{1}) \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \]

We clearly have \( a^n = a^2 \) for each integer \( n \geq 2 \). It follows that the monoid

\[ M_4 = \{0, a, a^2, 1\} \]

is a finite, commutative, anti-regular monoid without divisors of zero. (Since \( a^2 = a^2.a.a^2 \neq 0 \) the element \( a \) is anti-regular.) The elements \( a \) and \( a^2 \) are
not invertible in $M_4$. Further, this monoid is not regular, since there is no element $b$ for which $aba = a$ holds.

3.1.1K Bibliographical notes. Rings $R$ in which for each $a \notin J(R)$ (the Jacobson radical of $R$) is anti-regular (called as $I_\sigma$-rings in [N:75]) and related classes of $I$-rings and Zorn rings have been studied by Nicholson, Jacobson, Levitzki, Kaplansky and others; see [N:75] for detailed references.

The non-singularity of anti-regular modules and rings has been proved in [CDR:90] (Corollary 3.3); for a related result see Proposition 1.27(A) of [G3]; for a proof of the non-singularity of (Zelmanowitz) regular modules see [Z:72]. The purpose of 3.1.1F and 3.1.1G is to point out that the additive structure of a ring $R$ is not needed and the ring results extend to monoids.

3.1.2 Anti-regularity under localisation.

3.1.2A. Proposition. Let $\phi : M \rightarrow M'$ be a one-one monoid homomorphism. If $a$ is anti-regular in $M$ then $\phi(a)$ is anti-regular in $M'$.

Proof. The equation $bab = b \neq 0$ yields (since $\phi$ is one-one)

$$\phi(b)\phi(a)\phi(b) = \phi(bab) = \phi(b) \neq 0$$

showing that $\phi(b)$ is a non-zero 2-inverse of $\phi(a)$ in $M'$.

3.1.2B. Proposition. Let $M$ be a monoid (ring) and $T$ a central multiplicatively closed subset of $M$ such that every element of $T$ is a non-zero-divisor in $M$. Let $a$ be an anti-regular element in $M$. Then for each $t \in T$, $a/t$ is anti-regular in $T^{-1}M$. 73
Proof. This can be proved in the same manner as 3.1.2A. (Note that (by 0.3.1) the homomorphism \( M \rightarrow T^{-1}M \) is one-one and \( 1/t \) is a central element in \( T^{-1}M \).)

3.1.2C. Corollary. Let \( M \) and \( T \) as in the Proposition. If \( M \) is anti-regular then so is \( T^{-1}M \).

3.1.2D. Example. An example of a commutative anti-regular ring \( R \) and a multiplicatively closed set \( T \) such that \( T^{-1}R = \mathbb{Z} \) has been given in [CDR:91]. Thus 3.1.2C does not extend to arbitrary multiplicatively closed sets.

Note that by 0.4.12R, the above ring cannot be regular. For other examples of anti-regular, non-regular rings see [CDR: 90].

3.1.3. Definition. Let \( R \) be a commutative ring, \( S_0 \) the set of all non-zero-divisors of \( R \). Clearly, \( S_0 \) is a multiplicatively closed subset of \( R \). By the total quotient ring \( (T,Q,R) \) of \( R \), we mean the localisation \( S_0^{-1}R \) of \( R \) with respect to the set \( S_0 \).

3.1.4. Remark. As a special case of 3.1.2D, the total quotient ring of a commutative anti-regular ring is anti-regular.

In paragraphs 3.1.5 to 3.1.7 we recall basic definitions and results in the theory of rings of continuous functions. For unexplained concepts and results we refer to [FGL] and [GJ]

3.1.5. Definition. Let \( X \) be a topological space. The set \( C(X) \) of all continuous, real-valued functions on \( X \) can be made into a ring called the
ring of continuous functions by providing an algebraic structure on the set. Addition and multiplication are defined by the formulae

\[(f + g)(x) = f(x) + g(x),\]

and

\[(fg)(x) = f(x)g(x).\]

The zero element is the constant function 0, and the unity element is the constant function 1. The additive inverse \(-f\) of \(f\) is characterised by the formula

\[(-f)(x) = -f(x).\]

The multiplicative inverse \(f^{-1}\) (which exists when the function \(f\) does not vanish anywhere) is characterised by the formula

\[f^{-1}(x) = 1/f(x)\]

3.1.6. **Definition.** Consider the subsets of \(X\) of the form

\[f^{-1}(r) = \{x \in X : f(x) = r\} \quad f \in C(X), r \in \mathbb{R}\]

The set \(f^{-1}(0)\) is called the **zero-set** of \(f\). We shall denote this set by \(Z(f)\) or for clarity by \(Z_X(f)\):

\[Z(f) = Z_X(f) = \{x \in X : f(x) = 0\} \quad (f \in C(X))\]
3.1.7. Definition. Let \( f \in C(X) \). Then by 3.1.3. \( Z(f) \) is the zero set of \( f \). The \textit{cozero set of} \( f \) (denoted by \textit{coz}(f)), is the complement of the zero set (i.e., of the form \( X - Z(f) \)).
Chapter 3, section 2.

A characterisation.

In this section we give a necessary and sufficient condition for the anti-regularity of the ring $C(X)$.

Proposition 3.2.1 is a part of Exercise 4J of [GJ]. They call a topological space satisfying the conditions of (3.2.1) (and several other equivalent conditions) a $P$-space.

3.2.1. Proposition. The ring $C(X)$ is regular if and only if every zero-set is open; equivalently every cozero set is closed (and therefore, open-and-closed).

In the following proposition we give an analogous necessary and sufficient condition for the anti-regularity of $C(X)$. (Since the characteristic functions of clopen sets are idempotents in $C(X)$, the motivation of this result can be found in Proposition 3.1.1D; indeed 3.1.1D can be used in its proof. However for the sake of clarity we give a direct argument.)

3.2.2. Proposition. Let $X$ be a topological space. The ring $C(X)$ is anti-regular if and only if every non-empty cozero set contains a non-empty open-and-closed (clopen) subset.

Proof. Let $C(X)$ be anti-regular and $W$ a non-empty cozero set, so that $W = \text{coz}(f)$ for some $f \in C(X)$. Since $f \not= 0$, there exists $g \in C(X), g \not= 0$ such that $gf = g$. Now $e = gf$ is an idempotent in $C(X)$ and hence is the characteristic function of $\text{coz}(e)$. Since $e$ is non-zero and continuous, $\text{coz}(e)$ is a non-empty clopen subset of $W$. 

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To prove the converse, let \( f \) be a non-zero element of \( C(X) \). By hypothesis \( \text{coz}(f) \) contains a non-empty subset \( B \). We define a map \( g : X \rightarrow \mathbb{R} \) as follows. If \( b \in B \), we have \( f(b) \neq 0 \) and we set \( g(b) = 1/f(b) \); and if \( b \notin B \), we set \( g(b) = 0 \). Since \( B \) is non-empty and clopen \( g \) is a non-zero continuous map from \( X \) to \( \mathbb{R} \). Clearly \( gfg = g \). This proves the anti-regularity of \( C(X) \).

3.2.3. Corollary. Let \( X \) be a topological space, whose topology has a base consisting of clopen sets. Then \( C(X) \) is an anti-regular ring.

Proof. \( \text{Coz}(f) \) is open for every \( f \in C(X) \).
Chapter 3, section 3.

An example.

The results of section 2 will be applied in two distinct situations in this section.

3.3.1. Remark. Let $X$ be a topological space satisfying the hypothesis of Corollary (3.2.3) which is not a P-space. Then $C(X)$ is an anti-regular ring which is not regular. An example is the space $Q$ of rationals, since the zero-set of the inclusion map $j : Q \rightarrow R$ is not open in $Q$, being a singleton set (see 3.2.1).

However, for any metric space $X$, by §§2.6 and 3.3 of [FGL], the total quotient ring of $C(X)$ is regular. Question A cannot therefore be settled by considering $C(X)$ for a metric space $X$.

3.3.2. Example. The one-point compactification of an uncountable discrete space $\Delta$ is denoted by $\Delta*$; thus $\Delta* = \Delta \cup \{\infty\}$. It is easily seen that the topology on $\Delta*$ has a base $B$ consisting of clopen sets, namely $B = B_1 \cup B_2$ where $B_1 = \{\{x\} : x \in X\}$ and $B_2 = \text{all open sets containing the point at infinity}$. It follows from 3.2.3 that $C(\Delta*)$ is anti-regular. However as recorded in Beispiel 11.6 of Storrer [St:68], this is a non-regular ring in which every non-zero-divisor is a unit. Thus question A has a negative answer. (Notice that $\Delta*$ is non-metrizable, as was needed; see 3.2.4.)