CHAPTER 1

BIRTH DEATH PROCESSES AND RELATED TOPICS, RELEVANT CONCEPTS AND ORGANIZATION OF THE THESIS

1.0 INTRODUCTION

In this chapter we provide an introduction to birth death processes (BDPs) and related topics and provide certain definitions and concepts relevant to the thesis. This research work focuses its attention to the studies in BDPs and Related topics. BDPs are used frequently to model a variety of applications including queues, inventories, reliability, communication, production management, neutron propagation, optics, chemical reactions, epidemics and population dynamics. They are useful in the investigation of phenomena essentially concerned with a flow of events in time, especially those exhibiting such highly variable characteristics as birth, death, transformation, evolution, arrival, departure, etc. What makes BDPs so useful is that standard methods of analysis are available for determining numerous important quantities such as transient and stationary distributions and first passage times. Mathematics has long been intertwined with the biological sciences. The importance of mathematical approaches in many areas of biology is obvious but it is less appreciated that biological questions have stimulated the emergence of a variety of new directions in mathematics. Among many others, the areas of mathematics fully or partially developed in response to demands of biology include branching processes, traveling wave solutions of reaction-diffusion systems, diffusive instability, analysis of replicator equations, stochastic coalescent process, evolutionary game theory, and analysis of variance. Another fruitful and diverse mathematical field inspired by biology is the theory of BDPs. This theory was developed in the beginning of the 20th century as a result of attempts to model growth of a population taking into account
stochastic demographic factors. With time, the theory has been becoming increasingly sophisticated, spawning new branches of stochastic process analysis. Importantly, however, the first and simplest BDPs were considered by Yule (1924), Feller (1939) and Kendall (1948) provide a natural and useful theoretical framework for several areas of modern biology, such as estimation of the age of alleles, reconstruction of phylogenies, and modeling various aspects of genome evolution.

From practical point of view, when modeling a stochastic system by a BDP, there are instances in which long term behavior encapsulated by stationary distributions has no meaning. An important sub-class of Markov chains (MCs) with continuous time parameter space is BDPs, whose state space is the non-negative integers. These processes are characterized by the property that if a transition occurs, then this transition leads to a neighbouring state.

1.1 A PROLOGUE ON BIRTH – DEATH PROCESSES AND RELATED TOPICS:

The description of the process is as follows: The process sojourns in a given state $i$ for a random length of time following an exponential distribution with parameter $\lambda_i + \mu_i$. When leaving state $i$, the process enters state $(i + 1)$ or state $(i-1)$. The motion is analogous to that of a random walk except that transitions occur at random times rather than fixed time periods. Birth and death (BD) chains and processes, with a finite or countable number of states, play a central role in stochastic modeling for applied probabilists. The following books have complete chapters devoted entirely to different aspects of BD models: Andersen (1991), Bhattacharya and Waymire (1990), Gross and Harris (1998), Karlin and Taylor (1975), Kijima (1997) and Medhi (2003). Meanwhile, classical probability texts, over many generations, have popularized MCs and captivated the researchers with colourful
discussions of BD chains in terms of gambler's ruin probabilities (See Feller (1950) or Hoel, Port and Stone (1972)).

BDPs are frequently used as models of the growth of biological populations. A remarkable variety of dynamic behavior exhibited by many species of plants, insects and animals has stimulated great interest in the development of both biological experiments and mathematical models. In many ecological problems such as animal and cell populations, epidemics, plant tissues, learning processes, competition between species, growth patterns are influenced by population size. Such populations do not increase indefinitely, but are limited by, for example, lack of food and overcrowding. There are many deterministic models which describe such density-dependent logistic population growth. They effectively represent the development of a tumor, the growth of viral plaques or the population of spread in the theory of urban development.

Queueing theory is an important application area of BDPs. It has proved to be useful in a wide range of disciplines from computer networks and telecommunications to chemical kinetics and epidemiology. Suppose that customers arrive at a single server facility in accordance with a Poisson process. That is, times between successive arrivals are independent exponential variables having mean 1/\lambda. Upon arrival, each customer goes directly into service if the server is free, and if not, the customer joins the queue. When the server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service. The successive service times are assumed to be independent exponential variables having mean 1/\mu. The number of persons in the system at time t is a BDP with birth rate \lambda and death rate \mu independent of the number of customers present in the system at that time.
The presence of only one of the components, viz., birth or death also leads to important applications. For example, in the theory of radioactive transformation the radioactive atoms are unstable and disintegrate stochastically. Each of these new atoms is also unstable. By the emission of radioactive particles, these new atoms pass through a number of physical states with specified decay rates from one state to the adjacent state. Thus the radioactive transformations can be modeled as a birth process. Consider the enzyme reaction of blood clotting. In closing a cut, the gelation process of blood clotting is caused by an enzyme known as fibrin, which is formed by fibrogen. This conversion of fibrogen molecules into fibrin molecules follows a birth process.

BDPs on a finite state space cover a large spectrum of operations research and biological systems. Queuing models with finite state space have applications in production and inventory problems, for example, to optimize the size of the storage space, to determine the trade-off between throughput and inventory (or waiting time) and to exhibit the propagation of blockage. The performance of the produce-to-stock manufacturing facility can be obtained from the performance of the finite queuing systems. Network of queues with finite buffers occur widely in computer and telecommunications systems.

A BDP is a stochastic process in which jumps from a particular state (number of individuals, cells, lineages etc) are only allowed to neighboring states. A jump to the right, i.e., increase by one of the number of individual or similar quantities represents birth, whereas a jump to the left represents death. This property considerably simplifies the mathematical analysis, but the process remains applicable to numerous real-world systems. BD models allow one to address any questions formulated in terms of transition or state probabilities of the process, stationary
distribution, mean, variance and distribution of times of the first entrance to a particular set of states, probabilities of extinction, the mean time of existence, etc. The results obtained with these models can be compared with empirical data allowing one to either reject some of the initial assumptions, or accept the model as a useful tool for analysis and prediction of properties of the real system. Early models in population biology, including those based on the theory of BDPs, were largely deterministic. However, from the very beginning of population growth modeling, it is very much clear that a more refined analysis must take into account the role of stochastic factors in the evolution of the population. An early recognition of this fact is evident in the study of extinction of human families by Watson and Galton (1874). The classic deterministic theory of population growth treats the size of a population as a continuous variable. This means that the state space of the process under consideration is continuous in the deterministic setting. By contrast, the state space of the corresponding stochastic process is discrete. In this regard, the stochastic models are more realistic than the deterministic ones because counts of individuals (genes, cells, number of species or gene families, etc.) are discrete by definition. Loosely speaking, each deterministic model can be viewed as an approximation of the corresponding stochastic model. This, however, does not mean that deterministic models always yield qualitatively valid solutions. A case in point is the phenomenon of persistence and its complement, extinction: there are situations when a deterministic model predicts that a population approaches a positive stationary level, whereas the corresponding stochastic model shows that extinction occurs with certainty.

Another very vital area is to study the time-dependent behavior of BDPs and a wide variety of problems related to them by employing analytical and numerical
methods. In the analysis of BDPs, the emphasis is often laid on the steady state solutions while the transient or time-dependent analysis has received less attention. The assumptions required to derive the steady state solutions to queuing systems are seldom satisfied in the design and analysis of real systems. Also, steady state measures of system performance simply do not make sense for systems that never approach equilibrium. Moreover, the steady state results are inappropriate in situations wherein the time horizon of operations is finite. Hence, in many applications, the practitioner needs a knowledge of the time-dependent behavior of the system rather than easily obtainable steady state results. Further, transient solutions are available for a wide class of problems and contribute to a more finely tuned analysis of the costs and benefits of the systems.

BDPs have a huge mathematical literature discussing effective and interesting techniques to determine numerous important quantities like system size probabilities, their stationary behavior, first passage times, etc. The transition probabilities of finite BDPs, for example, can be expressed in terms of sums of exponentials. Its complete unified theory can be used for reference in general MC and other stochastic models. Recurrence relations play an important role in the transient analysis of BDPs. There is hardly a computational task which does not rely on recursive techniques at one time or another. The widespread use of recurrence relations can be ascribed to their intrinsic constructive quality and the great ease with which they are amenable to mechanization. In particular, “three-term recurrence relations” form the nucleus of continued fractions, orthogonal polynomials (OPs) and BDPs.

The study of the time dependent behavior of BDPs has given rise to many intricate and interesting OPs. Several interesting OPs occur in the study of BDPs with
linear and quadratic BD rates. This orthogonal representation leads to the spectral measure for BDPs which is important in the study of the transient behavior.

1.2 MATHEMATICAL BACKGROUND AND APPLICATIONS OF BIRTH AND DEATH PROCESSES:

The human body contains many cellular systems that require the continuous production of new fully functional, differentiated cells lacking. Common to each of these cellular systems is a primitive (undifferentiated) stem cell which replenishes cells through the production of offspring some of which proliferate and gradually differentiate until mature, fully functional cells are produced. In most cases the mature cells are incapable of further division, and after a variable period of time, they die. Much progress has been made on the identity of the various cell types that comprise the actors in the system. Understanding the dynamics of the interactions among the components of the system is difficult. Clearly, fine tuned feedback mechanisms must be involved in order to achieve the balance necessary for the maintenance of health. These theories of cellular differentiation and proliferation are based on BDPs. BDPs have been employed to uncover macro-evolutionary processes that have led to the biological diversity we observe today. Cancer arises from the stepwise accumulation of genetic changes that confer the unlimited, self-sufficient growth and resistance to normal homeostatic regulatory mechanisms upon an incipient neoplastic cell. It is now universally recognized that carcinogenesis is a multistage random process involving genetic changes and stochastic proliferation and differentiation of normal stem cells and genetically altered stem cells. In a two stage model, a malignant cancer cell is assumed to arise following the occurrence of two critical mutations in a normal stem cell. Initiated cells that have sustained the first mutation undergo a BDP. If the birth rate exceeds the death rate, this results in a colonel expansion of initiated cells.
Genomes of the four plant viruses of the genus nano virus consist of multiple circular single-stranded DNA components, each of which encodes a single protein. The phylogeny of replication proteins indicates that small viral multi-gene family has evolved by a process of duplication and subsequent loss of Rep-encoding genome components, analogous to the BDP process of evolution which has been described in eukaryotic multi-gene families. Bounded BDPs occur naturally in many areas of application, for example, in epidemic processes. In other applications, for example, in species interaction models, it is necessary to impose a bound on the number of potential births. The literature is replete with processes that can be modeled as bounded BDPs. These models have led invariably to rather intractable mathematics. The most frequent course of action has been to work with the partial differential equation for the generating functions. However, these are characteristically non-linear, multidimensional and subject to finite population constraints (See Gani’s (1965)).

The general study of temporally continuous, stochastic models of population growth apparently started with the work of Feller (1939). The cardinal assumption was that the growth of a population can be represented by a Markov process, i.e., the state of the population at time \( t \) can be described by the value of a random variable \( X(t) \) with the property

\[
P[X(t) = n \mid X(t_0) = m_0, X(t_1) = m_1, \ldots, X(t_k) = m_k] = P[X(t) = n \mid X(t_0) = m_0],
\]

for all \( t_i \leq t_0 \) and whenever \( t_0 < t \). The nature of the variable \( X(t) \) differs from model to model. In this review, we consider only models with continuous time although an analogous theory exists for stochastic BDPs with discrete time.

If we interpret \( X(t) \) as a population size, then a BDP is a Markov process
\{X(t), \ t \geq 0\} such that, in an interval \((t, t + \Delta t)\), each individual in the population has the probability \(\lambda_n \Delta t + o(\Delta t)\) of giving birth to a new individual in the (probability of transition from state \(n\) to state \(n + 1\)) and the probability \(\mu_n \Delta t + o(\Delta t)\) of dying (probability of transition from state \(n\) to state \(n - 1\)). The parameters \(\lambda_n\) and \(\mu_n\) are called the birth rate and death rate, respectively (\(n\) is the population size). For an intuitively plausible depiction of a BDP, it is useful to imagine a material particle, which moves from an integer to the neighboring integer, the path function \(X(t)\) being the position of particle at time \(t\).

The state probabilities \(P_n(t) = P[X(t) = n]\) of the process being in state \(n\) at time \(t\) satisfies the following system of differential equations, called Kolmogorov forward equations.

\[
\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t)
\]

\[
\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - (\lambda_n + \mu_n) p_n(t) + \mu_{n+1} p_{n+1}(t), \quad n \geq 1
\] (1.2.1)

Here, we consider BDPs whose state space consists on non-negative integers \(\{0, 1, \ldots, N, \ldots\}\). Generally, there are two types of random processes: one in which there are no restrictions on the allowed set of states and the other in which there are restrictions in the sense that some states have special properties. For example in a growing population, once the number \(n\) of individual is zero, the growth process stops (if there is no immigration). Thus the state \(n = 0\) is a special state, i.e., once the process reaches this state, it is trapped forever. Such states are called absorbing states. Another special state is the so-called reflecting state. Once the process reaches a reflecting state, it must return to the previously occupied state. Having in mind biological applications of BDPs, we consider here random processes that have either one or two special (absorbing or reflecting) states.
System of equations in (1.2.1) have to be solved subject to an initial condition and some boundary conditions. It is sufficient to solve them for the initial condition \( p_n(0) = \delta_{n,m} \), i.e., for the case when the process is initially in a define state \( m \). The state probabilities \( p_n(t) \) give full information about the analyzed process but it is usually difficult to solve the system (1.2.1). The simplest case when the solution of system (1.2.1) is straightforward is a pure birth process or the Poisson process. In this case, we have \( \lambda_n = \lambda, \mu_n = 0 \) and the solution of the system of equations in (1.2.1) subject to the initial condition \( p_0(0) = 1 \) is the Poisson distribution

\[
p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{with parameter } \lambda t.
\]

It is well known that the distribution of the time intervals between any two successive jumps in any Markov process with conditions time and discrete space of states is exponential (See Allen (2003)). More precisely, let \( W_i \) be the instant of the \( i^{th} \) jump of the BDP (1.2.1) and \( T_i = W_{i+1} - W_i \) be the sojourn time \( T_i \) in the state \( X(t) = n \) which the mean \( E[T_i] = 1/(\lambda_n + \mu_n) \). When a jump occurs, it will be a birth with the probability \( \lambda_n / (\lambda_n + \mu_n) \) or a death with the probability \( \mu_n / (\lambda_n + \mu_n) \).

If BD rate of the system (1.2.1) are linear functions of \( n \) then the so called probability -- generating functions technique can be applied for writing down the appropriate partial differential equations for the probability generating function (See Goel and Richter-Dyn (1974)). Historically this method had been the main tool for analyzing various BDPs (See Kendal (1948) and Kendal (1949)). It is straightforward to obtain results for a simple birth process \( (\lambda_n = \lambda, \mu_n = 0) \), simple death process \( (\lambda_n = 0, \mu_n = \mu) \), simple death and immigration process \( (\lambda_n = \lambda, \mu_n = \mu) \) (See Cox and Miller (1965), simple BDP \( (\lambda_n = \lambda, \mu_n = \mu) \), generalized simple BDP \( (\lambda_n = \lambda(t)n, \mu_n = \mu(t)n) \) and a simple BDP with immigration \( (\lambda_n = \lambda_n + \nu, \mu_n = \mu_n) \).
For example, for a simple birth process with the initial condition $p_m(0) = 1$, it can be shown that the state probabilities are

$$p_n(t) = \left( \begin{array}{cc} n & -1 \\ m & -1 \end{array} \right) e^{-m\lambda t} (1 - e^{-\lambda t})^{n-m}, \ n \geq m.$$  

This stochastic process was first studied by Yule (1924) in connection with the mathematical theory of evolution. The state of the process was thought of as a species within a genus and the creation of a new species by mutation was conceived as being a random event with the probability proportional to the number of species. Yule used this process to explain the observed power law distribution of genera of plants having $n$ species.

For a simple BDP, one can find $p_0(t) = p_0$.

$$p_n(t) = (1 - p_0) (1 - \dfrac{\lambda p_0}{\mu}) (\dfrac{\lambda p_0}{\mu})^{n-1}, \ n \geq 1,$$

where $p_0 = \mu \left( e^{(\lambda - \mu) t} - 1 \right) \left[ \lambda e^{(\lambda - \mu) t} - \mu \right]^{-1}$

Under the initial condition $p_0(t) = 1$. For other possible initial conditions, the solution of (1.2.1) is more complicated but still can be obtained. With this exact nature of this solution, it has many applications in current research, phylogeny reconstruction phylogenics (See Harvey et al(1994) and estimation of the age of rare alleles (See Slatkin (2002)).

As pointed out above, the method of probability generating functions generally works when birth and death rates are linear functions of $n$. Karlin and McGregor showed that the solution of (1.2.1) could be obtained with the help of a sequence of orthogonal polynomials, which are closely related to the BDP. The general linear case $\lambda_n = \lambda_n + \nu, \ \mu_n = \mu n + \rho$ was solved in (Karlin and McGregor (1957a)) for $\rho = 0$. The case of $\rho \neq 0$ was analyzed in Ismail et al (1988). The asymptotically symmetric quadratic case $\lambda_n = (N - n) (n+a), \ \mu_n = n (n+b)$ first
appeared in applications concerned with genetic models (See Karlin and McGregor (1962)) Some other special cases also have been described in Valent (1996) The problem with exact solutions of system (1.2.1) is that, in many cases, the expressions for the state probabilities, although explicit, are intractable for analysis and include special polynomials. In such cases, it may be sensible to solve more modest problems concerning the BDP under consideration, without the knowledge of the time dependent behaviour of state probabilities $p_n(t)$.

It is quit common to write down the differential equations for the first few moments of $X(t)$ (See Goel and Richter-Dyn (1974)). For example, for a simple birth process, the equation for the mean $E[X(t)]$ is

$$\frac{dE[X(t)]}{dt} = \lambda E[X(t)]$$

with the solution $E[X(t)] = e^{\lambda t}$ for the case $p_1(0) = 1$. It should be noticed that, in this case, the mean growth of the process follows the same exponential law as the one that appears in the simplest deterministic model of population growth, namely,

$$\frac{dN(t)}{dt} = \lambda N(t),$$

where $N(t)$ is the is the population size and $\lambda$ is a Malthusian parameter. Sometimes, this fact is used as justification for the assertion that the deterministic theory is simply an account of the expectation behaviour of the random variables, which occur in the stochastic formulation. This is nor generally true as first pointed out by Feller (1939). For example, this is not the case for the logistic stochastic process considered below.

The method for writing down equations for the moments of $X(t)$ works only when $\lambda_n$ and $\mu_n$ are linear functions of $n$. If $\lambda_n$ and $\mu_n$ include terms with $n$ of degree higher than 1, the equation for the mean involves, generally, the second moment the equation for the second moment involves the third moment and so on. This hierarchy
of equations can only be solved approximately, by making any of a variety of approximations. Recently, the moment closure approximation has become a popular technique for obtaining such solutions (See Levin and Pacala (1997)). Applications of this method to the logistic stochastic process can be found in Nasell (2003) and Newman et al (2004).

There is a steady increase in the applications of BDPs to natural sciences and practical problems including biological sciences, the theory of queues and inventories and other subjects such as reliability, production management, computer-communication systems, neutron propagation, optics, chemical reactions, construction and mining and compartmental models. Therefore, the broad field of applications of BD models amply justifies an intensive study of BDPs and related topics.

1.3 CERTAIN CONCEPTS AND DEFINITIONS ASSOCIATED WITH THE STUDIES IN BIRTH AND DEATH PROCESSES:

Stochastic processes occurring in most real life situations are such that for discrete set of parameters \((t_1, t_2, \ldots, t_n \in T)\) the random varieties \(x(t_1), x(t_2), \ldots, x(t_n)\) exhibit some sort of dependence. The dependence exhibited by the process \(\{X(t), t \in T\}\) is called Markov-dependence, if

\[
P[X(t) \leq x \mid X(t_n) = X_n, X(t_{n-1}) = X_{n-1}, \ldots, X(t_0) = X_0] = P[X(t) \leq x \mid X(t_n) = X_n]
\]

\[= P[X_n, x; t_n, t].\]

A stochastic process exhibiting this property is called Markov Process (MP). Depending on the nature of the state space and the parameter space MP can be divided into four different classes. Whenever the parameter space and the state space are discrete, we call such MP as Markov Chain (MC). A very important special class of MPs has come to be known as the birth and death processes (BDPs). They may be either discrete or continuous time processes in which the defining conditions in the state transitions take place between neighbouring states only. That is, one may choose the set of integers as the discrete state space (with no loss of generality) and then the BDPs requires that if \(X_n = i\), then \(X_{n+1} = i +1, i\) or \(i -1\) and no other.
BDPs in a queue: The queueing system that is treated as a BDPs will cause the state of arrivals to increase by one and departure cause the state to decrease by one.

A stochastic process \( \{X_n, n \geq 0\} \) is said to constitute a **Markov Chain** (MC) with state space \( S \) whenever,

\[
P[X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] = P[X_{n+1} = x_{n+1} | X_n = x_n]
\]

and the probability of the conditioning event \( [X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] \) is positive. Here \( x_0, x_1, \ldots, x_{n+1} \in S \).

A useful class of continuous time MPs when analyzing queueing systems are **BDPs**. The only possible state transitions in this kind of processes are from \( i \) to \( i-1 \) or from \( i \) to \( i+1 \) is designated \( \lambda_i \geq 0 \) for \( i \geq 1 \).

![State diagram of BDPs](image)

The state space of the BDP is \( \{0,1,2,3,\ldots\} \). The **intensity matrix** will be tri-diagonal type since there are only two ways of leaving a state. Hence we have the intensity matrix

\[
Q = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

As mentioned earlier, certain types of queueing systems are suitably modelled by BDPs. The numbers \( \{\lambda_i\} \) and \( \{\mu_i\} \) are interpreted as the arrival rate of the queue and service rate of the server, respectively.
Definition 1.3.1: A Pure Birth Process

A pure birth process is a BDP where $\mu_i = 0$ for all $i \geq 0$.

Definition 1.3.2: A Pure Death Process

A pure death process is a BDP where $\lambda_i = 0$ for all $i \geq 0$.

Definition 1.3.3: Ergodic state.

A positive recurrent aperiodic state is defined as an ergodic state.

Definition 1.3.4: Ergodic Process

A stochastic process is ergodic if all of its states are ergodic. A stochastic matrix $P = ((p_{ij}))$ (equivalently the corresponding MC $\{X_n\}$) is called ergodic if

$$\lim_{n \to \infty} P^n = \pi,$$

for all states $i$ and $j$ where the limiting vector $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ is called the stationary vector of $P$ and has the properties that $\pi P = \pi$, $\pi_j \geq 0$ for all $j$ and

$$\sum_{j=1}^{\infty} \pi_j = 1.$$

Definition 1.3.5: Renewal Process

Let $\{X_n, n \geq 1\}$ be a sequence of non-negative independent and identically distributed random variables with a common distribution function $F(\cdot)$. Again let $S_0 = 0$, and $S_n = X_1 + X_2 + \cdots + X_n$, $n \geq 1$.

Then the process $\{N(t), t \geq 0\}$, $N(t) = \text{Sup}\{n \mid S_n \leq t\}$ is called a renewal process.

Definition 1.3.6: Laplace - Stieltjes Transform

The Laplace - Stieltjes Transform of a function $g : R \to R$ is a function

$$\{L^*(g)\} = \int_0^\infty e^{-sx}dg(x), \ s \in \mathbb{C},$$

whenever the integral exists. The integral here is the Laplace - Stieltjes integral.
Definition 1.3.7: Secular Determinant

For a square matrix $A$, the determinant of the matrix whose off-diagonal components are equal to those of $A$, and whose diagonal components are equal to the difference between those of $A$ and a parameter $\lambda$; it is equal to the characteristic polynomial in $\lambda$ of the linear transformation represented by $A$.

Definition 1.3.8: Kronecker delta

The Kronecker delta or Kronecker’s delta is a function of two variables, usually integers, which is 1 if they are equal, and 0 otherwise. For example, $\delta_{12} = 0$, but $\delta_{33} = 1$. It is written as the symbol $\delta_{ij}$, and treated as notational shorthand rather than as a function.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Definition 1.3.9: Lotka – Volterra Model

Lotka-Volterra model is the simplest model of predator-prey interactions.

$$\frac{dH}{dt} = rH - aHP$$

$$\frac{dP}{dt} = bHP - mP$$

It has two variables (P, H) and several parameters:

$H =$ density of prey

$P =$ density of predators

$r =$ intrinsic rate of prey population increase

$a =$ predation rate coefficient

$b =$ reproduction rate of predators per 1 prey eaten

$m =$ predator mortality rate.
Definition 1.3.10: Hitting time

A hitting time (or first hit time) is a particular instance of a stopping time, the first time at which a given process "hits" a given subset of the state space. Exit times and return times are also examples of hitting times. Let $T$ be an ordered index set such as the natural numbers, $\mathbb{N}$, the non-negative real numbers, $[0, +\infty)$, or a subset of these; elements $t \in T$ can be thought of as "times". Given a probability space $(\Omega, \Sigma, P)$ and a measurable state space $S$, let $X : \Omega \times T \to S$ be a stochastic process, and let $A$ be a measurable subset of the state space $S$. Then the first hit time $\tau_A : \Omega \to [0, +\infty]$ is the random variable defined by

$$\tau_A(\omega) := \inf\{t \mid X_t(\omega) \in A\}.$$ 

The first exit time (from $A$) is defined to be the first hit time for $S \setminus A$, the complement of $A$ in $S$. Confusingly, this is also often denoted by $\tau_A$. The first return time is defined to be the first hit time for the singleton set $\{X_0(\omega)\}$, which is usually a given deterministic element of the state space, such as the origin of the coordinate system.

Definition 1.3.11: Lyapunov function

A Lyapunov function is a scalar function $V(y)$ defined on a region $D$ that is continuous, positive definite, $V(y) > 0$ for all $y \neq 0$, and has continuous first-order partial derivatives at every point of $D$. The derivative of $V$ with respect to the system $y' = f(y)$, written as $V'(y)$ is defined as the dot product

$$V'(y) = \nabla V(y) \cdot f(y)$$

Remark 1.3.1

The existence of a Lyapunov function for which $V'(y) \leq 0$ on some region $D$ containing the origin, guarantees the stability of the zero solution of $y' = f(y)$, while
the existence of a Lyapunov function for which \( V'(y) \) is negative definite on some region \( D \) containing the origin guarantees the asymptotical stability of the zero solution of \( y' = f(y) \).

For example, given the system
\[
y' = z; \quad z' = -y - 2z
\]
and the Lyapunov function \( V(y,z) = \frac{y^2 + z^2}{2} \), we obtain
\[
V'(y,z) = yz + z(-y-2z) = -2z^2
\]
which is nonincreasing on every region containing the origin, and thus the zero solution is stable.

**Definition 1.3.12 Fatou's Lemma**

If \( \{f_n\} \) is a sequence of nonnegative measurable functions, then
\[
\liminf_{n \to \infty} \int f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu
\]
An example of a sequence of functions for which the inequality becomes strict is given by
\[
f_n(x) = \begin{cases} 1 & \text{if } x \in [-n, n] \\ 0 & \text{otherwise.} \end{cases}
\]

**Definition 1.3.13 Markov decision process**

A Markov Decision Process is a tuple \( (S,A,P(\cdot,\cdot),R(\cdot)) \), where \( S \) is the State space, \( A \) is the action space, \( P_\alpha(S,S') = P(S_{t+1} = S' | S_t = s, a_t = a) \) is the probability that action \( a \) in state \( s \) at time \( t \) will lead to state \( s' \) at time \( t + 1 \). \( R(s) \) is the immediate reward (or expected immediate reward) received in state \( s \). The goal is to maximize some cumulative function of the rewards, typically the discounted sum under a discounting factor \( \gamma \) (usually just under 1). This would look like \( \sum_{t=0}^{\infty} \gamma^t R(S_t) \) where \( \gamma \) is the discount rate and satisfies \( \gamma < 1 \). The sum is to be maximised.
Markov decision processes are an extension of Markov chains; the difference is the addition of actions and rewards (giving motivation). If there were only one action, or if the action to take were somehow fixed for each state, a Markov decision process would reduce to a Markov chain.

**Definition 1.3.14 Recurrent state**

Let \( f_{ij}^{(n)} = P[X_n = j, X_{n-1}, \ldots, X_0 \neq j | X_0 = i] \) and \( F_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \). If \( F_{ii} = 1 \), that is return to state \( i \) is certain, then \( i \) is called recurrent.

**Definition 1.3.15 Transient State**

If \( F_{ii} < 1 \), that is return to state \( i \) is uncertain, then \( i \) is called transient.

**Definition 1.3.16 Null recurrent state**

The mean recurrence time from state \( i \) to state \( j \) is given by \( \mu_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)} \). A recurrent state \( i \) is said to be null recurrent if \( \mu_{ii} = \infty \), that is if the mean recurrence time is infinite.

**Definition 1.3.17 Favard’s Theorem**

Assume \( \{ \varepsilon_n \}_{n=1}^{n=\infty} \) is a sequence of complex numbers such that \( |\varepsilon_n| < 1 \) for \( n = 1, 2, \ldots \). Let \( \{ \phi_n \}_{n=0}^{n=\infty} \) satisfy Szegő recursion

\[
\phi_n(z) = z \phi_{n-1}(z) + \varepsilon_n \phi_{n-1}^*(z), \quad \phi_0(z) = 1
\]

and let \( \phi_n \) be defined by

\[
\phi_n(z) = k_n \phi_0(z) \text{ where } k_0 = 1 \text{ and }
\]

\[
k_n = \left( \prod_{k=1}^{n} \sqrt{1 - |\varepsilon_k|^2} \right)^{-1}, \text{ for } n = 1, 2, \ldots
\]
Then there exists a unique finite positive Borel measure \( \mu \) on \( T \) with infinite support such that we have \( \phi_n = \phi_n (d \mu) \) that is \( \{ \phi_n \}_{n=0}^{\infty} \) is orthonormal with respect to \( \mu \).

**Definition 1.3.18 Random Walk**

Let \((\Omega, F, P)\) be a probability space and \( \{ X_i \} \) a discrete-time stochastic process defined on \((\Omega, F, P)\), such that the \( X_i \) are independent and identically distributed real-valued random variables, and \( i \in \mathbb{N} \), the set of natural numbers. The random walk defined on \( X_i \) is the sequence of partial sums, or partial series

\[
S_n = \sum_{i=1}^{n} X_i.
\]

If \( X_i = \{-1, 1\} \), then the random walk defined on \( X_i \) is called a simple random walk. A symmetric simple random walk is a simple random walk such that \( P(X_i = 1) = \frac{1}{2} \) that the above defines random walks in one-dimension.

**1.4 ORGANIZATION OF THE THESIS:**

In **Chapter 1**, we provide an introduction to BDPs, review the relevant literature and stress the importance of BDPs and provide certain concepts and definitions which are relevant to the thesis.

In **Chapter 2** we obtain and analyze certain interesting results in BDPs in the context of transient probability functions of primal BDP and its dual process, determine the busy period distributions in a queueing system modeled by a queueing process and establishing an equivalent result by determining the transient probability functions in a related BDP. The results presented in this chapter are reported in Rajaram and Sampath Kumar (2006).
In Chapter - 3 we consider a class of BDPs and bounded processes and obtain certain interesting results. We derive the state probabilities for a more general bounded BDPs in which the transition rates are arbitrary and time-independent functions of the population sizes. Some applications in epidemic theory and species interaction processes are briefly described. Chapman-Kolomogorov equations are used directly and these equations are transformed into a system of lower-triangular equations through use of embedding techniques to obtain explicit solutions to the state probabilities of the underlying process and indicate certain results for a one dimensional bounded BDPs. The results presented in this chapter are reported in Rajaram and Sampath Kumar (2006a).

In Chapter - 4 we consider and study the deviation matrices of BDPs and derive an algorithm to compute the fundamental matrix for a special class of BDPs which includes certain queueing models. We also obtain a closed form expression for these queues. The results presented in this chapter are reported in Rajaram and Sampath Kumar (2006b).

In Chapter - 5 we obtain certain results and derive the reaction time distribution of BDPs and also study in detail, certain interesting properties of finite state BDP characterized by hyper exponential inter event times in the context of queues. The obtained results are reported in Rajaram (2007a).

In Chapter - 6 we obtain certain results and determining the necessary and sufficient conditions for the stability of a linear growth BDP and further study the problem of determining a necessary and sufficient condition for a BDP to be stochastically increasing or decreasing on an interval. Also, obtain some results.
pertaining to random walks, BDPs and queues. The results obtained in this chapter are reported in Rajaram (2007b)

This proposed research work may be considered as a modest contribution towards the advancement of theoretical and applied field especially in BDPs and related topics. The author is of the view that, the results presented in Chapters 2, 3, 4, 5 and 6 are original and new to the existing results.

We now proceed to the next chapter to provide certain interesting results in BDPs.