Chapter 7

Summary and Conclusion
7.1 Summary and Conclusion

Liquid helium-4 (LHE-4) undergoes a liquid to liquid phase transition which transforms it from its normal (N) liquid phase (helium-I) to superfluid (S) phase (helium-II) at $T = T_\lambda (~= 2.17 \, K)$. Helium-II exhibits several fascinating physical properties such as zero resistance to linear flow, zero entropy, very high thermal conductivity, thermomechanical and mechano-caloric effects etc. London proposed that this transition is a manifestation of Bose-Einstein condensation of $^4\text{He}$-atoms in $p = 0$ state which is commonly called as $p = 0$ condensate representing a well known state of a many body system where quantum effects dominates the macroscopic behaviour of the system. Numerous efforts have been made to investigate LHE-4 both by theoretical and experimental physicists, however, the microscopic origin of its unique properties has not been clearly understood.

Two microscopic approaches based on pseudo-potential and variational principle have been basically used to develop a viable theory of the system but none of these efforts has concluded the desired theory even though several hundreds of research articles have been published over the last eight decades. Microscopic theories based on widely different assumptions such as: (i) the existence of $p = 0$ condensate, (ii) $(q,-q)$ pair formation, (iii) pair condensation, and (iv) many particle condensation but none of these theories fully explains experimental properties of the liquid to an acceptable degree of accuracy. In fact no single theoretical framework has helped in explaining different aspects of helium-II. Similarly, hundreds of papers based on variation principle have been reported on the calculation of the amount of $p = 0$ condensate and the excitation spectrum but without satisfactory results. While excitation spectrum has not been found with desired agreement with experiments, amount of $p = 0$ condensate obtained by using sophisticated Monte Carlo methods is found to vary between 0 – 13%. Results of the neutron scattering experiments which have been performed with the basic objective of observing $p = 0$ condensate in LHE-4 are not conclusive on its existence. Reviewing the results of neutron scattering experiments and theoretical works on the estimation of $p = 0$ condensate, Sokol [1] remarked,

*If the underlying assumptions of the expressions used in the fits (neutron scattering data) are incorrect, then the results inferred will not have much meaning. It is interesting to note that the observed scattering in liquid helium can be fit extremely well simply by using the momentum*
distribution \( n(p) \) of the ideal Bose gas. This holds true in both the normal and super-fluid phase. The best description of the scattering in the super-fluid is obtained by using an ideal Bose gas \( n(p) \) with no condensate.

In a recent review article Leggett [2] also remarked,

_In the sixty years since London’s original proposal, while there has been almost universal belief that the key to super-fluidity is indeed the onset of BEC at \( T_\lambda \) it has proved very difficult, if not impossible, to verify the existence of the latter phenomenon directly. The main evidence for it comes from high-energy neutron scattering and, very, recently, from the spectrum of atoms evaporated from the liquid surface, and while both are certainly consistent with the existence of a condensate fraction of approximately 10%, neither can be said to establish it beyond all possible doubts._

Similar views have been expressed by different authors from time to time [3-5]. While these comments of the experts in the field reflect that the existence of \( p = 0 \) condensate in super-fluid helium is still doubted, most of the microscopic theories start with the basic assumption of the existence of \( p = 0 \) condensate. Consequently, it is not surprising that such theories fail to give a complete description of the system. Various shortcomings of these theories are also pointed out by Woods and Cowley [3], Rickayzen [6], Kleban [7] and, very recently, by Ettouhami [8] who questions the validity of the Bogoliubov theory [9] of weakly interacting bosons which is used as starting point of the field theoretical approach clubbed with the assumption of the existence of \( p = 0 \) condensate. As such Landau’s two fluid phenomenological theory [10,11] supplemented by the idea of quantized circulation presented by Onsager [12] and Feynman [13] remains the only way to understand the properties of LHE-4 despite its several shortcomings.

Absence of a proper microscopic theory of LHE-4 has been felt for a long time [3,6]. Only recently, Jain has successfully developed a microscopic theory of a system of interacting bosons which is well equipped to explain the properties of LHE-4. His theory vindicates (i) Landau two fluid phenomenology [10], (ii) London’s idea of macroscopic wave function of the S-state [14], (iii) the observation of Bogoliubov [9] that super-fluidity is the manifestation of an inter-particle
interaction, (iv) Ginzburg-Landau $\psi-$theory [15], etc. Jain’s theory explains different properties of LHE-4 within experimental errors. Consequently, it could be regarded as the long awaited theory of LHE-4. Important features of the theory can be summarised as follows:

1. The theory makes no assumption such as the existence of $p = 0$ condensate. Its all inferences are derived from the solutions of the Schrödinger equation of $N-$interacting bosons.

2. The theory successfully incorporates both components of the inter-particle interaction (repulsive and attractive components).

3. The $(q,-q)$ pairs with $q$ and $K$ motion form the basic unit of the system.

4. The $(q,-q)$ bound pairs of atoms (with $q = q_o$) around $T_\lambda$ arises from the quantum correlation potential and inter-particle attraction. This pair formation is energetically favourable and the binding of these pairs differs from the binding of two electrons in a Cooper pair in the sense that the necessary attraction is the inherent $^4He-^4He$ attraction.

5. Pair formation leads to the collective binding of all particles in the system for which the entire system behaves like a single macro-molecule. This binding serves as an energy gap between the super-fluid and normal phases of the system.

6. The theory shows that $p = 0$ condensate is not necessary for the superfluidity of the system.

7. The transition at $T_\lambda$ is characterised by two separate phenomena: (i) ordering of the particles in phase space with relative phase point separation of $\Delta \phi = 2n\pi$ ($n =$ integer number), and (ii) BEC of particles in $K = 0$ state of their $K-$motion.

8. The atoms in helium-I are randomly distributed in normal space, momentum space and phase space, while in helium-II they define an orderly arrangement in these spaces with a kind of crystalline arrangement in normal space, a momentum $q = \pi/d$ or its integral multiple, and relative phase position of $\Delta \phi = 2n\pi$; they cease to have relative motion and remain free to move in order of their positions on a closed path.
9. It has been successfully developed within the framework of the wave mechanics.

10. It is consistent with excluded volume condition as well as microscopic and macroscopic uncertainty.

11. Its mathematical formulations are exceptionally simple and its inferences have unparalleled clarity.

All these points motivated us to testify this theory by estimating numerical values of certain physical properties of superfluid helium-4 and examine their agreement with experiments. In this context we have studied logarithmic divergence of specific heat at constant pressure and related thermodynamic properties, excitations spectrum at low Q and its anomalous nature, energy gap and related properties such as superfluid density and critical velocities, transition temperature, surface tension and surface layer thickness which are presented in different chapters of this thesis. The fact that our results match closely with experiments indicates its accuracy and its great potential in explaining the behavior of LHE-4 type different systems of interacting bosons. Evidently, this fulfills one of the basic objectives of this thesis.

Macro-orbital theory estimates the energy associated with the \( \lambda \)–transition and concludes that this energy is responsible for the logarithmic divergence of specific heat (at constant pressure, \( C_p \)) at \( T_\lambda \). In this thesis, we have used this energy to estimate the numerical values of \( C_p \) as well as expansion coefficient (\( \alpha_P \)), isothermal compressibility (\( \kappa_T \)) and pressure coefficient (\( \beta \)) and compared them with the experimental values (cf. chapter 3). A close agreement of the estimated values with the experiments establishes the accuracy of the relations of Macro-orbital theory. We note that for the first time a microscopic theory (Jain’s Macro-orbital theory) provides a proper account of the origin of logarithmic divergence of specific heat and related thermodynamic properties.

Using the dispersion relations available in Macro-orbital theory, we have estimated excitation energy \( E(Q) \), group and phase velocities at different \( Q \) for different atomic arrangements (sc, bcc, fcc and hcp). Our results for \( E(Q) \) closely agree with experiments and we observe an anomalous dispersion region extended up to \( 0.43\text{Å}^{-1} \). Our results for fcc/hcp arrangements agree more closely with experiments (cf. chapter 4).
In Macro-orbital theory, the energy gap that exists between the normal and superfluid phase accounts for the various unique properties of the system, e.g. critical velocity \( v_c \), thermomechanical and mechanocaloric effect, etc. We used this energy gap to estimate superfluid and normal fluid density, critical velocity for linear and rotational flow, vortex line density, vortex size etc. Our results for superfluid and normal fluid density agrees closely with experiments (figure 5.2). Our results for critical velocity are higher than the measured one and this difference is expected because the measured \( v_c \) usually represents the velocity required for vortex formation and does not account for the loss of total super-fluidity, while critical velocity predicted by Macro-orbital theory accounts for the total loss of superfluidity of the system. Our results are very close to the measured values of Hess [16] and Huhn et al [17] who measured \( v_c \) in a small hole by filtering it near the hole with porous medium. The strong temperature dependence of the critical velocity observed by Hess and Hulin et al agrees with our findings.

Using the expression available from Macro-orbital theory we estimated \( T_\lambda = 2.271, 2.033, 1.964 \) and 1.964 \( K \) for sc, bcc, fcc and hcp arrangements respectively (cf. chapter 6). Our estimated values deviate 4 – 12% from the observed value \( (2.172K) \) as against 40% deviation of London’s estimation. Our investigation further indicates that the effective mass of helium atom plays an important role on the pressure dependence of \( T_\lambda \). While, as indicated from figure 6.1, this dependence is expected to be linear, its theoretical foundation may form a subject of a future investigation.

Our simple calculations for surface tension at \( T = 0 \) (\( \sigma (0) \)) based on Macro-orbital theory do not include the effects of parameters such as density variation and the change of atomic layer separation near the free surface of the liquid. Our estimated value \( \approx 495 \text{ mdynes/cm} \) is about 30% higher than experimental values \( 375 \text{ mdynes/cm} \). It may be stated that smaller density of helium atoms near the free surface could account for the difference. However, the so called sophisticates theories also render \( \sigma (0) \) values (Table 6.19) varying from 230 to 510 \text{ mdynes/cm}. We hope that a future course of study which incorporates the said fluctuations with Macro-orbital theory may help us in improving our results.

The estimated results of various physical properties of LHE-4 that are presented in chapters 3, 4, 5, and 6, closely agree with experiments and thus establish that Macro-orbital theory successfully explains these properties of the
system. In addition, the theory can be used to study entropy, thermal conductivity, specific heat, pressure dependent dispersion relation, static structure factor $S(Q)$ and dynamic form factor $S(Q,\omega)$ etc. in the quantitative level. Formalism of this theory can be extended to different systems like Helium – 3, superconductor, 2-D bosonic system etc.. Our group plans to investigate these aspects and hopes to open new frontier for future studies.


