CHAPTER - I

A BRIEF SURVEY ON GRAPH LABELINGS AND PRELIMINARIES

In this thesis we consider only finite undirected graphs without loops or multiple edges. For graph theoretic terminologies, we refer [11], [15], and [20].

Chapter 1 is an introductory one. In this chapter we give a brief survey on sum labeling, integral sum labeling and (a,d)-antimagic labelings and basic definitions and results on graph labelings which are required for the subsequent chapters.

1. Introduction

Graph labelings were first introduced in the late 1960s. Most graph labeling methods trace their origin to one introduced by Rosa [39] in 1967, or one given by Graham and Sloane [18] in 1980. Rosa [39] called a function \( f \) a \( \beta \)-valuation of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the set \( \{0, 1, \ldots, q\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are distinct. Golomb [16] subsequently called such labelings graceful and this is now the popular term. Rosa introduced \( \beta \)-valuations as well as a
number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, $\beta$-valuations originated as a means of attacking the conjecture of Ringel [38] that $K_{2n+1}$ can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with $n$ edges. This conjecture remains unresolved to this day despite massive attempts and hence transformed into a series of various vertex or edge or both labeling problems.

In excess of 600 papers have spawned a bewildering array of graph labeling methods over the past four decades. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments.

A **graph labeling** is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph Labeling problems are motivated by practical problems. Labeled graphs serve as useful mathematical models for a broad range of applications such as: coding theory, including the design of good type codes, Synch-set codes, missile guidance codes and convolutional codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encodings of integers. Labeled graphs have also been applied in determining ambiguities in
X-ray crystallographic analysis, to design a communication network addressing system, data base management, in determining optimal circuit layouts and radio astronomy problem etc. [15].

2. Sum and Integral Sum Labelings.

In 1990, Harary [21] introduced the notion of a sum graph. A graph $G(V, E)$ is called a sum graph if there is a bijection $f$ from $V$ to a set of positive integers $S$ such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel, and Jacobs [23] defined a real sum graph in an analogous way by allowing $S$ to be any finite set of positive real numbers. Bergstrand et al. [5] defined a product graph analogous to a sum graph except that 1 is not permitted to belong to $S$. They proved that every product graph is a sum graph and vice versa.

For a connected graph $G$, let $\sigma(G)$, the sum number of $G$, denote the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is a sum graph. A labeling that makes $G$ together with $\sigma(G)$ isolated vertices a sum graph is called an optimal sum graph labeling. Ellingham [14] proved the conjecture of Harary [21] that
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\( \sigma(T) = 1 \) for every tree \( T \neq K_1 \). Smyth [45] proved that there is no graph \( G \) with \( e \) edges and \( \sigma(G) = 1 \) when \( n^2/4 < e \leq n(n-1)/2 \). More generally, Kratochvil, Miller, and Nguyen [28] conjecture that \( \sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1 \). Hao [19] has shown that if \( d_1 \leq d_2 \leq \ldots \leq d_n \) is the degree sequence of a graph \( G \) then \( \sigma(G) > \max(d_i - i) \) where the maximum is taken over all \( i \). Bergstand et al. [4] proved that \( \sigma(K_n) = 2n - 3 \). Hartsfield and Smyth [25] claimed to have proved that \( \sigma(K_{m,n}) = \lceil 3m + n - 3 \rceil / 2 \) when \( n \geq m \) but Yan and Liu [53] found counter examples to this assertion when \( m \neq n \), Pyatkin [37], Liaw, Kuo, and Chang [29], Wang and Liu [50] and He et al. [27] have shown that for \( 2 \leq m \leq n \),\( \sigma(K_{m,n}) = \lceil n/p + (p+1)(m-1)/2 \rceil \) where \( p = \lceil \sqrt{2n/(m-1) + 1/4} - 1/2 \rceil \) is the unique integer such that \((p-1)p(m-1)/2 < n \leq (p+1)p(m-1)/2 \). Miller et al. [32] proved that \( \sigma(W_n) = n/2 + 2 \) for \( n \) even and \( \sigma(W_n) = n \) for \( n \geq 5 \) and \( n \) odd (see also [47]).

At a conference in 2000 Miller [31] posed the following two problems. Given any graph \( G \), does there exist an optimal sum graph labeling that uses the label 1? Find a class of graphs \( G \) that have sum
number of the order $|V(G)|^s$ for $s > 1$. (Such graphs were shown to exist for $s = 2$ by Gould and Rodl in [17]).

In 1994 Harary [22] generalized sum graphs by permitting $S$ to be any set of integers. He calls these graphs integral sum graphs. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [43] has shown that $C_n$ and $W_n$ are integral sum graphs for all $n \neq 4$. Chen [13] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. This conjecture was proved in 2004 by Sethuraman and Venkatesh [42]. Wu, Mao, and Le [51] proved that $mP_n$ are integral sum graphs. They also proved that the conjecture of Harary [22] that the sum number of $C_n$ equals the integral sum number of $C_n$ if and only if $n \neq 3$ or 5 is false and that for $n \neq 4$ or 6 the integral sum number of $C_n$ is at most 1.

Xu [52] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees $T$; $mK_3$ for all $m$; and the union of any number of integral sum trees. Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is $mG$ for all $m > 1$. Nicholas,
Vilfred and Somasundaram [34] proved that all banana trees and the union of any number of stars are integral sum graphs.

Liaw, Kuo, and Chang [29] proved that all caterpillars are integral sum graphs (see also [51] and [52] for some special cases of caterpillars). They also proved that all cycles except $C_4$ are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns $C_n \odot K_1$ for $n \geq 4$ [44] and that the subdivision graphs of $C_n \odot K_1$ for $n \geq 3$ are integral sum graphs [40].

The integral sum number, $\zeta(G)$, of $G$, is the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is an integral sum graph. Thus, by definition, $G$ is an integral sum graph if and only if $\zeta(G) = 0$. Harary, [22] conjectured that for $n \geq 4$ the integral sum number $\zeta(K_n) = 2n - 3$. This conjecture was verified by Chen [12], by Sharary [43], and by Xu [52]. Yan and Liu [54] proved:

- $\zeta(K_n - E(K_r)) = n - 1$ when $n \geq 6$, $n \equiv 0 \pmod{3}$ and $r = 2n/3 - 1$;
- $\zeta(K_{m,m}) = 2m - 1$ for $m \geq 2$;
- $\zeta(K_n - \text{edge}) = 2n - 4$ for $n \geq 4$ [49];
- if $n \geq 5$ and $n - 3 \geq r$, then $\zeta(K_n - E(K_r)) \geq n - 1$;
- if $\lceil 2n/3 \rceil - 1 > 2$, then $\zeta(K_n - E(K_r)) \geq 2n - r - 2$; and if $2 \leq m < n$, and $n = (i + 1)$
(im – i +2) / 2, then \( \sigma(K_{m,n}) = \zeta(K_{m,n}) = (m - 1) (i + 1) + 1 \) while if 

(i + 1) (im – i + 2)/2 < n < (i + 2) [(i + 1) m– i+ 1] / 2, then 

\[
\sigma(K_{m,n}) = \zeta(K_{m,n}) = \left[ \frac{(m - 1) (i + 1) (i + 2) + 2n}{(2i + 2)} \right].
\]

In [36] Nicholas and Vilfred defined the **edge reduced sum number** of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They proved that for \( K_n, n \geq 3 \), this number is 

\[
\frac{n(n-1)/2 + \lceil n/2 \rceil}{2}.
\]

They ask for a characterization of graphs for which the edge reduced sum number is same as its sum number. They conjectured that an integral sum graph of order \( p \) and size \( q \) exists if and only if \( q \leq 3 (p^2 - 1)/8 - \lfloor (p - 1)/4 \rfloor \) when \( p \) is odd and 

\( q \leq p (3p - 2)/8 \) when \( p \) is even. We discussed this conjecture in section 8 of chapter II. They also define the **edge reduced integral sum number** in an analogous way and conjectured that this number for \( K_n \) is 

\[
\frac{(n - 1) (n - 3)}{8} + \lfloor (n - 1)/4 \rfloor \text{ when } n \text{ is odd and } \frac{n (n - 2)}{8} \text{ when } n \text{ is even. The validity of this conjecture is established in section 8 of chapter II.}
\]

3. **(a, d)-Antimagic Labeling**

The concept of (a, d)-antimagic labeling was introduced by Bodendieck and Walther [6] in 1993. A connected graph \( G (V, E) \) is said
to be \((a, d)\)-antimagic if there exist positive integers \(a, d\) and a bijection \(f : E \rightarrow \{1, 2, \ldots, |E|\}\) such that the induced mapping \(g_f : V \rightarrow \mathbb{N}\), defined by \(g_f(v) = \sum \{f(u, v) : (u, v) \in E(G)\}\), is injective and \(g_f(V) = (a, a + d, \ldots, a + (|V| - 1)d)\). In [30] these are called \((a, d)\)-vertex antimagic edge labelings or \((a,d)\)-VAE.. They proved [8, 9] the Herschel graph is not \((a, d)\)-antimagic and obtained both positive and negative results about \((a, d)\)-antimagic labelings for various cases of graphs called parachutes \(P_{g,b}\) (\(P_{g,b}\) is the graph obtained from the wheel \(W_{g+p}\) by deleting \(p\) consecutive spokes.) In [2] Baca and Hollander proved that necessary conditions for \(C_n \times P_2\) to be \((a, d)\)-antimagic are \(d = 1\), \(a = (7n + 4)/2\) or \(d = 3\), \(a = (3n + 6)/2\) when \(n\) is even, and \(d = 2\), \(a = (5n + 5)/2\) or \(d = 4\), \(a = (n + 7)/2\) when \(n\) is odd. Bodendiek and Walther [7] conjectured that \(C_n \times P_2 (n \geq 3)\) is \(((7n + 4)/2, 1)\)-antimagic when \(n\) is even and is \(((5n + 5)/2, 2)\)-antimagic when \(n\) is odd. These conjectures were verified by Baca and Hollander [2] who further proved that \(C_n \times P_2 (n \geq 3)\) is \(((3n + 6)/2, 3)\)-antimagic when \(n\) is even. Baca and Hollander [2] conjectured that \(C_n \times P_2\) is \(((n + 7)/2, 4)\)-antimagic when \(n\) is odd and at least 7. Bodendiek and Walther [7] also conjectured that \(C_n \times P_2 (n \geq 7)\) is \(((n + 7)/2, 4)\)-antimagic. Baca and Hollander [3] proved
that the generalized Petersen graph $P(n, 2)$ is $((3n + 6)/2, 3)$-antimagic for $n \equiv 0 \pmod{4}$, $n \geq 8$. Bodendiek and Walther [10] proved that the following graphs are not $(a, d)$-antimagic: even cycles; paths of even order starting; $C_3^{(k)}$; $C_4^{(k)}$; trees of odd order of at least 5 that have a vertex that is adjacent to three or more end vertices; $n$-ary trees with at least two layers when $d = 1$; $K_{3,3}$; the Petersen graph; and $K_4$. They also proved: $P_{2k+1}$ is $(k, 1)$-antimagic; $C_{2k+1}$ is $(k + 2, 1)$-antimagic; if a tree of odd order $2k + 1$ ($k > 1$) is $(a, d)$-antimagic, then $d = 1$ and $a = k$; if $K_{4k}$ ($k \geq 2$) is $(a, d)$-antimagic, then $d$ is odd and $d \leq 2k(4k - 3) + 1$; if $K_{4k+2}$ is $(a, d)$-antimagic, then $d$ is even and $d \leq (2k + 1)(4k - 1) + 1$; and if $K_{2k+1}$ ($k \geq 2$) is $(a, d)$-antimagic, then $d \leq (2k + 1)(k - 1)$. Lin, Miller, Simanjuntak, and Slamin [30] showed that no wheel $W_n$ ($n > 3$) has an $(a, d)$-antimagic labeling.

The anti-prism of order $2n$ has vertex set $\{x_{1,i}, ..., x_{1,n}, x_{2,1}, ..., x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i+1}\}$ (subscripts are taken modulo $n$). For $n \geq 3$ and $n \not\equiv 0 \pmod{4}$ Baca [2] gives $(6n + 3, 2)$-antimagic labelings and $(4n + 4, 4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that for $n \equiv 2 \pmod{4}$, $n \geq 6$, the antiprism on $2n$
vertices has a \((6n + 3,2)\)-antimagic labeling and a \((4n + 4,4)\)-antimagic labeling.

Nicholas, Somasundaram, and Vilfred [35] proved the following: If \(K_{m,n}\) where \(m \leq n\) is \((a, d)\)-antimagic then \(d\) divides 
\[
\frac{(m - n)(2a + d(m + n - 1))}{4} + dm/2; 
\]
if \(m + n\) is prime, then \(K_{m,n}\) where \(n > m > 1\) is not \((a, d)\)- antimagic; if \(K_{n,n+2}\) is \((a, d)\)-antimagic, then \(d\) is even and \(n + 1 \leq d < (n + 1)^2/2\); if \(K_{n,n+2}\) is \((a, d)\)-antimagic and \(n\) is odd, then \(a\) is even and \(d\) divides \(a\); if \(K_{n,n+2}\) is \((a, d)\)-antimagic and \(n\) is even, then \(d\) divides \(2a\); if \(K_{n,n}\) is \((a, d)\)-antimagic, then \(n\) and \(d\) are even and \(0 < d < n^2/2\); if \(G\) has order \(n\) and is unicyclic and \((a,d)\)-antimagic, then \((a, d) = (2, 2)\) when \(n\) is even and \((a, d) = (2, 2)\) or \((a, d) = ((n + 3)/2, 1)\); a cycle with \(m\) pendant edges attached at each vertex is \((a, d)\)-antimagic if and only if \(m = 1\); the graph obtained by joining an endpoint of \(P_m\) with one vertex of the cycle \(C_n\) is \((2, 2)\)-antimagic if \(m = n\) or \(m = n - 1\); if \(m + n\) is even the graph obtained by joining an endpoint of \(P_m\) with one vertex of the cycle \(C_n\) is \((a, d)\)-antimagic if and only if \(m = n\) or \(m = n - 1\). They conjecture that for \(n\) odd and at least 3, \(K_{n,n+2}\) is \(((n + 1)(n^2 - 1)/2, n + 1)\)-antimagic and
they have obtained several results about \((a, d)\)-antimagic labelings of caterpillars.

4. Preliminaries

In this section we provide basic definitions and results which are required in the subsequent chapters so as to make the thesis self contained. For well known concepts, definitions and notations not explicitly given here, one may refer to [11], [15], [20].

Definition 4.1. A graph \(G\) consists of a pair \((V(G), E(G))\) where \(V(G)\) is a non-empty finite set whose elements are called points or vertices and \(E(G)\) is set of unordered pairs of distinct elements of \(V(G)\). The elements of \(E(G)\) are called lines or edges of the graph \(G\).

We write \(e = uv\) for an edge and say that \(u\) and \(v\) are adjacent vertices; vertex \(u\) and edge \(e\) are incident with each other as are \(v\) and \(e\). If two distinct edges \(e_1\) and \(e_2\) are incident with a common vertex, then they are adjacent edges. A graph with \(p\) vertices and \(q\) edges is called a \((p, q)\) graph. In this case \(p\) is called the order of the graph and is denoted by \(\text{order}(G) = |V(G)| = O(G)\) and \(q\) is size of the graph and is denoted by \(q(G) = |E(G)|\). If \(G\) is a \((p, q)\) graph, then \(q \leq \frac{p(p - 1)}{2}\).
An edge incident with only one vertex is called a loop. Edges joining the same pair of vertices are called parallel edges or multiple edges.

A graph without loops and parallel edges is called a simple graph.

Unless otherwise specified, by a graph we mean a simple graph.

When there is no possibility of confusion we write $V(G) = V$ and $E(G) = E$. Each vertex is represented by a small dot and each edge is represented by a line segment joining the two vertices with which the edge is incident. Thus a diagram of the graph depicts the incidence relation holding between its vertices and edges.

**Definition 4.2** Two graphs $G$ and $H$ are isomorphic, written $G \cong H$, if there exists a bijection between their vertex sets which preserves adjacency.

**Definition 4.3** A graph $H$ is said to be a subgraph of a graph $G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A subgraph $H$ of $G$ is called an induced subgraph of $G$, if $H$ contains all the edges of $G$ that connects vertices of $H$.

A subgraph $H$ of a graph $G$ is called a spanning subgraph of $G$, if $V(H) = V(G)$. 
Definition 4.4 Let $G(V, E)$ be a graph. Let $v \in V$. The subgraph of $G$ obtained by removing the vertex $v$ and all the edges incident with $v$ is called the subgraph obtained by the removal of the vertex $v$ and is denoted by $G-v$. Clearly $G-v$ is an induced subgraph of $G$.

The subgraph of $G$ obtained by the removal of the edge $e$ is called the subgraph obtained by the removal of the edge $e$ and is denoted by $G-e$. Clearly $G-e$ is a spanning subgraph of $G$ which contains all the edges of $G$ except $e$.

Definition 4.5 Let $G(V, E)$ be a graph. Let $u, v$ be two vertices which are not adjacent in $G$. Then $G + uv = (V, E \cup \{uv\})$ is called the graph obtained by the addition of the edge $uv$ to $G$. Clearly $G + uv$ is the smallest super graph of $G$ containing the edge $uv$.

Definition 4.6 A walk of a graph $G$ is an alternating sequence of vertices and edges $u_0, e_1, u_1, e_2, \ldots, u_{n-1}, e_n, u_n$ beginning and ending with vertices, in which each edge is incident with two points immediately preceding and succeeding it. It is called a $u_0 - u_n$ path if all the vertices and edges are distinct. If the two end vertices in a path are the same, that is, if $u_0 = u_n$, $n \geq 3$, in a path, then it is called a cycle. The graph consisting of a cycle of length $n$ is denoted by $C_n$ and by $P_n$ a path with $n$ vertices.
(and n-l edges). The cycle $C_3$ is called a triangle. $P_n$ is also called a path of order $n$ and length $n-1$.

**Definition 4.7** A graph is connected if every pair of vertices are joined by a path.

A maximal connected subgraph of $G$ is called a connected component or simply a component of $G$. A graph is called disconnected if it is not connected. Clearly a graph $G$ is disconnected if and only if $G$ has more than one component.

**Definition 4.8** The distance $d(u, v)$ between two points $u$ and $v$ in $G$ is the number of lines (called ‘length’) of a shortest path joining $u$ and $v$ if any; otherwise $d(u, v) = \infty$.

Two sub graphs $H$ and $K$ of a graph $G$ are said to be edge-disjoint, if $E(H) \cap E(K) = \emptyset$. A collection of subgraphs of $G$ are said to be edge disjoint if they are pair wise edge-disjoint.

**Definition 4.9** The degree (or valency) of a vertex $v$ in a graph $G$ is the number of edges incident with $v$. The degree of $v$ is denoted by $d_G(v)$ or $\deg v$ or simply $d(v)$.

A vertex $v$ in $G$ is called isolated vertex if $\deg(v) = 0$ and it is called an end vertex (or pendent vertex) if $\deg(v) = 1$. An edge incident
to a pendent vertex is called a pendent edge or end edge. In a (p, q)-graph, 0 \leq \deg(v) \leq p-1 for every vertex v.

**Definition 4.10** The minimum degree among the vertices of G is denoted \( \delta(G) \) and is defined as \( \min \{ \deg(v) : v \in V(G) \} \) and the maximum degree is denoted \( \Delta(G) \) is defined as \( \max \{ \deg(v) : v \in V(G) \} \).

**Definition 4.11** A graph G is called regular graph of degree r if \( \delta(G) = \Delta(G) = r \). In other words, a graph G is regular of degree r if all vertices of G have the same degree r. A regular graph of degree 3 is called a cubic graph.

**Theorem 4.12** The sum of the degrees of the vertices of a (p, q) graph G is twice the number of edges. That is, the degrees \( d_1, d_2, \ldots, d_p \) of the points of a graph is a sequence of non-negative integers whose sum is 2q. 
\[ \sum \deg v = 2q. \]

**Definition 4.13** A partition of a non-negative integer n is a finite set of non-negative integers \( d_1, d_2, \ldots, d_p \) whose sum is n. We denote this partition by \( (d_1, d_2, \ldots, d_p) \).
Definition 4.14 Let $G$ be a $(p, q)$ graph. The partition of $2q$ as the sum of the degrees of its vertices is called the **partition** or the **degree sequence** of the graph $G$.

Definition 4.15 A partition $p = (d_1, d_2, ..., d_p)$ of $n$ into $p$ parts is said to be a **graphical partition** or a **graphic sequence** if there exists a graph $G$ whose vertices have degree $d_i$, $i = 1, 2, ..., p$ and $G$ is called a **realization** of $p$.

Definition 4.16 The **complement** $G$ or $G^c$ of a graph $G$ is a graph having $V(G)$ as its vertex set, but two vertices are adjacent in $G$ if and only if they are not adjacent in $G^c$. If $G \cong G^c$, then $G$ is said to be a **self-complementary graph**.

Definition 4.17 The **complete graph** $K_p$ has every pair of its $p$ vertices adjacent. Thus $K_p$ is a regular graph of degree $p-1$ and it has $p(p-1)/2$ number of edges. $K_3$ is a triangle. The graph $K_m^c$ is totally disconnected with $m$ isolated vertices, some times denoted as $mK_1$.

Definition 4.18 A **bipartite** graph (or **bigraph**) $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every line of $G$ joins $V_1$ with $V_2$. 
If $G$ contains every line joining $V_1$ and $V_2$, then $G$ is called **complete bipartite** graph. If $V_1$ and $V_2$ have $m$ and $n$ points, we write $G = K_{m,n}$. Obviously $K_{m,n}$ has $m + n$ vertices and $mn$ edges. $K_{1,n}$ is called a **star** for $m \geq 1$.

**Definition 4.19** Graphs $G$ and $H$ are said to be **comparable graphs** if and only if either $G$ is a subgraph of $H$ or $H$ is a subgraph of $G$. Otherwise they are called **non-comparable graphs**. $K_m$ and $K_n$ are comparable graphs for every $m, n \in \mathbb{N}$ but $K_{2,5}$ and $K_{3,4}$ are non-comparable graphs.

**Definition 4.20** A connected graph without cycle is called a **Tree**.

**Definition 4.21** A connected graph with a unique cycle is called a **unicyclic graph**. If $G$ is a unicyclic graph, then $|V(G)| = |E(G)|$.

**Definition 4.22** A spanning cycle of a graph is called a **Hamilton cycle**. A graph is **Hamiltonian**, if it contains a Hamilton cycle.

**Definition 4.23** Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then their **union** $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

**Definition 4.24** The **sum** $G_1 + G_2$ is $G_1 \cup G_2$ together with all edges joining vertices of $V_1$ to vertices of $V_2$. 
The graph \( P_n + K_1 \) is called a **Fan** and \( P_n + 2K_1 \) is called the **Double fan**.

The graph \( C_n + K_1 \) is called a **cone** or a wheel \( W_n \) with \( n \) spokes and the graph \( C_n + 2K_1 \) is called the **Double cone**.

**Definition 4.25** The **cross product** \( G_1 \times G_2 \) has its vertex set \( V_1 \times V_2 \) and two points \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent in \( G_1 \times G_2 \) whenever \((u_1 = v_1 \text{ and } u_2 \text{ adjacent to } v_2) \) or \((u_2 = v_2 \text{ and } u_1 \text{ adjacent to } v_1)\).

The **product** \( P_m \times P_n \) is called **Planar grids** and \( P_2 \times P_n \) is called **Ladder**. The product \( C_m \times P_n \) is called **Grids on cylinder** of order \( mn \). In particular \( D_n = C_n \times P_2 \) is called a **prism**. \( B_m = K_{1,m} \times P_2 \) is called a **Book**.

**Definition 4.26** [21] A graph \( G \) is called a **sum graph** if the vertices of \( G \) can be labeled with distinct positive integers so that \( e = uv \) is an edge of \( G \) if and only if the sum of the labels on vertex \( u \) and vertex \( v \) is also a label on a vertex of \( G \).

If \( G \) is not a sum graph, adding a finite number of isolated vertices to it always yields a sum graph and the **sum number** \( \sigma(G) \) of \( G \) is the smallest number of isolated vertices so added. If \( G \) is a sum graph with
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respect to a label set $S$, then $G$ can be denoted as $G = G^*(S)$. A labeling that realizes $G + \sigma(G)K_1$ as a sum graph is said to be an **optimal sum graph labeling**.

An **integral sum graph** is also defined just as sum graph, the difference being that the label set $S$ is a subset of $\mathbb{Z}$ [22]. The **integral sum number** $\zeta(G)$ is the smallest non-negative integer $s$ such that $G \cup sK_1$ is an integral sum graph ($\Sigma$-graph). By definition, it is clear that $\zeta(G) \leq \sigma(G)$ for all graph $G$, and $G$ is an $\Sigma$-graph if and only if, $\zeta(G) = 0$.

**Definition 4.27** [21] A family of sum graph $G_n$ that are defined by $G_n = G^*(N_n)$ where $N_n = \{1, 2, \ldots, n\}$. For example, $G_1 = K_1$, $G_2 = 2K_1$, $G_3 = K_1 \cup K_2$ and $G_4 = K_1 \cup P_3$. For example the graph $G_7$ is given in figure 1.

![Fig 1. $G_7 = G^*(\{1, 2, 3, 4, 5, 6, 7\})$.](image-url)
Theorem 4.28 [33] For any graph $G_n$, $|E(G_n)| = \frac{1}{2} \left( \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right)$ where $\left\lfloor n \right\rfloor$ denotes the greatest integer less than or equal to $n$. ■

Definition 4.29 [22] A family of integral sum graph $G_{n,n}$ that are defined by $G_{n,n} = G^+(S)$, where $S = \{-n, \ldots, -2, -1, 0, 1, 2, \ldots n\}$.

Theorem 4.30 [33] Let $G$ be any connected graph and $x, u, v, w \in V(G)$. If $f$ is an $\Sigma$-labeling with $f(x) = -f(u)$, then for each $v \in N(u)$ such that $f(u) + f(v) = f(w)$, $w = x$ or $xw \in E(G)$. ■

Theorem 4.31 [13] Let $G$ be a non trivial graph with an integral sum labeling $f$. Then $f(x) \neq 0$ for every vertex $x$ of $G$ if and only if the maximum degree $\Delta(G) < |V(G)| - 1$. ■

Theorem 4.32 [13] If $G_1$ and $G_2$ are two node disjoint graphs and $\Delta(G_i) < |V(G_i)| - 1$ (i = 1, 2), then $\zeta(G_1 \cup G_2) \leq \zeta(G_1) + \zeta(G_2)$. ■

Definition 4.33 [33] Let $G$ be a connected graph with maximum degree $\Delta(G) = |V(G)| - 1$. We take $V_\Delta(G) = \{ x \in V(G) / \deg(x) = |V(G)| - 1 \}$.

Theorem 4.34 [33] Let $f$ be an $\Sigma$-labeling of a connected graph $G$ with $\Delta(G) = |V(G)| - 1$. If $|V_\Delta(G)| \geq 2$, then

(i) There exists a vertex $x \in V_\Delta(G)$ such that $f(x) = 0$ and
(ii) For every vertex $y \in V_\Delta(G) \setminus \{x\}$, there exists a vertex $y' \in V(G) \setminus V_\Delta(G)$ such that $f(y) + f(y') = 0$. 

**Theorem 4.35** [33] Let $f$ be an $\Sigma$-labeling of a connected graph $G$ of order $k+2$ with $\Delta(G) = |V(G)| - 1$ and $|V_\Delta(G)| \geq 2$. If $y \in V_\Delta(G)$ such that $f(y) \neq 0$, then for every vertex $u \in V(G)$, $f(u) = r \cdot f(y)$ where $r \in \{0, 1, -1, -2, \ldots, -k/k \geq 1\}$. 

**Theorem 4.36** [33] If $G \neq K_3$ is a connected $\Sigma$-graph with $\Delta(G) = |V(G)| - 1$, then $|V_\Delta(G)| < 3$. 

**Theorem 4.37** [33] Integral sum graph $G \neq K_3$ of order $n$ with $|V_\Delta(G)| = 2$ is unique up to isomorphism. (This graph of order $n$ is denoted by $G_{\Delta_n}$.) 

**Definition 4.38** [36] Let $G$ be any graph. The minimum number of edges whose removal from $G$ results in a sum graph is called **Edge Reduced Sum number** and is denoted by $ER_\sigma(G)$. 

**Definition 4.39** [36] Let $G$ be any graph. The minimum number of edges whose removal from $G$ results in an integral sum graph is called **Edge Reduced Integral Sum number** and is denoted by $ER_\zeta(G)$. 

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Obviously, \( \text{ER} \zeta(G) \leq \text{ER}_\sigma(G) \) for any graph \( G \).

**Definition 4.40** [48] A graph \( G \) is called a **anti-sum graph** or **anti-\( \Sigma \)-graph** if the vertices of \( G \) can be labelled with distinct positive integers so that \( e = uv \) is an edge of \( G \) if and only if the sum of the labels on vertices \( u \) and \( v \) is not a label in \( V(G) \).

An **anti-integral sum graph** or **anti-\( \int \Sigma \)-graph** is also defined just as anti-sum graph, the difference being that the labels may be any distinct integers.

**Definition 4.41** A graph \( G \) is **decomposable** into the subgraphs \( G_1, G_2, ..., G_n \) of \( G \), if no \( G_i \) (\( i = 1, 2, ..., n \)) has isolated vertices and the edge set of \( G \) can be partitioned into the subsets \( E(G_1), E(G_2), ..., E(G_n) \).

If \( G_i \equiv H \) for every \( i, i = 1, 2, ..., n \) then \( G \) is said to be **H-decomposable**.

If \( G \) is H-decomposable, then we say that \( H \) divides \( G \) and we write \( H / G \).

**Definition 4.42** [55] Suppose \( G_1, G_2, ..., G_k \) are graphs of order at most \( n \). We say that there is a **packing** of \( G_1, G_2, ..., G_k \) into the complete graph \( K_n \) if there exist injections \( \alpha_i : V(G_i) \to V(K_n) \), \( i = 1, 2, 3, ..., k \) such that \( \alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset \) for \( i \neq j \), where the map \( \alpha_i^* : E(G_i) \to E(K_n) \) is induced by \( \alpha_i \).
Similarly suppose $G$ is a graph of order $m$ and $H$ is a graph of order $n \geq m$ and there exists an injection $\alpha : V(G) \rightarrow V(H)$ such that $\alpha^* (E(G)) \cap E(H) = \emptyset$ then we say that there is a packing of $G$ into $H$, and in case $n = m$, we also say that there is a packing of $G$ and $H$ or $H$ and $G$ are packable.

Clearly decomposition and packing on graphs are reverse processes and so we give results on decomposition of graphs. It is easy to get the corresponding results on packing.

**Definition 4.43** [49] An $\Sigma$-graph ($\Sigma$-graph) $G$ is said to be a maximal $\Sigma$-graph (maximal $\Sigma$-graph) if $G$ is not a spanning subgraph of any other $\Sigma$-graph (sum graph).

The maximal integral sum graph (s) of a given order, say $n$, is a maximal integral sum graph of the same order with the maximum number of edges.

**Definition 4.44** [21] In the sum labeling of a graph, vertices whose labels correspond to an edge $uv$ are said to be working vertices. It has been realized that certain graphs can only be labeled in such a way that all the working vertices are also isolates. Such graphs are called exclusive otherwise it is called inclusive.
Definition 4.45 A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. The set of all vertices with any one colour is independent and is called a colour class. An n-colouring of a graph G uses n colours; it thereby partitions V into n colour classes.

Definition 4.46 The chromatic number $\chi(G)$ is defined as the minimum n for which G has n-coloring. A graph G is n-colorable if $\chi(G) \leq n$ and is n-chromatic if $\chi(G) = n$.

Definition 4.47 An edge coloring of a graph is an assignment of colours to its edges so that no two incident edges have the same colour.

The edge chromatic number $\chi'(G)$ is defined as the minimum n for which G has n-edge coloring. A graph G is n-edge colorable if $\chi'(G) \leq n$ and is n-edge chromatic if $\chi'(G) = n$.

Definition 4.48 [10] Let G (V, E) ∈ $\Gamma$ be a graph of order $|V| \geq 3$, and $a, d \in \mathbb{N}$. A bijective mapping $f : E \rightarrow \{1, 2, \ldots, |E|\}$ with induced mapping $g_f : V \rightarrow \mathbb{N}$ defined by $g_f(v) = \sum_{e \in I(v)} f(e)$, $v \in V$, where $I(v) = \{e \in E / e$ is incident to v}$, for $v \in V$ is called $(a,d)$-arithmetic antimagic labeling if and only if $g_f(V)$ forms an arithmetic progression with initial value a and step size d. That is, $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. 
Bodendiek and Walther [10] noted the following relation for (a,d)-antimagic graph.

**Lemma 4.49** [10] Let $G (V,E)$ with $|V| = p \geq 3$ and $|E| = q \geq 2$, be an (a, d)–antimagic graph. Then $a$ and $d$ satisfy the following conditions a) and b)

(a) $a, d \in \mathbb{N}$ are positive solutions of the linear Diophantine equation

$$2ap + p(p - 1)d = 2q(q + 1)$$

and

(b) $a \geq \delta(\delta + 1)/2$ where $\delta$ denotes the minimum degree in $G$. ■