INTRODUCTION

This thesis embodies the work done by the author under the guidance of Dr. V. Swaminathan. The Connection between topologies and graphs has been the subject matter of study for quite a long time. In 1967, J. W. Ewans, F. Harary and M. S. Lynn in their paper "on computer enumeration of finite topologies" established a one-to-one correspondence between the set of all topologies with n points and the set of all transitive digraphs with n points. Then in 1968, T. N. Bhargav and T. J. Ahlborn in their paper "on topological spaces associated with digraphs", associated with each digraph D (which is not necessarily transitive), a unique topology on the vertex set having the property of completely additive closure (That is, the intersection of any collection of open sets is open). The topology is defined as follows. A subset A of V(D) is open if and only if for every pair of points i, j \in V with j in A and i not in A, (i, j) is not a line in D. In 1973, E. Sampath Kumar and K. H. Kulkarni extended the result of Harary et al to any arbitrary set (not necessarily finite). That is, there is a one-to-one correspondence between the set of all transitive digraphs on a vertex set V (not necessarily finite) and the set of all topologies on V having the property of completely additive closure. In their paper the topology corresponding to a digraph is generated by the closed inward
neighbourhoods and digraph corresponding to the topology is got by drawing a line from \( u \) to \( v \) if and only if \( u \) is in every open set containing \( v \). It has also been proved in this paper that the topology associated with a digraph \( D \) as per the definition of Bhargav induces a digraph which is the transitive closure of \( D \). Later in 1992, the relationship between comparability graphs and compatible topologies was proved by Pascal Prea in his Paper "Graphs and topologies on discrete sets". Pascal Prea had the aim of defining a topology on the vertex set of a graph such that an induced subgraph is connected if and only if it is connected for the topology. Such a topology has been termed by Prea as compatible topology. Prea has found that a topology \( \mathcal{S} \) on the vertex set of a graph is compatible if and only if (i) \( xy \) is an edge if and only if every open set containing \( x \) contains \( y \) or every open set containing \( y \) contains \( x \) and (ii) for every vertex \( x \), there exists an open set containing \( x \) and contained in \( N[x] \). It has also been prove that a graph admits a compatible topology on the vertex set if and only if it is a comparability graph. Azriel Rosenfeld has written a paper "Digital topology" in 1979 [6]. Digital pictures are rectangular arrays of non-negative numbers. The analysis of a digital picture usually involves segmenting it into parts and measuring the various properties of and relationship among the parts. In particular, one often wants to separate
out the connected components of a picture subset to determine the
adjacency relationships among those components, to track and encode
their borders or to thin them down to skeletons that have no interiors,
without changing their connectedness properties. There are standard
algorithms for doing all of these tasks. But to prove that they work, one
needs to establish some basic topological properties of digital picture
subsets. The aim of digital topology is to the study the topological
properties of digital picture subsets. While studying topological
properties, one gives a graphic representation of digital picture subsets.
The adjacency or surroundedness relations among the regions can be
compactly represented by a graph whose nodes are the regions and in
which two nodes are joined by an edge if these two regions are
adjacent. There are many algorithms for constructing the adjacency
graph of a partition of a picture.

In the papers of Harary and Bhargav, a common thread can be
seen namely establishing a one to one correspondence between
topologies on a set and digraphs on the same set. Perhaps their main
aim was enumerating finite topologies on a set. Much water has flown
under the bridge since then. Many new areas have emerged in graph
theory. Domination in graphs is one such area. If we find methods of
constructing graphs from topologies and topologies from graphs, we may try to find answers to the following questions.

(i) Do the concepts of dense subsets in topology and domination sets in graphs correspond?

(ii) What are the graphs for which the sequence created in graph dynamics is of period two?

(iii) What are the topologies for which the sequence created is of period two?

Given a graph G we can generate a topology (called a graphic topology) in many ways by specifying either a basis or a subbasis of subsets of the vertex set of G. We give below some of the ways. In each case, the topology is generated from the subbasis specified.

(i) The subbasis consists of all doubletons whose elements are adjacent vertices in the graph.

(ii) The subbasis \( S = \{\{N(v)\} : v \in G, \deg v \geq 2\} \cup \{\{N(v)\} : v \in G, \deg v = 1 \text{ and adjacent point of } v \text{ has degree } \geq 2 \text{ or } |V(G)| = 2\}. \) Also if \( K_2 \) is a component of G then, \( V(K_2) \) is an element of \( S \).

(iii) The subbasis consists of the closed neighbourhoods of each vertex.

(iv) The subbasis consists of open neighbourhoods of each vertex.
(v) The subbasis consists of sets $A$ such that $x, y \notin A$, implies $x$ and $y$ are not adjacent.

(vi) The subbasis consists of sets $A$ such that $x, y \in A$ implies $x$ and $y$ are adjacent.

In the reverse direction, given a topology on a nonempty set $X$ there are many ways of defining a graph with $X$ as the vertex set (called topological graphs). Some of the ways are given below.

(i) $v_1, v_2 \in X$ are adjacent if and only if $\{v_1, v_2\} \in \mathcal{Z}$.

(ii) $v_1, v_2 \in X$ are adjacent if and only if $\{v_1, v_2\}$ is either open or closed in $(X, \mathcal{Z})$. Also if $v \in X$ is such that $\{v\}$ is clopen then $v$ is adjacent to every element of $X$ other than itself.

(iii) $v_1$ and $v_2$ are adjacent if and only if every open set containing $x$ contains $y$ and vice versa.

(iv) $v_1$ and $v_2$ are adjacent if and only if every open set containing $x$ contains $y$ or every open set containing $y$ contains $x$.

Suppose $G$ is a given graph and topology is generated by all doubletons whose elements are adjacent, as subbasis. If $\delta(G) \geq 2$ then the topology associated with the graph will be discrete. This drawback can be rectified to some extent when we take the
definition (ii) for generating a topology. Definition (ii) and (iv) are almost the same except the fact that if $K_2$ is a component, then we do not take singletons containing the end vertices of $K_2$ as subbasic open sets in definition (ii) but they are taken as subbasic open set in definition (iv). Obviously, in definition (i) to (v), the intersection of two subbasic open sets need not be a subbasic open set. But in definition (vi), the intersection of two subbasic open sets is open. Hence the subbasis that can be considered in definition (vi) is actually a basis.

A graph $G$ is said to be a topological graph if there exists a topology $\mathcal{I}$ on $V(G)$ such that $G$ is isomorphic to $\psi(\mathcal{I})$. A topology $\mathcal{I}$ on a set $V$ is said to a graphic topology if there exists a graph $G$ with vertex set $V$ such that $\mathcal{I}$ is homeomorphic with $\phi(G)$. A graph $G$ is said to be selfmorphic if $G \cong \psi \phi(G)$. A topology $\mathcal{I}$ is selfmorphic if $\mathcal{I}$ is homeomorphic with $\phi \psi(\mathcal{I})$. For associating a topology with a given graph or a graph with a given topology we have several options. Each of the associations leads to a problem of determining topological graphs, graphic topologies and selfmorphic graphs.
In chapter I, we collect the basic definitions and theorems on graphs which are needed for the subsequent chapters. For graph theoretic terminology we refer to Harary [14]. By a graph we mean a finite undirected graph without loops and multiple edges.

In chapter II, we characterize topological graphs, graphic topologies and selfmorhpic graphs by taking different types of associations of topologies with graphs and graphs with topologies. We also find a relation between dense subsets and dominating subsets. Further the relation between $\gamma(G)$, $\gamma(S_\psi)$, $\gamma(G(3))$ are determined for a graphoidal cover $\psi$. Given a graph $G$ and the set of all minimal dominating sets of $G$, the intersection graph formed by these sets is studied with respect to domination number. A similar probe is also made by considering the intersection graphs formed by minimum dominating sets, strong dominating sets and proper dominating sets.

Difference labelings and common weight decomposition were first introduced and studied by Bloom and Ruiz [9], [10]. In difference labeling non-negative integer labels are used for vertices. Instead, we use open sets of a topology as labels and set difference is used to label the edges. A decomposition of a $\mathcal{S}$-labeled digraph $D$ into parts, each
part containing the arcs having a common weight is called \( \mathcal{Z} \)-common weight decomposition. Given an undirected graph \( G \), with edge set \( E(G) \) and decomposition \( R \) of \( G \) into subgraphs, does there exists a \( \mathcal{Z} \)-labeling of the vertices of \( G \) such that for a suitable orientation \( D \) of \( G \), \( R \) is a \( \mathcal{Z} \)-common weight decomposition? This problem is called specified edge decomposition problem.

B.D. Acharya [1], [2] has introduced set-indexer of a graph \( G = (V, E) \) as an injective set assignment \( f: V \rightarrow 2^X \) such that the induced set assignment \( f_\Delta: E \rightarrow 2^X \) defined by \( f_\Delta(uv) = f(u) \Delta f(v) \) is also injective. Instead of allowing \( X \) to be arbitrary, we consider topologies on \( V(G) \). Suppose \( \mathfrak{T} \) is a topology on \( V(G) \) induced by \( \{N(u) : u \in V(G)\} \) as subbasis. A \( \mathfrak{T} \)-indexer of \( G = (V, E) \) is an injective set assignment \( f: V \rightarrow \mathfrak{T} \) such that the induced set assignment \( f^*: E \rightarrow \mathfrak{T} \) defined by \( f^*(uv) = f(u) \cdot f(v) \) is one to one where \( \cdot \) denotes \( \cup \cap \Delta \).

Let \( f: V(G) \rightarrow \mathfrak{T} \) be an injective map. Let \( f^*: E(G) \rightarrow \mathfrak{T} \) be defined by \( f^*(uv) = f(u) \cap f(v) \). If range of \( f = (O_0, O_1, \ldots, O_{p-1}) \) and range of \( f^* = (O_1, O_2, \ldots, O_q) \), then \( f \) is said be Strongly indexable by \( \mathfrak{T} \).
Let $G$ be a $(p, q)$ graph. Let $\mathcal{I}$ be a topology on $V(G)$. $\mathcal{I}$ can be taken as the topology induced by $\{N(u) : u \in V(G)\}$ as subbasis. A graph $G$ is said to be $\mathcal{I}$—graceful if there exist an injection $f : V(G) \rightarrow \mathcal{I}$ such that $f^+ : E(G) \rightarrow \mathcal{I}$ defined by $f^+(e) = f(u) \cup f(v)$ where $e = uv$ is $1 - 1$ and the range of $f^+$ contains $q$ distinct sets with each one a strict subset of another and covered by it. ($f^+(e)$ can also be defined as $f(u) \cap f(v)$ or $f(u) \triangle f(v)$). That is, $f^+(E(G)) = O_1 \subset O_2 \subset \ldots \subset O_q$ where $O_i$ is an open set and each $O_i$ covers $O_{i-1}$ ($2 \leq i \leq q$).

In chapter III, we answer specified edge decomposition problem in affirmative where the decompositions consists of matchings. We also find certain classes of graphs which are $\mathcal{I}$—indexable. We prove that $C_n$ and $K_n$ are not $\mathcal{I}$—graceful, a $(p, q)$ graph with $q \geq p$ is not $\mathcal{I}$—graceful and $K_{1,n}$ is $\mathcal{I}$—graceful.

Highly irregular graphs (HI) have been introduced and studied by Yousef Alavi et al [30, [31, [32]. In the HI graphs, points in a neighbourhood, have distinct degrees. It has been proved that HI graphs are quite rare in the sense that if HI(n) and G(n) denote respectively the
number of non isomorphic HI graphs with $n$ vertices and the number of all non isomorphic graphs on $n$ vertices, then $\frac{\text{HI}(n)}{G(n)} \to 0$ as $n \to \infty$.

Let $G$ be a connected graph with vertex set $V$. $G$ is said to be *Neighbourhood Highly Irregular* (NHI) if for any $v \in V$ and for any $u, w \in N(v)$, $u \neq w \Rightarrow N[u] \neq N[w]$.

In chapter IV, we introduce the class of NHI graphs and show that it is strictly wider than the class of HI graphs. Further these graphs are dense in the sense that $\frac{\text{NHI}(n)}{G(n)} \to 1$ as $n \to \infty$. Also in this chapter we investigate problems concerning the existence and enumeration of NHI graphs.

Let $\mathcal{G}_n$ denote the set of all graphs with order $n$. Let $G_1, G_2 \in \mathcal{G}_n$. If there exists an ordering of vertices of $G_1, G_2$ such that $G_1, G_2 > \Rightarrow$ \text{Trace}(A(G_1).A(G_2)) = 0 then $G_1$ is said to be orthogonal to $G_2$ and this is denoted by $G_1 \perp G_2$.

In chapter V, we investigate the orthogonality among several classes of graphs of the same order.
Let $G$ be a graph. Let $f: V \cup E \rightarrow \{1, 2, \ldots, v + \varepsilon\}$ be a $1-1$ map. $f$ is called a weak neighbourhood magic labeling if

$$\sum_{v \in N[u]} f(v)$$

is constant.

Let $G$ be a graph. Let $f: V \cup E \rightarrow \{1, 2, \ldots, v + \varepsilon\}$ be a $1-1$ map. $f$ is called a neighbourhood magic labeling if

$$\sum_{v \in N[u]} \{f(v) + f(uv)\}$$

is constant where $f(uv) = |f(u) - f(v)|$. This constant number is called the magic strength of $f$.

Let $G$ be a graph. $f$ is called a neighbourhood magic graceful labeling if $f: V \rightarrow \mathbb{N}$ is a $1-1$ map and

$$\sum_{v \in N[u]} \{f(v) + f^+(uv)\}$$

is constant where $f^+(uv) = |f(u) - f(v)|$ and $f^+$ must be $1-1$ from $E \rightarrow \{1, 2, \ldots, q\}$.

In chapter VI, we prove that $K_n, C_3$ have weak neighbourhood magic labeling and $C_n, n \geq 4, K_{m,n}, m$ or $n \geq 2, P_n, n \geq 3$ have no weak neighbourhood magic labeling. $K_n, n \geq 4$, has no neighbourhood magic labeling and $C_4$ has neighbourhood magic labeling. Also we prove that $K_{1,n}, K_3$ and $C_4$ have no neighbourhood magic graceful labeling.
For each point $v$ of a graph $G$, take a new point $v'$. Join $v'$ to all points of $G$ adjacent to $v$. The graph $S(G)$ thus obtained is called the splitting graph of $G$.

In chapter VII, we study the domination number of splitting graph and we relate the domination number of splitting graph $S(G)$ with total domination number of $G$. Also we prove that if $S(G)$ is HI then $G$ is HI but the converse is not true. Also, $G$ is NHI if and only if $S(G)$ is NHI.

We conclude with some open problems.