CHAPTER - 2

CERTAIN RESULTS IN BIRTH AND DEATH PROCESSES

2.0 INTRODUCTION

BD processes, with a finite or countable number of states, play a central role in stochastic modeling for applied probabilities. For different aspects of BD models, one may refer to Andersen (1991), Bahattacharya and Waymire (1990), Gross and Harris (1985), Karlin and Taylor (1975), Kijima (1997) and Medhi (2003). Over several years, the researchers have popularized the theory and applications of MCs with several discussions on BDPs in terms of gambler’s ruin probabilities, (See Feller (1950)).

This chapter presents certain interesting results in primal BDPs and their dual processes. The relationship between BDPs have been analysed. In Section 2.1 the relationship between the transient probability functions of a primal BDP and its dual has been developed. In Section 2.2, the problem of determining busy period distributions in a queueing system modeled by a BDP is shown to be equivalent to finding the transient probability functions in a related BDP. This equivalence of problems is then independently verified for the classical single server queueing system. Then, the two celebrated problems of determining ruin probabilities and steady state distributions on a finite BDP are shown to be linked the obtained results on dual processes. The equivalence of these two problems and an example for the classical gambler’s ruin problem are presented in Section 2.3. Finally, a characterization of the n – step transition probability within a finite state, diagonalizable MC has been derived in Section 2.4. The obtained result uses Cayley-Hamilton theorem to determine the recurrence relations. As an application of this characterization, a method is outlined to determine the transient
probability function of an arbitrary BDP on a finite state space. Section 2.5 provides a solution method associated with BD chain.

2.1 DUAL BIRTH AND DEATH PROCESSES AND CERTAIN RESULTS

For the sake of convenience, we describe in this chapter the state rate transitions in the form of figures rather than presenting them in the form of tables. Consider a general recurrent BDP having transition birth rate $\lambda_i$, for $i = 0, 1, 2 \ldots$ and transition death rates $\mu_i$ for $i = 1, 2, 3 \ldots$ as shown in the state rate transition diagram, Figure 2.1.1. All these rates are assumed to be positive numbers. The state space may be finite or

![State rate transition diagram](image)

Figure 2.1.1

Throughout this chapter, it is assumed that the transition rates are uniformly bounded. The transient probability function $P_{ij}(t)$, where $i, j = 0, 1, 2, 3, \ldots$ may be found by solving the Kolmogorov backward or forward system of differential equations, (See Anderson (1991), Bhattacharya and Waymire (1990) and Gross and Harris (1985)). This system of differential equation may be written in matrix form as

$$P'(t) = Q \cdot P(t) = P(t) \cdot Q$$

where

$$P(t) = \begin{bmatrix}
P_{0,0}(t) & P_{0,1}(t) & \ldots & P_{0,n}(t) \\
P_{1,0}(t) & P_{1,1}(t) & \ldots & P_{1,n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
P_{n,0}(t) & P_{n,1}(t) & \ldots & P_{n,n}(t)
\end{bmatrix}$$

25
is the matrix of transition probability functions and

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \vdots \\ \mu_1 & - (\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \vdots \\ 0 & \mu_2 & - (\lambda_2 + \mu_2) & \lambda_2 & 0 & \vdots \\ 0 & 0 & \mu_3 & - (\lambda_3 + \mu_3) & \lambda_3 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \end{bmatrix}$$

is the transition rate matrix. For BDPs having birth rates $\lambda_i, i = 0, 1, 2, \ldots$ and death rates $\mu_i, i = 1, 2, 3, \ldots$ uniformly bounded, the solution of the Kolmogorov backward equation may be written as $P(t) = \exp(Qt)$.

The dual process of the BDP of Figure 2.1.1 has a state rate transition diagram as shown in Figure 2.1.2.

![Figure 2.1.2](image)

The transient probability functions of this dual process are denoted by $P_{i,j}^*(t)$. The following relationship holds between a general BD system and its dual BDP.

**Remark 2.1.1**

If $P_{i,j}(t)$ and $P_{i,j}^*(t)$ are the transient probability functions of the BDPs corresponding to Figure 2.1.1 and Figure 2.1.2 respectively, then

$$P_{i,j}(t) = \sum_{k=0}^{\infty} [P_{j,k}^*(t) - P_{j-1,k}(t)] \text{ and } P_{i,j}^*(t) = \sum_{k=0}^{j} [P_{j,k}(t) - P_{j+1,k}(t)]$$

for all states $i, j = 0, 1, 2, 3, \ldots$ with the convention that $P_{i,k}^*(t) = 0$ if $k > -1$. 

26
The proof of this result appears as Proposition 2.3 in Anderson (1991). The proof depends on the forward and backward equations and some algebraic simplifications. Consequently, if the transient probability functions in either the primal BDP or dual BD system are known, then the transient probability function in the other system are known as well.

We shall now proceed to consider the finite recurrent BD chain having transition probabilities given in Figure 2.1.3.

![Figure 2.1.3](image)

In addition to the usual assumptions that all letters in Figure 2.1.3 represent fractions inclusively between 0 and 1 and that

- \( r_0 + p_0 = 1 \)
- \( q_1 + r_1 + p_1 = 1 \)
- \( q_2 + r_2 + p_2 = 1 \)
- \( \ldots, q_{H-1} + r_{H-1} + p_{H-1} = 1 \)
- and \( q_H + r_H = 1 \).

We also assume that

- \( p_0 + q_1 \leq 1 \)
- \( p_1 + q_2 \leq 1 \)
- \( p_2 + q_3 \leq 1, \ldots, p_{H-1} + q_{H} \leq 1 \)
This implies that the $\rho'$ s in the following dual BD Figure 2.1.4 is nonnegative fractions less than 1.

\[
\begin{array}{cccccccc}
& & & & & & & \\
& 0 & 1 & 2 & \cdots & H & & \\
& & & & & & & \\
\rho_0 & q_1 & \rho_1 & q_2 & \rho_2 & q_3 & \cdots & \rho_{H-1}
\end{array}
\]

Figure 2.1.4

where

\[
\begin{align*}
\rho_0 + \rho_0 + q_1 &= 1, \\
\rho_1 + \rho_1 + q_2 &= 1, \\
\rho_2 + \rho_2 + q_3 &= 1 \\
& \quad \vdots \\
\rho_{H-1} + \rho_{H-1} + q_H &= 1
\end{align*}
\]

This absorbing BD chain shown in Figure 2.1.4 is the dual BD chain of the BD chain given in Figure 2.1.3.

The Remark 2.1.1 holds good for infinite BDPs also. We shall now present this new result.

**Theorem 2.1.1**

If $P_{i,j}^{(n)}$ and $P^{*}_{i,j}^{(n)}$ are the $n$-step transient probabilities of the BD chains corresponding to Figures 2.1.3 and 2.1.4 respectively, then

\[
\begin{align*}
P_{r,s}^{(n)} &= \sum_{k=r}^{H} \left[ P_{s,k}^{* (n)} - P_{s-1,k}^{* (n)} \right] \\
and \quad P^{*}_{r,s}^{(n)} &= \sum_{k=0}^{r} \left[ P_{s,k}^{(n)} - P_{s+1,k}^{(n)} \right]
\end{align*}
\]

for $n \geq 0$ and for all states $r, s = 0, 1, 2, 3, \ldots H$ with the convention $P_{s,k}^{(n)} = 0$ if $k > -1$
Proof

We use mathematical induction to prove this result. For \( n = 0 \),

\[
P_{r,s}^{(0)} = \sum_{k=r}^{H} \left[ P_{s,k}^{(0)} - P_{s-1,k}^{(0)} \right]
\]

holds good by substituting the initial conditions that

\[
P_{r,s}^{(0)} = \delta_{r,s} = P_{r,s}^{(0)} \quad \text{for all states } r,s = 0,1,2,...,H
\]

where \( \delta_{r,s} \) is the Kronecker delta and using the convention \( P_{-1,k}^{(n)} = 0 \) if \( k > -1 \).

Suppose, \( 2 \leq s \leq H-1 \). As the induction hypothesis, we assume that

\[
P_{r,s}^{(n)} = \sum_{k=r}^{H} \left[ P_{s,k}^{(n)} - P_{s-1,k}^{(n)} \right]
\]

holds and show that

\[
P_{r,s}^{(n+1)} = \sum_{k=r}^{H} \left[ P_{s,k}^{(n+1)} - P_{s-1,k}^{(n+1)} \right].
\]

By Chapman – Kolmogorov equations for the BD chain of Figure 2.1.3 we find that

\[
P_{r,s}^{(n+1)} = \sum_{j=0}^{H} P_{r,j}^{(n)} P_{j,s}^{(l)}.
\]

Since, the chain in Figure 2.1.3 is a BD chain and hence the one step transition probabilities are zeros except for the above mentioned three transitions. This simplifies to

\[
P_{r,s}^{(n+1)} = P_{r,s-1}^{(n)} P_{s-1,s}^{(l)} + P_{r,s}^{(n)} P_{s,s}^{(l)} + P_{r,s+1}^{(n)} P_{s+1,s}^{(l)}.
\]

More specifically we have

\[
P_{r,s}^{(n+1)} = P_{r,s-1}^{(n)} P_{s-1,s} + P_{r,s}^{(n)} + P_{r,s+1}^{(n)} Q_{s+1}.
\]

However

\[
P_{r,s}^{(n+1)} = \sum_{k=r}^{H} \left[ P_{s-1,k}^{(n)} - P_{s-2,k}^{(n)} \right] P_{s-1} + \sum_{k=r}^{H} \left[ P_{s,k}^{(n)} - P_{s-1,k}^{(n)} \right] P_{s} + \sum_{k=r}^{H} \left[ P_{s+1,k}^{(n)} - P_{s,k}^{(n)} \right] P_{s+1}
\]

by the induction hypothesis. Replacing \( s \) by \([1 - p_s - q_s] \) and rearranging terms produces

\[
P_{r,s}^{(n+1)} = \sum_{k=r}^{H} \left[ P_{s-1,k}^{(n)} - P_{s-2,k}^{(n)} \right] P_{s-1} + \sum_{k=r}^{H} \left[ P_{s,k}^{(n)} - P_{s-1,k}^{(n)} \right] [1 - p_s - q_s] + \sum_{k=r}^{H} \left[ P_{s+1,k}^{(n)} - P_{s,k}^{(n)} \right] q_{s+1}
\]

by the induction hypothesis. Replacing \( s \) by \([1 - p_s - q_s] \) and rearranging terms produces

\[
\sum_{k=r}^{H} (p_s P_{s-1,k}^{(n)} + [1 - p_s - q_{s+1}] P_{s,k}^{(n)} + q_{s+1} P_{s+1,k}^{(n)}) = \sum_{k=r}^{H} (P_{s-1,k}^{(n)} + P_{s,k}^{(n)} + P_{s+1,k}^{(n)}).
\]

\[\]
By the definition of $p_s$ from Figure 2.1.4

$$P_{r,s}^{(n+1)} = \sum_{k=r}^{H} (p_{s} P_{s+1,k}^{(n)} + p_{s+k} P_{s,k}^{(n)} + q_{s+1} P_{s+1,k}^{(n)}) - \sum_{k=r}^{H} (p_{s-l} P_{s-2,k}^{(n)} + p_{s-l} P_{s-k}^{(n)} + q_{s} P_{s,k}^{(n)})$$

Substituting in the transition probabilities of the chain of Figure 2.1.4 and writing this last result in Chapman – Kolmogorov equation form gives

$$P_{r,s}^{(n+1)} = \sum_{k=r}^{H} (P_{s-l}^{(l)} P_{s-k}^{(n)} + P_{s,k}^{(l)} P_{s+1,k}^{(n)} - \sum_{k=r}^{H} (P_{s-l}^{(l)} P_{s-k}^{(n)} + P_{s,l-s}^{(l)} P_{s-l,k}^{(n)} + P_{s-l,a}^{(l)} P_{a,k}^{(n)}))$$

Thus, we have

$$P_{r,s}^{(n+1)} = \sum_{k=r}^{H} (P_{s,k}^{(n+1)} - P_{s-l,k}^{(n+1)})$$

by the Chapman – Kolmogorov equations of the BD chain of Figure 2.1.4. This completes the induction step and establishes the first equality in Theorem 2.1.1 whenever $2 \leq s \leq H - 1$. If $s = 0, 1$ or $H$, the preceding argument may be suitably modified to establish the desired result.

We then show that,

$$P_{r,s}^{(n)} = \sum_{k=0}^{r} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}]$$

Let us consider the sum $\sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}]$. By what we have just proved we find that

$$P_{r,s}^{(n)} = \sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}]$$

and

$$P_{s+1,k}^{(n)} = \sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}]$$

Hence, by substitution

$$\sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}] = \sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}] - \sum_{k=0}^{H} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}]$$

However, by cancelling terms, this equation simplifies to the following telescoping series
which in turn reduces to $P_{r,s}^{(n)} - P_{s+1,s}^{(n)}$ which equals $P_{r,s}^{(n)}$ since $P_{r,s}^{(n)} = 0$ for $0 \leq s \leq H$ because state $-1$ is an absorbing state in the BD chain of Figure 2.1.4. Thus, we have shown that

$$\sum_{k=0}^{s} [p_{s,k}^{(n)} - p_{s+1,k}^{(n)}] = P_{r,s}^{(n)}$$

which is the second equality of Theorem 2.1.1. This completes the proof of Theorem 2.1.1.

2.2 BUSY PERIOD DISTRIBUTIONS AND TRANSIENT PROBABILITY FUNCTIONS:

We now proceed to relate the busy period distribution and transient probability functions of Theorem 2.1.1. For a given queueing system, the problems of determining transient probability functions and busy period distributions are usually regarded as two different, but related, questions. In this section, it is shown that these two problems for queueing systems modeled by BDPs are more closely linked than previously realized. An elegant connection between these problems is described in terms of dual BDPs.

Let us recall the busy period distribution of the BDP described in Figure 2.1.1, $P_{b,	heta}^{b}(t)$. We observe that $P_{b,	heta}^{b}(t)$ satisfies the system of forward or backward equations associated with Figure 2.2.1, below (See Gross and Harris (1985)).

![Figure 2.2.1](image-url)
Now let us consider the BDP having transient rates as shown in Figure 2.2.2.

![Figure 2.2.2](image)

Let $P(t)$ represent the transient probability functions of the BDP of Figure 2.2.2. Then by section 2.1, the dual of the BDP given in Figure 2.2.2 is given in Figure 2.2.3.

![Figure 2.2.3](image)

and has transient probabilities $\tilde{P}_{i,j}(t)$. We may observe that Figure 2.2.3 is almost the same as the busy period distribution transitions as given in Figure 2.2.1. But, the states are offset by 1. Hence, we have

$$\tilde{P}_{i+1,-1}(t) = P_{i,0}(t) \text{ for } i = 1, 2, 3, \ldots$$

**Theorem 2.2.1**

The busy period distribution $P_{i,0}(t)$ of Figure 2.2.1 is related to the transient probability functions of Figure 2.2.2 according to

$$P_{i,0}(t) = 1 - \sum_{k=0}^{i-1} \tilde{P}_{0,k}(t) \text{ for } i = 1, 2, 3, \ldots$$

(2.2.1)

**Proof**

By Remark 2.1.1 we have

32
\[ \tilde{P}_{i,j}(t) = \sum_{k=0}^{j} [\tilde{P}_{j,k}(t) - \tilde{P}_{j+1,k}(t)] \]

for \( i, j = 0, 1, 2, \ldots \). Sum over \( j = 0, 1, 2 \ldots \) and subtract from 1 to get

\[ \tilde{P}_{i-1}^*(t) = 1 - \sum_{k=0}^{i-1} \tilde{P}_{0,k}(t) \]

Replace \( i \) by \( i-1 \) and substitute \( \tilde{P}_{i-1}^*(t) = P_{i,0}^b(t) \) to obtain the desired result.

\[ P_{i,0}^b(t) = 1 - \sum_{k=0}^{i-1} \tilde{P}_{0,k}(t) \text{ for } i = 1, 2, 3 \ldots \] (2.2.2)

This completes the proof.

We may observe that a special case of interest occurs when \( i = 1 \). This gives

\[ P_{1,0}^b(t) = 1 - \tilde{P}_{0,0}(t) \] (2.2.3)

We shall now provide an example

**Example 2.2.1: M/M/1 Queueing Model**

It is instructive to verify equation (2.2.3) directly for the classical single server queueing system. From Gross and Harris (1985), we know that

\[ \frac{d}{dt} P_{1,0}^b(t) = t^{-1} (\mu/\lambda)^{1/2} e^{-(\lambda+\mu)t^2/2} \]

A convenient expression for \( \tilde{P}_{0,0}(t) \) is found in Parthasarathy (1987) and is restated below

\[ \tilde{P}_{0,0}(t) = \int_0^t \left[ I_2(2y\sqrt{\lambda \mu}) - I_0(2y\sqrt{\lambda \mu}) e^{-(\lambda+\mu)y} \right] dy \]

where \( I_0, I_1, I_2 \), are modified Bessel functions of the first kind.

Now, it follows from the Fundamental Theorem of Calculus and elementary properties of Bessel functions, (See Gross and Harris (1985)). We have

\[ \frac{d}{dt} P_{1,0}^b(t) = -\frac{d}{dt} \tilde{P}_{0,0}(t) \]
This is equivalent to equation (2.2.3). This example provides a link between sections 2.10 and 2.11 of Gross and Harris (1985).

The problem of determining busy period distributions of a queueing system modeled by a given BDP is equivalent, by equation (2.2.1), to finding transient probability functions in a related BDP. This connection unifies transient analysis for queues that are modeled by BDPs and provides practitioners an alternative way to calculating time dependent probabilities.

2.3. STEADY STATE DISTRIBUTIONS AND RUIN PROBABILITIES IN BIRTH–DEATH CHAINS

Suppose, we consider a BD chain of a finite state space having transition probability as in Figure 2.1.3. For convenience, we reproduce this diagram as Figure 2.3.1.

![Figure 2.3.1](image)

Let us assume that the BD chain in Figure 2.3.1 has a steady distribution, \( \pi(j) \), for \( j = 0, 1, 2... H \). It follows from a well known formula for the stationary distribution, (See Hoel Port and Stone (1972)) that

\[
\pi(j) = \pi_j / \sum_{k=0}^{H} \pi_k \quad \text{for } j = 0, 1, 2, \ldots H \\
\]

(2.3.1)

where

\[
\pi_j = \begin{cases} 
1 & \text{if } j = 0 \\
\frac{p_0 p_1 p_2 \cdots p_{j-1}}{q_1 q_2 q_3 \cdots q_j} & \text{if } 1 \leq j \leq H 
\end{cases}
\]

This formula comes from solving a system of linear equations. The ruin probability is also derived in Hoel, Port and Stone (1972) as,
\[ P_j(T_H < T_0) = \frac{\sum_{k=0}^{\infty} \gamma_k}{\sum_{k=0}^{\infty} \gamma_k} \quad \text{for } j = 0, 1, 2, \ldots, H \]  

(2.3.2)

where \( T_k \) represents the time to first reach state \( k \) and where

\[
\gamma_k = \begin{cases} 
1 & \text{if } k = 0 \\
\frac{q_1 q_2 \ldots q_k}{p_1 p_2 \ldots p_k} & \text{if } 1 \leq k < H 
\end{cases}
\]

This formula calculates the probability of reaching state \( H \) before we reach state 0 assuming we start at state \( j \). Expression (2.3.2) is determined by a one-step backwards analysis and skillful manipulation of recurrence relations.

Comparison of expressions (2.3.1) and (2.3.2) reveal the formulas are similar but with the p's and q's switched. It turns out that the link between the original BD chain and its dual BD chain is the key to realizing how these two problems are related.

Suppose the original BD chain follows Figure 2.1.3 and its dual BD chain is diagramed in Figure 2.1.4. Let us assume that all the conditions on \( p_0, p_1, \ldots, p_{H-1}; q_1, q_2, \ldots, q_H; r_0, r_1, r_2, \ldots, r_H \) and \( p_0, p_1, \ldots, p_{H-1} \) stated in Figure 2.1.3 and Theorem 2.1.1 are still valid. This ensures that both the original and dual BD chains have nonnegative transition probabilities as pictured. We again assume the chain of Figure 2.1.3 has a steady state distribution, \( \pi(j) \), for \( j = 0, 1, 2, \ldots, H \).

By Theorem 2.1.1 we have

\[
P_{r,s}^{(n)} = \sum_{k=0}^{H} [p_{s,k}^{*} - p_{s-1,k}^{*}] z
\]

for \( n \geq 0 \) and for all states \( r, s = 0, 1, 2, 3, \ldots H \). Substitute \( r = i \) and sum the preceding expression over \( s \) from \( j \) to \( H \), obtaining

\[
\sum_{s=j}^{H} P_{r,s}^{(n)} = \sum_{s=j}^{H} \sum_{k=s}^{H} [p_{s,k}^{*} - p_{s-1,k}^{*}] = \sum_{k=0}^{H} \sum_{s=j}^{H} [p_{s,k}^{*} - p_{s-1,k}^{*}] = \sum_{k=i}^{H} [p_{i,k}^{*} - p_{i-1,k}^{*}] = 1 - \sum_{k=i}^{H} p_{j,k}^{*}
\]
by simplifying the telescoping series and noting that H is an absorbing state. Now suppose \( n \to \infty \). On the left hand side, there is convergence to the steady state probabilities of the original BD chain. On the right hand side, all but one term converges to 0 since all states except \( k = H \) are transient states and therefore the n-step probabilities vanish as \( n \to \infty \). So

\[
\sum_{s=j}^{H} \pi(s) = 1 - \lim_{n \to \infty} P_{j \rightarrow H}^{(n)} = 1 - P_{j \rightarrow 1}[T_{H} < T_{-1}].
\]

By considering the complement of this equation, \( \sum_{s=0}^{j-1} \pi(s) = P_{j \rightarrow H}[T_{H} < T_{-1}] \)

This last expression is nothing but the probability of hitting state H first before hitting state -1 in the dual BD chain of Figure 2.1.4 equals a sum of the steady state probabilities in the original BD chain of Figure 2.1.3. This explains why the p's and q's are reversed. Therefore, a solution of the steady distribution on Figure 2.1.3 provides a way to determine the ruin probabilities and vice versa. Note a similar unification of these two problems also occurs for BDPs as well. Limit arguments, such as those in Hoel, Prot and Stone (1972) may now be use to extend these results to the countable state space setting.

### 2.3.1: RUIN PROBABILITIES IN BIRTH AND DEATH PROCESSES

Consider the BD chain in Figure 2.3.2. Suppose \( p + q = 1 \) where \( p, q > 0, p \neq q \). We which to determine \( P_{j \rightarrow 1}(T_{11} < T_{-1}) \) where \( j = 1, 2, 3, \ldots, H \). This is the classical gambler's ruin problem except with an extra state, -1 where the usual roles of p and q have been reversed.

![Figure 2.3.2](image-url)
By the preceding argument, we have \( \sum_{s=0}^{j-1} \pi(s) = P_{j-1}(T_H < T_{-1}) \), where \( \pi(j) \) is the steady state distribution of the following BD chain.

\[
\begin{align*}
q & \quad p \quad p \quad p \quad p \quad p \\
0 & \quad 1 & \quad 2 & \quad 3 & \cdots & \quad H-1 & \quad H
\end{align*}
\]

**Figure 2.3.3**

By (2.3.1)

\[
\pi(j) = \pi_j \left( \sum_{k=0}^{H} \pi_k \right)^{-1} \text{ where } j = 0, 1, 2, \ldots, H \text{ and }
\]

\[
\pi_j = \begin{cases} 1 & \text{ if } j = 0 \\ \frac{p^j}{q^j} & \text{ if } 1 \leq j \leq H \end{cases}
\]

By summing the finite geometric sequence

\[
\pi(j) = \pi_j \left( 1 - \frac{p}{q} \right) \left( 1 - \left[ \frac{p}{q} \right]^{H+1} \right)^{-1}
\]

Again, summing the geometric sequence simplifies \( \sum_{s=0}^{j-1} \pi(s) \) as

\[
\sum_{s=0}^{j-1} \pi(s) = \sum_{s=0}^{j-1} \frac{\pi_s \left( 1 - \frac{p}{q} \right) \left( 1 - \left[ \frac{p}{q} \right]^{H+1} \right)}{\left( 1 - \left[ \frac{p}{q} \right] \right) \left( 1 - \left[ \frac{p}{q} \right] \right) \left( 1 - \left[ \frac{p}{q} \right] \right) \left( 1 - \left[ \frac{p}{q} \right] \right)} = P_{j-1}(T_H < T_{-1})
\]

This produces the desired ruin probabilities which are seen to agree with the expression (2.3.2) once the values of \( p \) and \( q \) are known and \( H \) is replaced by \( H+1 \).
2.4. TRANSITION PROBABILITIES OF FINITE, DIAGONALIZABLE
MARKOV CHAINS

In this section, we characterize the n-step transition probability of any Markov
chain (MC) on a finite state space having a 1-step transition probability matrix \( P \), which
is diagonalizable over the real numbers. We refer to these chains as finite, diagonalizable
MC. Let \( S = \{1, 2, 3, \ldots, N\} \) be the state space.

By Cayley–Hamilton Theorem, we know that \( f(P) = 0 \) where \( O \) is the zero
matrix, that is \( O \) is the \( N \) by \( N \) matrix of zeros and \( f(x) = \det(P-xI) \) and \( I \) is the usual \( N 
by N \) identity matrix. Recall a minimal polynomial, \( m(x) \) (with regards to \( P \)), is a
polynomial with real coefficients of lowest degree such that \( m(P) = 0 \). By Theorem 7.4 on
page 114 of Nering (1970), \( P \) is diagonalizable if and only if \( m(x) \) factors into distinct
linear factors with real coefficients. So suppose the distinct real roots of \( m(x) \) are
\( z_1, z_2, z_3, \ldots, z_k \) where \( 1 \leq k \leq N \). By Theorem 4.2 on page 101 of Nering (1970), the
\( z_1, z_2, \ldots, z_k \) are also roots of \( f(x) \), that is, they are eigen values of \( P \).

We assume that \( z_1 < z_2 < \ldots < z_k \). From Theorem 7.10 (e) of Nobel and
Daniel (1988) we have

\[
|z_h| \leq \|P\|_\infty \quad \text{for } 0 \leq h \leq N
\]

where \( \|P\|_\infty = \max_{i,j} |P_{i,j}| \). For \( P \) a stochastic matrix, \( \|P\|_\infty = 1 \). Hence,

\[
|z_h| \leq 1 \quad \text{for } 0 \leq h \leq N
\]

Further we observe that the vector \((1,1,\ldots,1)\) is an eigenvector of \( P \) with eigen
value of 1, we know \( z_k = 1 \). Suppose the minimal polynomial \( m(x) \) has the form

\[
m(x) = a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \ldots + a_{k-1} x + a_k
\]

with real coefficients and \( a_0 \neq 0 \).

But \( m(P) = 0 \) or

\[
0 = m(P) = a_0 P^k + a_1 P^{k-1} + a_2 P^{k-2} + \ldots + a_{k-1} P + a_k.
\]
Then, for \( n \geq k \)

\[ a_0P^n + a_1P^{n-1} + a_2P^{n-2} + \ldots + a_{k-1}P^{n-k+1} + a_kP^{n-k} = 0 \]

Thus,

\[ p^n = \frac{-\left(\frac{a_{k-1}}{a_0}\right)p^{n-1} - \left(\frac{a_{k-2}}{a_0}\right)p^{n-2} - \ldots - \left(\frac{a_{k-1}}{a_0}\right)p^{n-k+1} - \left(\frac{a_k}{a_0}\right)p^{n-k}}{a_0} \]

But \( p^n \) are \( P_{ij}^{(n)} \), the n-step transition probabilities. Hence,

\[ p_{ij}^n = \frac{-\left(\frac{a_{k-1}}{a_0}\right)p_{ij}^{(n-1)} - \left(\frac{a_{k-2}}{a_0}\right)p_{ij}^{(n-2)} - \ldots - \left(\frac{a_{k-1}}{a_0}\right)p_{ij}^{(n-k+1)} - \left(\frac{a_k}{a_0}\right)p_{ij}^{(n-k)}}{a_0} \]

for all \( i, j \) in \( S = \{1, 2, 3, \ldots, N\} \) and \( n \geq k \). That is, \( P_{ij}^{(n)} \) satisfies the same linear, constant coefficient recurrence relation for any \( i, j \) in \( S = \{1, 2, 3, \ldots, N\} \). Hence, the characteristic equation of these recurrence relations is \( m(x) \) for each \( i, j \) in \( S = \{1, 2, 3, \ldots, N\} \). Thus,

\[ P_{ij}^{(n)} = A_{ij}z_1^n + A_{ij}^2z_2^n + \ldots + A_{ij}^kz_k^n \quad (2.4.1) \]

for \( n \geq k \) where the coefficients \( A_{ij}^h \) for \( 1 \leq h \leq k \) may be determined in terms of \( z_h \) for \( 1 \leq h \leq k \) and \( P_{ij}^{(n)} \) for \( k \leq n \leq 2k - 1 \) by solving a system of linear equations. Here, we assume that \( z_h \) for \( 1 \leq h \leq k \) and \( P_{ij}^{(n)} \) for \( k \leq n \leq 2k-1 \) are known.

We now proceed to establish the following characterization of n-step transitional probabilities in finite, diagonalizable MCs.

**Theorem 2.4.1**

Consider \( P \) the one-step transition probability matrix of a MC on a finite state space. Also, suppose that \( P \) is diagonalizable over the real numbers. Then

\[ P_{ij}^{(n)} = A_{ij}z_1^n + A_{ij}^2z_2^n + \ldots + A_{ij}^{k-1}z_{k-1}^n + A_{ij}^kz_k^n \quad (2.4.2) \]

for \( n \geq k \) where \(-1 \leq z_1 < z_2 < \ldots < z_k = 1\) and \( 1 \leq k \leq N \) and where the coefficients, \( A_{ij}^h \) for \( 1 \leq h \leq k \) does not depend on \( n \).
Proof

If \( z_1 = -1 \), then the limit of the right hand side of the equation (2.4.2) as \( n \to \infty \) exists and is nonzero because the chain has a steady state distribution, (See theorem 7 of Hoel, Port and Stone (1972)) while the limit of the right hand side of the equation (2.4.2) does not exists unless \( A_{i,j} = 0 \).

Remark 2.4.1: Suppose, \( P \) is the 1-step transition probability matrix of a finite, diagonalizable recurrent, aperiodic Markov chain, then \(-1 < z_1 \) or \( A_{i,j} = 0 \) in equation (2.4.2).

Theorem 2.4.1 reveals that on a finite, diagonalizable MC, the n-step transition problems are all described by the same constant coefficient, linear recurrence relation and therefore the n-step transition probabilities can each be expressed as a linear combination of powers of roots corresponding to a single characteristic equation. These roots take values in the interval \([-1,1]\). Moreover, one of these roots is always 1. If the steady state distribution exists, the roots are in the interval \((-1,1]\) and the coefficient of root 1 is the steady state probability of the MC. A similar result has been derived by a different method for BD chains (See Mohanty (1991)). Theorem 2.4.1 may be generalized by relaxing the condition that \( P \) is diagonalizable. Here, we would replace (2.4.2) with an expression having coefficients that would vary with \( n \) as described by the theory of linear, constant coefficient, recurrence relations having multiple roots.

We shall now proceed to provide

2.5 A SOLUTION METHOD ASSOCIATED WITH BIRTH-DEATH CHAIN

The transient probability functions of a finite BDP have been solved using a variety of techniques over the years, (See Kijima (1997), Mohanty et al (1993) and
Rosenlund (1978)). We now provide a new solution method using Theorem 2.4.1 and the obtained result in Theorem 2.5.1.

Suppose, we consider a recurrent BDP on a finite state space having rate transition given in Figure 2.5.1.

![Figure 2.5.1](image)

From this finite BDP of Figure 2.5.1, the associated BD chain of Figure 2.5.2 is considered below.

![Figure 2.5.2](image)

where \( p_i = \lambda_i b^{-1} \), for \( i = 0, 1, 2, 3, \ldots, H-1 \) and \( q_i = \mu_i b^{-1} \), for \( i = 1, 2, 3, \ldots, H \) and \( r_i = 1 - (\lambda_i + \mu_i) b^{-1} \) for \( i = 1, 2, 3, \ldots, H-1 \) and \( r_0 = 1 - \lambda_0 b^{-1} \) and \( r_H = 1 - \mu_H b^{-1} \) with \( b = \max_{i=0,\ldots,H} |\lambda_i + \mu_i| \) where \( \lambda_i, \mu_0 \) are taken to be 0.

### Remark 2.5.1

The following theorem, called uniformization is a well-known result used primarily for the numerical computation of the transition probability functions \( P_{ij}(t) \) of a
Markov process (MP). It applies for the preceding finite BDP of Figure 2.5.1 and more generally, for any MP with countable state space having uniformly bounded diagonal transition rates in the Q matrix (See Gross and Harris (1985) and Medhi (2003)). Here, it suffices to state the uniformization result in terms of the BDP of Figure 2.5.1 and its associated uniformization BD chain of Figure 2.5.2.

**Theorem 2.5.1**

Suppose $P_{i,j}(t)$ is the transition probability function of the BDP of Figure 2.5.1. Then $P_{i,j}(t)$ may be written as

$$P_{i,j}(t) = e^{-bt} \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} P_{i,j}^{(n)}$$

for $i, j = 0, 1, 2, 3 \ldots$

where $P_{i,j}^{(n)}$ is the n-step transition probability of the associated BD chain of Figure 2.5.2.

**Proof:**

We may observe that $P_{i,j}(t)$ is completely determined once $P_{i,j}^{(n)}$ is known. That is one can find the analytic solution for $P_{i,j}(t)$ by finding $P_{i,j}^{(n)}$.

Using the above result we have

$$P_{i,j}(t) = e^{-bt} \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} P_{i,j}^{(n)} = e^{-bt} \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} P_{i,j}^{(n)} + e^{-bt} \sum_{n=H+1}^{\infty} \frac{(bt)^n}{n!} P_{i,j}^{(n)}$$

The BD chain in Figure 2.5.2 is a finite MC. It is also known that, the one-step probability matrix $P$ of such a BD chain is diagonalizable with $H + 1$ distinct real eigen values $z_1 \leq z_1 < z_2 < \ldots < z_{H+1}$ (See Buchanan and Turner (1992)).

Hence, Theorem 2.4.1 applies by giving

$$P_{i,j}(t) = e^{-bt} \sum_{n=0}^{H} \frac{(bt)^n}{n!} P_{i,j}^{(n)} + e^{-bt} \sum_{n=H+1}^{\infty} \frac{(bt)^n}{n!} \left( A_{i,j}^1 z_1^n + A_{i,j}^2 z_2^n + \ldots + A_{i,j}^H z_H^n + A_{i,j}^{H+1} \right)$$

$$= e^{-bt} \sum_{n=0}^{H} \frac{(bt)^n}{n!} P_{i,j}^{(n)} + e^{-bt} \sum_{n=H+1}^{\infty} \frac{(bt)^n}{n!} \left( A_{i,j}^1 (btz_1)_n + A_{i,j}^2 (btz_2)_n + \ldots + A_{i,j}^H (btz_H)_n + A_{i,j}^{H+1} (bt)_{n} \right)$$

42
\[ P_{i,j}(t) = e^{-bt} \sum_{n=0}^{N} \frac{(bt)^n}{n!} P_{i,j}^{(n)} + \left( A_{i,j}^{(1)} e^{bt(z_i-1)} + A_{i,j}^{(2)} e^{bt(z_i-1)} + \ldots + A_{i,j}^{(N)} e^{bt(z_i-1)} + A_{i,j}^{(H+1)} \right) \]

\[ - e^{-bt} \sum_{n=0}^{N} \frac{A_{i,j}^{(1)} (btz_i)^n}{n!} + A_{i,j}^{(2)} (btz_i)^n + \ldots + A_{i,j}^{(H+1)} (bt)^n \]

In this expression, \( b, h \) and \( z_h \) are all considered to be known numerical quantities.

**Remark 2.5.2**

This approach does not require the use of Laplace transforms.

We now proceed to the next chapter to present certain results in BDPs with a transformation procedure and applications.