CHAPTER IV

(A, D)—ANTIMAGIC LABELING OF GRAPHS

1. Introduction

In 1990 Ringel [24] introduced the concept of antimagic graph. Each edge labeling \( f \) of a graph \( G = (V, E) \) from 1 through \( |E| \) induces a vertex labeling \( g_f \) where \( g_f(v) \) is the sum of the labels of all edges that are incident upon vertex \( v \). Labeling \( f \) is called antimagic if and only if the values \( g_f(v) \) are pairwisely distinct for all vertices \( v \) of \( G \). Graph \( G \) is called antimagic if and only if it has an antimagic labeling.

Let \( \mathcal{G} \) denote the set of all finite, connected undirected graphs \( G = (V, E) \) without loops and multiple edges. The main problem in the theory of antimagic graphs is the determination of all antimagic graphs in \( \mathcal{G} \). This problem still remains open. Ringel [24] conjectured that every graph \( G \in \mathcal{G} \) of order \( \geq 3 \) is antimagic.

In 1993 Bodendieck and Walther [6] introduced the concept of an (a,d)—arithmetic antimagic labeling.
Definition 1.1. [10] Let $G(V, E)$ be a graph of order $\geq 3$, and $a, d \in \mathbb{N}$. A bijective mapping $f : E \rightarrow \{1, 2, \ldots, |E|\}$ with induced mapping $g_f : V \rightarrow \mathbb{N}$ defined by $g_f(v) = \sum_{e \in I(v)} f(e)$, where $I(v) = \{ e \in E : e \text{ is incident to } v \}$, for $v \in V$ is called $(a, d)$–arithmetic antimagic labeling if and only if $g_f(v)$ form an arithmetic progression with initial value $a$ and step width $d$. That is, $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. See figures 4.1 and 4.2.

Figure 4.1  

Figure 4.2

$G$ is called $(a, d)$–antimagic if and only if $G$ admits an $(a, d)$-antimagic labeling. Obviously every $(a, d)$-antimagic graph is also antimagic. But the converse need not be true. For example, $C_4$ is antimagic as shown in figure 4.3. But $C_4$ does not admit an $(a, d)$-antimagic labeling for any pair $a, d \in \mathbb{N}$. 
The weight $w_f(v)$ (sometimes denoted as $w(v)$) of a vertex $v$ in $V(G)$ under an edge labeling $f$ is the sum of values $f(e)$ assigned to all edges $e$ incident to $v$. Let $W$ denote the set of all vertex weights of the graph $G$.

Bodendiek and Walther [8,9] proved the existence of two very interesting sub sets of set of all $(a,d)$-antimagic parachutes. Baca and Hollander [2] characterized all $(a,d)$-antimagic graphs of prisms $D_n = C_n \times P_2$ when $n$ is even. They showed that when $n$ is odd, the prism $D_n$ are $((5n+5)/2, 2)$-antimagic. They also conjectured that prisms with odd cycles of length $n$, $(n \geq 7)$, are $((n+7)/2, 4)$-antimagic. Bodendiek and Walther [10] proved that the following graphs are not $(a,d)$-antimagic; even cycles; paths of even order, stars; $C_3^{(k)}$; $C_4^{(k)}$; for $n \geq 2$ trees $T_{2n+1}$ that have a vertex that is adjacent to three or more pendant vertices; $n$-ary trees with at least two layers when $d = 1$; $K_{3,3}$; the Petersen graph; and $K_4$. They also proved that $P_{2k+1}$ is $(k, 1)$-antimagic; $C_{2k+1}$ is $(k + 2, 1)$-antimagic; if a tree of odd order $2k + 1$ $(k > 1)$ is
(a, d)-antimagic, then \( d = 1 \) and \( a = k \); if \( K_{4k} \) \((k \geq 2)\) is (a, d)-antimagic, then \( d \) is odd and \( d \leq 2k(4k - 3) + 1 \); if \( K_{4k+2} \) is (a, d)-antimagic, then \( d \) is even and \( d \leq (2k + 1)(4k - 1) + 1 \); and if \( K_{2k+1} \) \((k \geq 2)\) is (a, d)-antimagic, then \( d \leq (2k + 1)(k - 1) \). Lin, Miller, Simanjuntak, and Slamin [28] showed that no wheel \( W_n \) \((n > 3)\) has an (a, d)-antimagic labeling.

Bodendieck and Walther [10] noted the following relation of (a, d)-antimagic graph.

**Lemma 1.2 [10]** Let \( G(V, E) \) with \(|V| = p \geq 3\) and \(|E| = q \geq 2\), be an (a, d)-antimagic graph. Then a and d satisfy the following conditions a) and b)

a) \( a, d \in \mathbb{N} \) are positive solutions of the linear Diophantine equation

\[
2ap + p(p - 1)d = 2q(q + 1).
\]

b) \( a \geq \delta (\delta + 1)/2 \) where \( \delta \) denotes the minimum degree in \( G \).

Partitions seem very simple but play important roles in combinatorics, lie theory, representation theory, mathematical physics and theory of special functions. Euler, Ramanujan, Rademacher and Erdos revealed the beauty and uses of partitions [41]. In this chapter, we use a simple technique of partition in order to find a given graph is (a,d)-antimagic graph or not. In section 2 we use the technique of
partition to prove that graphs $K_1 + (K_1 \cup K_2)$, $C_4$ and $P_{2n}$ are not $(a,d)$-antimagic and prove that (i) the 1-sided infinite path $P_1$ is $(1,2)$-antimagic (ii) path $P_{2n+1}$ is $(n,1)$-antimagic and (iii) $(n+2,1)$-antimagic labeling is the unique $(a,d)$-antimagic labeling of $C_{2n+1}$. We also give a new relation, $a + (p - 1) \leq q(q + 1)/2 - (q - \Delta)(q - \Delta + 1)/2$ for any $(a, d)$-antimagic graph and give a direct relation between $(a, d)$-antimagic labeling and degree sequence of a $(a, d)$-antimagic graph.

We introduce the definition of odd $(a, d)$-antimagic in section 3 and prove that for an odd $(a,d)$-antimagic graph $G$ (i) $(2a + (p - 1) \leq p = 4q^2$ and (ii) $\delta^2 \leq a$ and $a + (p - 1) \leq (2q - \Delta) \Delta$ where $\delta$ and $\Delta$ are the minimum and maximum degrees of $G$ respectively. Also we prove that path $P_{2n+1}$ and cycle $C_{2n}$ are not odd $(a, d)$-antimagic graphs whereas odd $(2n+2, 2)$-antimagic is the unique odd $(a, d)$-antimagic labeling of cycle $C_{2n+1}, n \in \mathbb{N}$.

2. $(a,d)$-Antimagic Graphs Using Partition Technique.
   
   \( (a) \) and \( (b) \)

   We know that the Diophantine equations are a necessary condition for $(a,d)$-antimagic graphs. We give one more relation in the following theorem.
Theorem 2.1

Let $\delta$ and $\Delta$ denote the minimum and maximum degrees in $G$ and $p \geq 3, q \geq 2$ respectively. If $G (V, E)$ is an $(a, d)$-antimagic graph, then $a, d \in \mathbb{N}$ satisfy the following conditions.

a) $a, d \in \mathbb{N}$ are positive solutions of the linear Diophantine equation

$$2ap + p(p - 1)d = 2q(q + 1)$$

(b) $a \geq \delta(\delta + 1)/2$ and

(c) $a + (p - 1)d \leq q(q + 1)/2 - (q - \Delta)(q - \Delta + 1)/2$

Proof.

(a) We have $\sum_{v \in V} g_f(v) = 2(\sum_{e \in E} f(e))$.

This implies $a + (a + d) + \ldots + (a + (p - 1)d) = 2(1+2+\ldots+q)$ which implies $ap + p(p - 1)d / 2 = 2q(q + 1)/2$.

Hence we get $2ap + p(p - 1)d = 2q(q + 1)$.

(b) Since $\delta$ is the maximum degree of the graph $G$, at each vertex at least $\delta$ number of edges incident and hence the possible minimum value of the edges incident at a vertex is $\geq 1 + 2 + \ldots + \delta$. This implies, for every vertex $v \in V(G)$, $g_f(v) = \sum_{e \in I(v)} f(e) \geq 1 + 2 + \ldots + \delta = \delta(\delta + 1)/2$.

Hence $a \geq \delta(\delta + 1)/2$. 
Similarly the maximum possible induced vertex label is the sum of $\Delta$ distinct label of edges with possible maximum value each. Thus the possible maximum value of $\Delta$ edges is $q + (q - 1) + (q - 2) + \ldots + (q - \Delta + 1) = q(q + 1)/2 - (q - \Delta)(q - \Delta + 1)/2$. Since the maximum vertex value of the $(a,d)$—antimagic labeling of $G$ is $a + (p - 1)d$. We get $a + (p - 1)d \leq q(q + 1)/2 - (q - \Delta)(q - \Delta + 1)/2$. Hence we get the result.

**Theorem 2.2** [Relation between $(a,d)$—antimagic and degree sequence]

Let $G$ be a $(p, q)$ graph with degree sequence $d_1, d_2, \ldots, d_p$. Then $G$ is an $(a,d)$-antimagic if and only if the set of $p$ numbers formed by taking sum of exactly $d_i$ distinct numbers at a time out of $\{1, 1, 2, 2, \ldots, q, q\}$ is $\{a, a+d, \ldots, a+(p-1)d\}, i = 1, 2, \ldots, p$.

**Proof:** Let $G$ be an $(a,d)$- antimagic graph with degree sequence $d_1, d_2, \ldots, d_p$. Then the vertex labels of $G$ are $a, a+d, a+2d, \ldots, a+(p-1)d$ where $1, 2, \ldots, q$ are the edge labels on $G$. In $G$ each edge label is added exactly to its end vertices while considering the vertex labels of the graph and hence $\{a, a+d, \ldots, a+(p-1)d\}$ is the set of $p$ numbers formed by taking sum of exactly $d_i$ distinct numbers at a time out of $\{1, 1, 2, 2, \ldots, q, q\}, i = 1, 2, \ldots, p$. 


The converse part is obvious from the definition of \((a,d)\)-antimagic graph. 

The following problems illustrate the application of theorem 2.2.

**Problem 2.3** Show that the graph \( G = K_1 + (K_1 \cup K_2) \) is not \((a, d)\)-antimagic, using partition technique.

**Solution:**

![Figure 4.4. G = K₁+(K₁∪K₂)](image)

Consider the graph \( G = K_1 + (K_1 \cup K_2) \). See figure 4.4. Let 1, 2, 3, 4 be the labels of the four edges \( e_1, e_2, e_3, e_4 \) of \( G \). Using theorem 2.2, possible \((a,d)\)-antimagic sequences, if exist, are subsequences each of length 4 of \(1, 2, \ldots, 9 = 4 + 3 + 2\). Thus the possible \((a, d)\)-antimagic sequences are

a) 1, 2, 3, 4 with \( a = 1; d = 1 \);

b) 2, 3, 4, 5, with \( a = 2; d = 1 \); \ldots ,

f) 6, 7, 8, 9; g) 1, 3, 5, 7; h) 2, 4, 6, 8; and i) 3, 5, 7, 9 .

Consider each of the above \((a,d)\)-antimagic sequences and their corresponding possible vertex labels. On each sequence partitioning of vertex labels is done by one vertex labels with three distinct parts, two
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vertex labels each of two distinct parts and one vertex label without partitions.

Now the label of the vertex with degree 3 should be $1+2+3=6$ and so the sequences a) and b) are not possible. c) In this case the only possibility of partitioning scheme is $6=1+2+3$, $5=1+4$, $4$, $3=2+1$. d) (i) $7=4+2+1$, $6=4+2$, $5=1+4$ or $2+3$, $4$. (ii) $7=4+3$, $6=1+2+3$, $5=4+1$, $4$. e) $8=4+3+1$, $7=4+3$, $6=2+4$, $5$. f) $9=4+3+2$, $8$, $7$ and $6$ cannot be partitioned into the vertex labels $1, 2, 3$ and $4$ for $G$. g) (i) $1$, $3$, $5=2+3$, $7=1+2+4$ (ii) $1, 3, 5=1+4$, $7=1+2+4$ (iii) $1$, $3=1+2$, $5=2+3$, $7=1+2+4$ h) $8=4+3+1$; $7=4+3$, $6=2+4$, $5$ and i) $9=4+3+2$, $7=4+3$, $5=2+3$, $3$.

Thus in all these cases the vertex labels $1$, $2$, $3$ and $4$ are not occurring twice exactly. Hence ($a$, $d$)-arithmetic antimagic labeling does not exist for graph $K_1 + (K_1 \cup K_2)$, $a,d \in \mathbb{N}$.

Hence the graph $K_1 + (K_1 \cup K_2)$ is not ($a$, $d$)-antimagic, $a,d \in \mathbb{N}$.

**Problem 2.4** Show that $C_4$ is not ($a$, $d$)-antimagic, using partition technique.
**Proof:** Let 1, 2, 3, 4 be the edge labels of the four edges $e_1$, $e_2$, $e_3$, $e_4$ of $C_4$. The possible induced vertex labels are 3 ($= 1 + 2 = \text{minimum}$), 4, 5, 6, 7 ($= 3 + 4 = \text{maximum possible value}$).

Therefore 3, 4, 5, 6 and 4, 5, 6, 7 are the only possible (a, d)-antimagic sub-sequences of length 4 of the sequence 3, 4, 5, 6, 7.

Now consider the two cases separately. Since each vertex of $C_4$ is of degree two, in the partitioning of vertex labels each vertex label should be bipartitioned and the bipartitioned numbers should be 1, 1, 2, 2, 3, 3, 4, 4, exactly.

**Case 1.** Let 3, 4, 5, 6 be the vertex labels of $C_4$.

The possible bipartition of the vertex labels are $3 = 1 + 2$; $4 = 1 + 3$ and thereby 4 has to occur in the bipartition of 5 and 6 and so $5 = 1 + 4$ and $6 = 2 + 4$. This is not possible since 3 occurs only once and 1 occurs three times. Hence this case is not possible.

**Case 2.** Let 4, 5, 6, 7 be the vertex labels of $C_4$.

The possible bipartition of the above numbers are $7 = 3 + 4$, $6 = 2 + 4$. Here 4 occurred twice and so its chance is over and hence $5 = 2 + 3$, $4 = 1 + 3$. Thus 3 occurs 3 times. Hence this case is not possible.
Thus the graph $C_4$ is not an $(a,d)$-antimagic for any $a,d \in \mathbb{N}$. ■

**Problem 2.5** Show that the 1-sided infinite path $P_1$ is $(1, 2)$-antimagic labeling, using partition technique.

**Proof:** The starting and end vertices of the 1-sided infinite path $P_1$ is of degree one and all other vertices are of degree 2. If $P_1$ is $(1, 2)$-antimagic labeling, then the vertex labels of $P_1$, are $1, 3, 5, 7, \ldots$. Since 1 a vertex label in $P_1$ implies it is the label of a pendant vertex of $P_1$. Let the starting edge label be 1.

A partition scheme of the numbers is $a = 1$, $3 = 1 + 2$, $5 = 2 + 3$, $7 = 3 + 4$, $9 = 4 + 5$ and so on. The last number of the $(a,d)$-antimagic is $a + (\lvert V(G) \rvert - 1)d = 1 + (\lvert V(G) \rvert - 1) \times 2 = 2 \times (\lvert V(G) \rvert - 1)$. Thus the 1-sided infinite path $P_1$ is $(1, 2)$-antimagic. See figure 4.5.

![Figure 4.5](image)

**Problem 2.6** Show that every path $P_{2n+1}$, $n \geq 1$, of order $2n+1$ is exactly $(n,1)$-antimagic, using partition technique.
Proof: For \( n \geq 1 \), assume that \( P_{2n+1} \) be \((a, d)\)-antimagic, \( a, d \in \mathbb{N} \). Here \( p = 2n+1 \) and \( q = 2n \). The Diophantine equation (a) becomes, 
\[
2a (2n + 1) + (2n + 1) \cdot 2nd = 4n (2n + 1).
\]
This implies \( a = n(2 - d) \), \( a, d, n \in \mathbb{N} \). This equation has the unique solution \( d = 1 \) and \( a = n \).

Thus \((n, 1)\)-antimagic is the only possible \((a, d)\)-antimagic labeling of \( P_{2n+1} \), \( n \geq 1 \). The maximum value among the vertex label is \( a+(p-1) \cdot d = 3n \). And so the induced vertex sequence is \( n, n+1, \ldots, 3n-1, 3n \) and the edge labels are \( 1, 2, \ldots, 2n \). Consider the following bipartition scheme for edge labels \( n = 0 + n, n + 1 = 0 + (n + 1), n + 2 = 1 + (n + ), n + 3 = 1 + (n + 2), n+4 = 2+(n+2), \ldots, 3n-3 = (n-2) + (2n-1), 3n-2 = (n-1) + (2n-1), (3n-1) = (n-1) + 2n, 3n = n+2n \). In figure 4.6 we get the corresponding \((n, 1)\)-antimagic labeling of \( P_{2n+1} \).
Problem 2.7 Show that for \( n \geq 1 \), path \( P_{2n} \) is not \((a,d)\)-antimagic, \( a, d \in \mathbb{N} \).

Proof: If possible, assume that for \( n \geq 1 \), \( P_{2n} \) be an \((a, d)\)-antimagic, for some \( a, d \in \mathbb{N} \). Thus the Diophantine equation (a) becomes,

\[
2a (2n) + 2n (2n - 1) d = 2 (2n - 1) (2n).
\]

This implies \( 2a = (2n-1)(2-d) \). Hence \( d = 1 \) is the only possible value of \( d \) and the corresponding value of \( a \) is \((2n-1)/2\), which is not a natural number.

Hence the path \( P_{2n} \) is not an \((a, d)\)-antimagic for \( a, d \in \mathbb{N}, n \geq 1 \) \( \blacksquare \).

Problem 2.8 Show that for \( n \geq 1 \), \((n+2, 1)\)-antimagic labeling is the unique \((a,d)\)-antimagic labeling of \( C_{2n+1} \), using partition technique.

Proof Assume for \( n \geq 1 \), \( C_{2n+1} \) be an \((a,d)\)-antimagic. Applying the Diophantine equation (a) on \( C_{2n+1} \), we get,

\[
2a (2n+1) + (2n+1) \cdot 2nd = 2(2n+1) (2n+2).
\]

This implies \( a + nd = 2n+2, a, d, n \in \mathbb{N} \). This implies \( a = n (2-d) + 2, a, d, n \in \mathbb{N} \). This equation has two solutions \( d = 1, a = n+2 \) and \( d=2, a=2 \).

Case -1. Let \( a = 2, d = 2 \).

Therefore the vertex labels are \( 2, 4, 6, \ldots, 2 + (2n) \times 2 = 2 (2n+1) \).

This is not possible since \( 2 = 1+1 \), is the only bipartition of \( 2 \) which is not possible. Hence this case is not possible.
Case- 2. Let $a = n+2, d = 1$.

The possible vertex labels are $n+2, n+3, \ldots, (n+1) + (2n+1) = 3n+2$ and by considering a bipartition of each of them we have two sub cases.

Case -2.1. Consider $C_{2n+1}$ when $n$ is odd.

The partition scheme of the numbers are

$$a = n+2 = 1+(n+1), n+3 = 1+(n+2), n+4 = 2+(n+2), \ldots, \ldots, \ldots$$

$$2n+1 = (3n+1)/2 + (n+1)/2, 2n+2 = (n+1)/2 + 3(n+1)/2, \ldots, \ldots,$$

$$3n = 2n+n, 3n+1 = n+(2n+1), 3n+2 = (2n+1) + (n+1).$$

In figure 4.7(c), we show that the above partition scheme is a possible $(a,d)$-antimagic labeling. Also see figures 4.7.(a) and 4.7.(b).
Case - 2.2. Consider $C_{2n+1}$ when $n$ is even.

The partition scheme of the numbers are

\[ a = n+2 = 1+(n+1), \quad n+3 = 1+(n+2), \quad n+4 = 2+(n+2), \ldots, \]

\[ 2n+1 = (n+1)/2+(3n+1)/2, \quad 2n+2 = 3(n+1)/2+(n+1)/2, \ldots, \]

\[ 3n = 2n+n, \quad 3n+1 = n+(2n+1), \quad 3n+2 = (2n+1) + (n+1). \]

In figure 4.8.(c), we show that the $(a,d)$-antimagic labeling of $C_{2n+1}$.

Also see figures 4.8.(a) and 4.8.(b). Hence the result is proved.
Figure 4.8.(a). $C_7$

Figure 4.8.(b). $C_9$

Figure 4.8.(c). $C_{2n+1}$, n even
3. Odd (a,d)–Antimagic Graphs

In this section, we define a numbering which is slightly different from (a,d)-antimagic numbering. While dealing with (a,d)-arithmetic antimagic graph G, the edge labels are expected to be the consecutive numbers 1,2, ..., |E(G)|. In stead of consecutive integers, if we prefer consecutive odd numbers 1,3,5, ..., 2 |E(G)|-1 as the edge labels, we get the new definition.

**Definition 3.1** A connected graph G(V, E) of order ≥ 3 is said to be **odd (a, d)-antimagic** if and only if there exist positive integers a and d, and a bijective mapping f: E→ {1, 3, 5, ... , 2 |E| -1} such that the mapping g_f induced by f and defined by

\[ g_f : V \rightarrow \mathbb{N}, such that g_f(v) = \sum_{e \in I(v)} f(e), \]

is injective and

\[ g_f(V) = \{a, a+d, a+2d, ..., a+ (|V|-1)d\} \] where I(v) = \{e\in E(G)/e is incident to v\} v \in V.

**Example 3.2** C_5 is an odd (6,2)-antimagic. See figure 4.9.
Theorem 3.3 Let $G(V, E)$ be an odd $(a, d)$-antimagic graph, $a, d \in \mathbb{N}$, $p, q \geq 3$. Then it satisfies the following conditions.

(i) $a, d \in \mathbb{N}$ are positive solutions of the linear equation, $(a + L)\ p = 4q^2$
where $L = a + (p - 1)\ d$ is the last term of the arithmetic progression.

(ii) $a \geq \delta^2$ where $\delta$ is the minimum degree of $G$ and

(iii) $a + (p - 1)\ d \leq (2q - \Delta)\ \Delta$ where $\Delta$ is the maximum degree of $G$.

Proof. Let $G (V, E) \in \mathbb{E}$, $|V| = p \geq 3$, $|E| = 2q - 1 \geq 3$, be an odd $(a,d)$-antimagic graph.

(i) We have $2 \left( \sum_{e \in E} f(e) \right) = \sum_{v \in V} g_f(v)$ which implies

$$2(1+3+5\ldots+(2q-1)) = a + (a+d) + \ldots + (a+(|V|-1)d)$$

This implies $2q^2 = pa + (1+2+\ldots+(p-1))\ d$

$$= pa + p (p-1)\ d/2$$
This implies \[ 4q^2 = (2a + (p-1) d) p \]
\[ = (a + L) p \]

Where \( L = a + (p-1) d \) is the last term of the arithmetic progression.

ii) We have, \( g_f(v) = \sum_{e \in (v)} f(e) \geq 1 + 3 + 5 + \ldots + (2\delta - 1) = \delta^2, \forall v \in V \) which implies \( a \geq \delta^2 \) since \( \delta^2 \) is the least possible vertex label in \( G \).

iii) The maximum value among vertex labels is \( a + (p - 1) d \) of the \((a,d)\)-antimagic labeling of \( G \) and the possible maximum value of labels of \( \Delta \) number of edges is \( 2q - 1, 2q - 3, \ldots, 2q - (2\Delta - 1) = 2q\Delta - \Delta^2 = (2q - \Delta) \Delta \). This implies \( a + (p-1) d \leq (2q - \Delta) \Delta \). Hence the result follows.

\[ \square \]

**Theorem 3.4** For \( n \geq 1 \), path \( P_{2n+1} \) is not odd \((a,d)\)-antimagic.

**Proof:** Here \( p = 2n+1 \) and \( q = 2n \). If possible let \( P_{2n+1} \) be an odd \((a,d)\)-antimagic graph for some \( a,d \in N \). Then equation (i) of theorem 3.3 becomes \( 2a(2n+1) + (2n+1)2nd = 16n^2 \) which implies

\[ a + nd = 8n^2/(2n+1). \]

For, \( a,d,n \in N \), \( a + nd \) is an integer which implies \( 8n^2/(2n+1) \) is an integer. \( 8n^2/(2n+1) \) is an integer if and only if \( n^2/(2n+1) \) is an integer

if and only if \( 2n+1 \) divides \( n^2 \)
if and only if $2n+1$ divides $n$ which is a contradiction, $n \in \mathbb{N}$. Hence $a+nd \neq 8n^2/(2n+1)$ and so $P_{2n+1}$ is not an odd $(a,d)$-antimagic graph.

**Theorem 3.5** For $n \geq 1$, odd $(2(n+1), 2)$-antimagic labeling is the unique odd $(a,d)$-antimagic labeling of $C_{2n+1}$.

**Proof.** Assume $C_{2n+1}$ be an odd $(a,d)$-antimagic labeling, $n \geq 1$. Using equation (i) of theorem 3.3, we get,

$$(2a + 2nd) (2n + 1) = 4 (2n+1)^2$$

which implies

$$a+nd = 4n+2, \ a, \ d, \ n \in \mathbb{N}.$$ Here $a, a+d, a+2d, \ldots, a+2nd$ are even numbers and so $a$ and $d$ are even numbers.

This implies $n (4-d) = a-2, \ a, d, n \in \mathbb{N}$ and $a \geq 4$ and $d \geq 2$.

Thus the equation $a - 2 = n (4 - d), \ a \geq 4, \ d \geq 2$ and $a, d, n \in \mathbb{N}$ has the unique solution, $d = 2$ and $a = 2 (n+1), n \in \mathbb{N}$. The following is the odd $(2n + 2, 2)$-antimagic labeling on $C_{2n+1}$.

Let $C_{2n+1} = (e_1, e_2, e_3, \ldots, e_{2n+1})$. Define edge labeling $f$ such that

$f(e_{2k+1}) = 2k+1, k=0, 1, \ldots, n$ and $f(e_{2k}) = 2n+1 + 2k, k=1, 2, \ldots, n$. See figures 4.10.(a) to 4.10.(d). Clearly $f$ is a $(2n+2, d)$-antimagic labeling on $C_{2n+1}$.
Thus $(2n+2,2)$-antimagic labeling is the only odd $(a,d)$- antimagic labeling of $C_{2n+1}$.

**Figure 4.10.(a).** $C_3$

**Figure 4.10.(b).** $C_5$

**Figure 4.10.(c).** $C_7$
Figure 4.10.(d). $C_{2n+1}$

**Note 3.6** We proved that path $P_{2n+1}$ is exactly $(n,1)$-antimagic, see problem 2.6 but it is not an odd $(a, d)$-antimagic, theorem 3.4, $n \in \mathbb{N}$. It seems that every odd $(a,d)$-arithmetic graph is $(a,d)$-antimagic but the converse need not be.