2.1 Introduction

This chapter explains the different aspects of the statistical theory, its predictions and Nilsson model as well as some related properties of the excited nuclear system.

Bethe and Ericson [1-3] used statistical method for obtaining the nuclear properties, while Moretto[4] and Ramamurthy et.al[5] further developed the method to include the microscopic effects[6] like pairing correlations[7], deformation, rotation and temperature. The statistical model due to Fong [8] was successful in reproducing the mass distribution curve for the fissioning of the compound system $^{236}\text{U}$ but it does not take into account the dependence on the shell structure and the nuclear temperature. Ignatyuk [9] included the pairing effects [10] in his calculations but he used constant values for the Nilsson parameter [11, 12] for all the shells.

In the case of degenerate Fermi gas model [3, 4, 13 - 17] we have a structureless system which is represented by constant level density parameter. This description cannot be extended to excited states, where the detailed structures and shape of the nuclear systems at finite temperatures[18] play an important role. In Ref.[4] Moretto has given an expression for the thermodynamical potential of rotating nuclei at a finite temperature. The angular momentum is generated by the Lagrange multiplier which conserve the total angular momentum of the system. In a recent work Civitarese et.al
[19,20] have extended the Darwin Fowler method[21,22] to include the temperature dependence of the level density parameter but neglected the effect of the angular momentum degrees of freedom on the level density. Bardeen Cooper Schrieffer (BCS) formalism [23] which takes into account the effect of pairing and temperature is used in the present work.

2.2 The partition function method

This method is a very powerful method and is widely used in Nuclear Physics because of its generality and flexibility. The grand canonical partition function[24] for a system of \( N \) neutrons and \( Z \) protons in a state of total angular momentum \( M \) along the direction of the rotation axis is given by

\[
e^\Omega = Q(\alpha_z, \alpha_n, \beta, \gamma) = \sum_{N',Z',E',M'} \exp(-\beta E' + \alpha_z Z' + \alpha_n N' + \gamma M'), \tag{2.1}
\]

where \( \alpha_z, \alpha_n \) and \( \gamma \) are the Lagrange multipliers conserving proton number, neutron number and total angular momentum along the \( Z \) axis for a given temperature \( T = 1/\beta \) called the statistical temperature. The total energy is the sum of the single particle energies occupied by the fermions. The summation is over all nuclei with \( N' \) neutrons and \( Z' \) protons and over all energy eigenvalues \( E' \) and the angular momentum \( M' \) of each nucleus.
The summation over the energy can be replaced by an integration as the nuclear levels exhibit a continuous spectrum even for a few MeV of excitation energy.

\[ Q(\alpha, \alpha_n, \beta, \gamma) = \sum \rho(E', Z', N', M', T) \exp(-\beta E' + \alpha Z' + \alpha_n N' + \gamma M') dE' \quad (2.2) \]

The quantity \( \rho(E, Z, N, M, T) \) represents the nuclear density of states of the nucleus \( (N, Z) \) at energy \( E \) with angular momentum \( M \) at finite statistical temperature \( T \). The total angular momentum is a function of deformation coordinates \( \delta \) and \( \theta \).

From equation (2.2) it is obvious that the grand canonical partition function \( Q \) can be considered as Laplace transform of the nuclear level density \( \rho(E, Z, N, M, T) \). Consequently \( \rho(E, Z, N, M, T) \) can be obtained by taking the inverse Laplace transform of the above equation. Williams[26,27] has used recursion relation for finding the level density. However, the more general method makes use of the inverse Laplace transform for the eqn. (2.2) is given by

\[ \rho(E, Z, N, M, T) = (1/2\pi)^3 \int d\alpha_n \int d\alpha \int d\gamma \int d\beta e^S, \quad (2.3) \]

where the quantity \( S = \ln Q + \beta E - \alpha Z - \alpha_n N - \gamma M \) is called the entropy of the system. The above integrals are known as the Darwin Fowler integrals[21,22]. The only approximation so far introduced into the formalism is the continuous approximation in which the level density is considered as a continuous function. 

12
The corresponding conservation equations are

\[-\partial \ln Q / \partial \beta = <E>\]

\[\partial \ln Q / \partial \alpha_z = <Z>\]

\[\partial \ln Q / \partial \alpha_n = <N>\]

\[\partial \ln Q / \partial y = <M>\] \hspace{1cm} (2.4)

Where, \( \ln Q = \sum \ln[1 + \exp(\alpha_n \beta \epsilon_i^n + \gamma m_i^n)] \)

\[+ \sum \ln[1 + \exp(\alpha_n \beta \epsilon_i^n + \gamma m_i^n)] \] \hspace{1cm} (2.5)

The single particle energy levels for the protons \( \epsilon_i^z \) with spin projection \( m_i^z \) and for neutrons \( \epsilon_i^n \) with spin projections \( m_i^n \) are generated using triaxially deformed Nilsson Hamiltonian. The particle number equation for protons, neutrons and the corresponding equations for energy \( E \) and angular momentum \( M \) are given below.

\[<Z> = \sum n_i^z = \sum [1 + \exp(-\alpha_n + \beta \epsilon_i^n + \gamma m_i^n)]^{-1}\]

\[<N> = \sum n_i^n = \sum [1 + \exp(-\alpha_n + \beta \epsilon_i^n + \gamma m_i^n)]^{-1}\]

\[<E> = \sum (n_i^n \epsilon_i^n, n_i^z \epsilon_i^z)\]

\[<M> = \sum (n_i^n m_i^n, n_i^z m_i^z)\] \hspace{1cm} (2.6)
where \( n_i \) is the occupation probability of the \( i \)th shell. These equations are solved for a given temperature \( T \) to obtain the Lagrange multipliers \( \alpha_z, \alpha_n \) and \( \gamma \). The applicability of thermodynamical concepts at very high temperatures is known from Ref.[3]. It is assumed that all the states with the same excitation energy \( E^* \) are equally populated.

The entropy of the system is obtained as
\[
S = S_z + S_n
\] (2.7)

where
\[
S_n = \sum \ln \left[ 1 + \left( \exp(\alpha_n \beta - \gamma m_i^n) \right) \right]
\]
\[
\sum (\epsilon_i^n - \alpha_n - \gamma m_i^n) [1 + \left( \exp \left\{ (\alpha_n - \beta) \epsilon_i^n + \gamma \right\} \right]
\]
\[
S_n = - \sum \left[ n_i^n \ln n_i^n + (1-n_i^n) \ln (1-n_i^n) \right]
\] (2.8)

\[
S_z = \sum \ln \left[ 1 + \left( \exp(\alpha_z \beta - \gamma m_i^z) \right) \right]
\]
\[
\sum (\epsilon_i^z - \alpha_z - \gamma m_i^z) [1 + \left( \exp \left\{ (\alpha_z - \beta) \epsilon_i^z + \gamma \right\} \right]
\]
\[
S_z = - \sum \left[ n_i^z \ln n_i^z + (1-n_i^z) \ln (1-n_i^z) \right]
\] (2.9)

The chemical potential \( \lambda = \alpha/\beta \) for a particular value of \( \beta \) is determined by the particle conservation.

\[
N = \sum n_i
\]

with \( n_i = \sum \left[ 1 + \exp(-\alpha + \beta \epsilon_i + \gamma m_i) \right]^{-1} \) (2.10)

The total energy \( E \) is given by the equation
\[
E(M,T, \delta, \theta) = \sum n_i \epsilon_i
\] (2.11)
The excitation energy as a function of angular for equilibrium deformation is calculated using the relation

\[ E^*(M,T,\delta,\theta) = E(M,T,\delta,\theta) - E_0 \]

\[ = \sum n_i \varepsilon_i - \sum \varepsilon_i \]  

(2.12)

with a similar equation for neutron. The summation in the first term in eq. (2.12) is carried over all the levels generated up to \( N=8 \).

Where \( E_0 = E(0,0,\delta,\theta) \) is the ground state energy of the system of \( N \) particle system denoted by

\[ E_0 = \sum \varepsilon_i \]

(2.13)

The rotational energies are calculated using the eqn.

\[ E_{\text{rot}} = E(M,T) - E(0,T) \]

(2.14)

for minimised parameters \( \delta \) and \( \theta \) for different angular momenta at a particular \( T \). By subtracting rigid body rotational energy (which increases as a function angular momentum) from the total energy of the system the excitation energy is calculated.

We define the excitation energy without rotational energy as

\[ E_{\text{exc}} = E(M,T) - E(M,0) \]

(2.15)

The occupation probability for the \( i \)th shell is given by

\[ n_i = \sum \left[ 1 + \exp(-\alpha + \beta \varepsilon_i + \gamma m_i) \right]^{-1} \]
The level density[29] of the system $\rho(M,E^*)$ as obtained in Ref.[30] is given by

$$\rho(M,E,\delta,\phi) = \beta \exp \left( \frac{S(M,E^*,\delta,\phi)}{S_{\text{max}}} \right)$$  \hspace{1cm} (2.16)

The normalization factor $S_{\text{max}}$ depends on the dimensionality of phase space which is the number of eigenstates used.

The single particle level density parameter 'a' is calculated using the expression

$$a(M,T,\delta,\phi) = S_{a}(M,T,\delta,\phi)/E^*(M,T,\delta,\phi)$$  \hspace{1cm} (2.17)

In our calculations the temperature is varied from 0.2 to 3 MeV and for each temperature the excitation energy, single particle level density parameter, nuclear level density parameter and the entropy are computed for the deformation $\delta$ and $\phi$ and for various spins by minimising free energy.

The shell structure and the statistical theory are successfully used to study the nuclear properties of the excited system. The statistical theory, with the single particle level structure as the only the input, can be used to extract information on many complex nuclear phenomena associated with changes in nuclear shape.

2.3 Nilsson Model

The Nilsson [12] model represents the self consistent potential by an axially symmetric oscillator potential with spin orbit coupling. The single particle wavefunctions, (the Nilsson
orbitals) are obtained by solving the Schroedinger equations with this potential. Nilsson introduced [31] the term proportional to $l_1 l_1$ and a spin orbit term proportional to $l_1 s$ in the Hamiltonian of an anisotropic oscillator in order to obtain the detailed characteristics of heavy, strongly deformed nuclei. The $l^2$ term favours large $l$ values and the $l_1 s$ term takes into account the strong spin orbit coupling of the nucleons. The interaction of one nucleon with the nuclear field is then represented by a single particle Hamiltonian of the following form.

$$H = H_0 + C l_1 s + D l^2$$  \hspace{1cm} (2.18)

where $C$ is the spin orbit coupling strength factor and $D$ is the centrifugal term which reduces the total energy and,

$$H_0 = \left( -\hbar^2 / 2m \right) \nabla^2 + V$$  \hspace{1cm} (2.19)

The $C$ and $D$ values in terms of the two new constants $k$ and $\mu$ are given as

$$k = - C / (2\hbar \omega_n) \quad \text{and} \quad \mu = 2 \left( D/C \right)$$

It is well known that more substantial changes in nuclear shape are expected as a consequence of fast nuclear rotations. In this section we give the description of Nilsson model and nuclear shape in terms of several deformation
parameters. Since the inputs for the statistical theory are the microscopic single particle level [32] corresponding to the triaxially deformed Nilsson harmonic oscillator potential [33], the results exhibit the effect of shell structure of the system at different deformations. The deformed potential [34,35] used here is given by

\[ V = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) - k\hbar \omega_o \left[ 2l_l S + (l^2_l - <l^2_l>_N) \right] \] (2.20)

The three harmonic oscillator frequencies \( \omega_x, \omega_y, \omega_z \) corresponding to the basic vibrations in the three principal axes \( x, y \) and \( z \) are given as

\[ \omega_x = \omega_o \left[ 1 - \sqrt{5/4\pi} \beta \cos (\theta - 4\pi/3) \right] \quad \text{where} \quad \beta = \sqrt{16\pi/5} \cdot \delta/3 \]
\[ = \omega_o \left[ 1 - \sqrt{5/4\pi} \sqrt{16\pi/5} \cdot \delta/3 \cos (\theta - 4\pi/3) \right] \]
\[ = \omega_o \left[ 1 + (1/3) \delta \cos \theta + (1/\sqrt{3}) \delta \sin \theta \right] \] (2.21)

\[ \omega_y = \omega_o \left[ 1 - \sqrt{5/4\pi} \beta \cos (\theta - 2\pi/3) \right] \]
\[ = \omega_o \left[ 1 - \sqrt{5/4\pi} \sqrt{16\pi/5} \cdot \delta/3 \cos (\theta - 2\pi/3) \right] \]
\[ = \omega_o \left[ 1 + (1/3) \delta \cos \theta - (1/\sqrt{3}) \delta \sin \theta \right] \] (2.22)

\[ \omega_z = \omega_o \left[ 1 - \sqrt{5/1\pi} \beta \cos \theta \right] \]
\[ = \omega_o \left[ 1 - \sqrt{5/1\pi} \sqrt{16\pi/5} \cdot \delta/3 \cos \theta \right] \]
\[ = \omega_o \left[ 1 - (2/3) \delta \cos \theta \right] \] (2.23)
where $\delta$ and $\theta$ are the two deformation parameters. Here the quantity $\omega$ ($\delta$, $\theta$) may be determined from the requirement that constant volume be enclosed by the equipotential surfaces. In the absence of any other deformation components, this condition may be written as

$$\omega_x \omega_y \omega_z = \omega_0^3 = \text{constant}$$

(2.24)

The constant $\omega_0$ is the value of $\omega$ ($\delta$) at $\delta = 0$ and is determined from the fit to nuclear size. The parametrization defined by equations (2.21), (2.22) and (2.23) is referred to as the Nilsson parametrization. The parameter $\delta = (a-b)/R_0$ corresponds to the elongation or flattening of the potential, while $\theta$ describes its nonaxiality, where $a$ is the semi major axis, $b$ is the semi minor axis and $R_0$ is the radius of equivalent sphere.

For $a > b$; $\delta > 1$ prolate shape

For $a < b$; $\delta < 1$ oblate shape

For $a = b$; $\delta = 0$ spherical shape

The range of variation of this last parameter extends from $\theta = 0$ to $2\pi$. However in the absence of rotation one can always change the meaning of the three principal axes so that it is sufficient to consider only one sector of the polar co-ordinate $\delta$ and the angular coordinate lying between $\theta = 0^\circ$ prolate elliptical shape and $\theta = 60^\circ$ oblate elliptical shape. In chapter III and IV we have not included paring correlations in the calculations and
considered the cranking frequency $\omega$ to be zero and so we have used the range of deformation parameter $\theta$ mentioned above.

If nuclear rotation is involved, and x-axis is the axis of rotation, we have to consider at least three sectors from $\theta = -120^\circ$ to $-60^\circ$ prolate noncollective to oblate collective, from $\theta = -60^\circ$ to $0^\circ$ oblate collective to prolate collective and from $\theta = 0^\circ$ to $60^\circ$ prolate collective to oblate noncollective [36]. In chapter V we have included pairing correlations in $^{122}$Xe and $^{124}$Ba in our calculations and considered the range of variation of deformation parameter $\theta$ described above.

The significance of various regions of deformation with respect to the rotation axis is illustrated in fig (2.1). In fact, only the limiting cases $\theta = -120^\circ, -60^\circ, 0^\circ$ and $60^\circ$ are indicated because the other half is exactly the same. Shapes corresponding to points inside each sector are of the triaxial type. The above parametrization has been used by Lund-Warsaw group. It is to be emphasized that some of the groups calculating nuclear properties at high angular momentum, for example, Dubna and Julich groups employ a somewhat different definition of the nonaxiality parameter $\theta$ [37].

The intrinsic nuclear spin is represented by $S$ while $k_l$ represents the orbital angular momentum in the stretched coordinate basis. The $k, \mu$ dependence is related to the main oscillator quantum number $N$ rather than $A$. The $k, \mu$ pair used in generating the single particle levels and spins are as in
Ref.[34] and they are different for different oscillator shells. These parameters are appropriate since they reproduce the experimental proton bandhead energies[35] more accurately than the standard parameter set given in Ref.[38]. However for neutrons the standard parameter set given in Ref.[38] is used. Two different sets are used for protons and neutrons. Table 2.1 shows the $\kappa,\mu$ parameter sets used in our calculations for protons and neutrons following the Ref.[38].

The deformation parameter $\delta$ is varied[39] in the range $\delta = 0.0$ to 0.6 in steps of 0.1 for $\theta = 60^\circ$ (oblate shape rotating around the symmetry axis). The cranking frequency $\omega$ is taken to be zero. The required angular momenta are generated using statistical theory[4,39] by introducing Z projection of the angular momentum as a constant of motion by means of the Lagrange multiplier corresponding to the single particle spins. This procedure is valid for rotation around the symmetry axis where the spin projection $m_i$ is a good quantum number. The accumulating total number of particles for all levels upto N shells is given by the equation

$$\Sigma \Delta \Lambda = \frac{1}{3} (N+1)(N+2)(N+3)$$  \hspace{1cm} (2.25)

The levels generated are upto $N = 8$ which is found to be sufficient for the range of temperatures used in these calculations. We have used 330 energy levels for protons and neutrons separately. From the calculations performed for $\theta = 60^\circ$ corresponding to oblate shape rotating around the symmetry axis, we find that the shape of Xenon nucleus does not seem to
vary with spin remaining oblate [40]. Alternately very high spins can be generated by adjusting and fine tuning the cranking frequency term $\omega$ in the Nilsson hamiltonian itself. This method of generating high spins is discussed and results presented in Chapter V.
Table 2.1
κ, λ Values as a function of the oscillator shell N for protons and neutrons

<table>
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<tr>
<th>Shell</th>
<th>Proton</th>
<th>Neutron</th>
</tr>
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<tbody>
<tr>
<td>N</td>
<td>κ</td>
<td>λ</td>
</tr>
<tr>
<td>0</td>
<td>0.120</td>
<td>0.00</td>
</tr>
<tr>
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<td>0.00</td>
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<td>0.69</td>
</tr>
<tr>
<td>8</td>
<td>0.054</td>
<td>0.69</td>
</tr>
</tbody>
</table>
Figure 2.1
The significance of various regions of deformation with respect to rotation axis
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