CHAPTER – VII

SPLITTING GRAPHS

In this chapter, we study the domination number of splitting graph and we relate the domination number of splitting graph $S(G)$ with total domination number of $G$. Also we prove that if $S(G)$ is HI then $G$ is HI but the converse is not true. Further, it is shown that $G$ is NHI if and only if $S(G)$ is NHI.

7.1. INTRODUCTION: Splitting graphs were first introduced and studied by Sampathkumar and Walikar [21]. Given a graph $G$, for each point $v$ of $G$ take a new point $v'$ and join $v'$ to all points of $G$ adjacent to $v$. The graph thus obtained is called the splitting graph of $G$ and is denoted by $S(G)$. If $G$ is a $(p, q)$ graph then $S(G)$ is a $(2p, 3q)$ graph and degree of $v$ in $S(G)$ is twice the degree of $v$ in $G$ and degree of $v'$ in $S(G)$ is equal to the degree of $v$ in $G$. A graph $G$ is a splitting graph if and only if $V(G)$ can be partitioned into two sets $V_1, V_2$ such that there exists a bijectition $f : V_1 \rightarrow V_2$ satisfying $N( f(v_1)) = N( v_1 ) \cap V_1$ for every $v_1 \in V_1$. In this chapter we relate the domination number of splitting graph $S(G)$ with total domination number of $G$. We also find the domination number of $S(G)$ for certain specific graphs $G$. 

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7.1.1. Definition: For each point \( v \) of a graph \( G \), take a new point \( v' \). Join \( v' \) to all points of \( G \) adjacent to \( v \). The graph \( S(G) \) thus obtained is called splitting graph of \( G \).

7.1.2. Example:

\[ \begin{align*}
    \text{Fig. 3} & & \text{Fig. 4}
\end{align*} \]

7.1.3. Theorem: If \( S(G) \) is HI then \( G \) is HI.

Proof: Suppose \( S(G) \) is HI. To show that \( G \) is HI.
Let \( u \in V(G) \). Let \( v, w \in N(u) \cap V \), then \( d_{S(G)}(v) = 2d_G(v) \), \( d_{S(G)}(w) = 2d_G(w) \). Since \( d_{S(G)}(v) \neq d_{S(G)}(w) \) We get, \( 2d_G(v) \neq 2d_G(w) \). Therefore \( d_G(v) \neq d_G(w) \). Therefore \( G \) is HI.

7.1.4. Remark: The converse is not true. For in fig. 5.

\[
N(u_2) = \{u_1, u_1', v_2, v_2', u_3, u_3'\}, \quad d_{S(G)}(u_1) = 2, \quad d_{S(G)}(u_3') = 2.
\]

![Fig. 5](image)

7.1.5. Theorem: \( G \) is NHI if and only if \( S(G) \) is NHI.

Proof: Let \( G \) be NHI. Let \( u \in V(G) \). Let \( v, w \in N(u) \).

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Case(i) : \(v, w \in V(G)\). Then \(N[v] \cap V \neq N[w] \cap V\).

Therefore \(N[v] \neq N[w]\).

Case(ii) : \(v \in V, w \in V'\) (similar proof if \(v \in V'\) and \(w \in V\)). Let \(w = x'\) (\(x \neq u\) since if \(x = u\) then \(x' = u'\) is not adjacent to \(u\). That is \(w\) is not adjacent to \(u\), a contradiction). Therefore \(x \in N(u)\). Since \(x' = w\) is adjacent to \(u\). Now \(N[v] \cap V \neq N[x] \cap V\) Therefore there exist a point \(y \in N[v] \cap V\) such that \(y \notin N[x] \cap V\). Therefore \(u\) is not adjacent to \(x\) and hence \(y\) is not adjacent to \(x'\). Thus \(y\) is not adjacent to \(w\). Consequently, \(y\) is not in \(N[w] \cap V\). Therefore \(N[v] \cap V \neq N[w] \cap V\). Therefore \(N[v] \neq N[w]\).

Case(iii) : \(v, w \in V'\). Let \(v = x'\), \(w = y'\). Clearly \(x \neq u\), \(y \neq u\).

Therefore \(x, y \in N(u)\) and hence \(N[x] \cap V \neq N[y] \). This implies that there exist a point \(z \in N[x] \cap V\) such that \(z \notin N[y] \cap V\). So, \(z\) is not adjacent to \(y\) which shows that \(z\) is not adjacent to \(y'\). Therefore \(z\) is not adjacent to \(w\). That is \(z \notin N[w] \cap V\). Since \(z \in N[x] \cap V\), \(z\) is adjacent to \(x\) and so \(z\) is adjacent to \(x'\). That is \(z \in N[x'] \cap V = N[v] \cap V\).
V. Hence $N[v] \cap V \neq N[w] \cap V$. That is $N[v] \neq N[w]$. Conversely, suppose $S(G)$ is NHI. Let $u \in V(G)$. Let $v, w \in N(u) \cap V$. Given $N[v] \neq N[w]$. To show that $N[v] \cap V \neq N[w] \cap V$.

Suppose $N[v] \cap V = N[w] \cap V$ ------- (1)

Let $x' \in N(v) \cap V$ then $x'$ is adjacent to $v$. Therefore $x$ is adjacent to $v$. and so $x \in N[v] \cap V = N[w] \cap V$. That is $x$ is adjacent to $w$. Consequently $x'$ is adjacent to $w$. Therefore $x' \in N[w] \cap V'$. That is $N[v] \cap V' \subseteq N[w] \cap V'$ Interchanging the role of $v$ and $w$ we get,

$N[w] \cap V' \subseteq N[v] \cap V'$.

Therefore $N[v] \cap V' = N[w] \cap V'$ ------- (2)

From (1) and (2), $N[v] = N[w]$ in $S(G)$, a contradiction. Therefore $N[v] \cap V \neq N[w] \cap V$. Therefore $G$ is NHI.

7.2. Domination in splitting graphs:

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7.2.1. Observation: If $G$ is a graph with $\gamma(G) = 1$ then $\gamma(S(G)) = 2$.

Hence $\gamma(S(K_n))$, $\gamma(S(K_{1,n}))$, $\gamma(S(w_n))$ are all 2.

7.2.2. Observation: $\gamma(S(K_{mn})) = 2$.

7.2.3. Observation: For $n \geq 2$,

$$\gamma(S(P_n)) = \begin{cases} 2 \frac{ln}{4} + 1 & \text{if } n \equiv 1 \pmod{4} \\ 2 \frac{n}{4} - 1 & \text{otherwise} \end{cases}$$

For: If $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ then $\{v_{4i-1}, v_{4i-2} : i = 1, 2, \ldots, k\}$, $\{v_{4i-1}, v_{4i-2} : i = 1, 2, \ldots, k\} \cup \{v_{4k}\}$ and $\{v_{4i-1}, v_{4i-2} : i = 1, 2, \ldots, k\} \cup \{v_{4k+1}, v_{4k+2}\}$ are dominating sets of $S(P_n)$ when $n = 4k$, $4k + 1$, $(4k + 2$ and $4k + 3$) respectively.

Let $D$ be a dominating set of $S(p_n)$. Then without loss of generality, $D \subseteq V(P_n)$. For any 4 consecutive points, $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ we have to take a minimum of two points to dominate these four points and their corresponding points. If $n = 4k$ then $D$ must contain a minimum of $2k$ points. Therefore $|D| \geq 2k$ for any dominating set of $D$. That is, $\gamma(S(P_n)) \geq 2k = 2 \frac{n}{4} - 1$. 

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Therefore \( \gamma(S(P_n)) = 2 \sqrt{n/4} \) if \( n \equiv 0 \) (mod 4)

When \( n = 4k + 1 \) by the above argument \( |D| \geq 2k + 1 = 2 \sqrt{n/4} + 1 \).

Therefore \( \gamma(S(P_n)) = 2 \sqrt{n/4} + 1 \).

When \( n = 4k + 2 \) or \( 4k + 3 \), \( |D| \geq 2k + 2 = 2 \sqrt{n/4} + 1 \).

7.2.4. Observation: \( \gamma(S(C_n)) = \begin{cases} 
2 \sqrt{n/4} + 1 & \text{if } n \equiv 0 \text{ (mod 4)} \\
2 \sqrt{n/4} & \text{Otherwise}
\end{cases} \)

For: \{ v_{4k+1}, v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2} \}, \{ v_1, v_2, v_5, v_6, \ldots, v_{4k+1}, v_{4k+2} \} are dominating sets of \( S(C_n) \) when \( n = 4k+1 \) or \( n = (4k+2, 4k+3, 4k+4) \) respectively.

Let \( D \) be a dominating set of \( S(C_n) \). Then without loss of generality \( D \subseteq V(C_n) \). For any 4 consecutive points \( v_i, v_{i+1}, v_{i+2}, v_{i+3} \) we have to take a minimum of two points to dominate these four points and their corresponding points. If \( n = 4k \) then \( D \) must contain a minimum of \( 2k \) points.

Therefore \( |D| \geq 2k \) for any dominating set of \( D \). (ie) \( \gamma(S(C_n)) \geq 2k = 2 \sqrt{n/4} \). Therefore \( \gamma(S(C_n)) = 2 \sqrt{n/4} \) if \( n \equiv 0 \) (mod 4).
When \( n = 4k + 1 \) by the above argument \( |D| \geq 2k + 1 = 2 \lfloor n/4 \rfloor + 1 \).

Therefore \( \gamma(S(C_n)) = 2 \lfloor n/4 \rfloor + 1 \).

When \( n = 4k + 2 \) or \( 4k + 3 \), \( |D| \geq 2k + 2 = 2 \lceil n/4 \rceil \).

Therefore \( \gamma(S(C_n)) = 2 \lceil n/4 \rceil \).

**7.2.5. Lemma:** Let \( G \) be a graph. Let \( V(S(G)) = V_1 \cup V_1' \) where \( V_1 = V(G) \). Let \( D \) be a minimum dominating set of \( G \). Then \( x \in D \) is an independent point of \( D \) if and only if \( x' \) is not adjacent to any point of \( D \).

**Proof:** For \( u \in D \), Define \( N'(u) = \{ v' \in V_1' : v \in N(u) \} \). Let \( x' \in V_1' \) such that \( x \) is not adjacent to any point of \( D \). Therefore \( x' \notin N'(u) \) for every \( u \in D \). Consider the corresponding point \( x \in v_1 \).

Therefore \( x \notin N(u) \) \( \forall u \in D \). Therefore \( x \in D \) and \( x \) is not adjacent to any point of \( D \). The converse is Obvious.

**7.2.6. Lemma:** \( \gamma(S(G)) \leq \gamma(G) + q \leq 2 \gamma(G) \) where \( q \) is the number of independent points in a minimum dominating set with least number of independent points.
**Proof:** Let $D$ be a minimum dominating set of $G$ with least number of independent points say $q$. Let $\{x_1, x_2, \ldots, x_q\}$ be the set of independent points of $D$. Then $D_1 = D \cup \{x_1', x_2', \ldots, x_q'\}$ is a dominating set of $S(G)$. (For if $y \in V_1$ and $y \notin \{x_1', x_2', \ldots, x_q'\}$ then by lemma 2.5, $y$ is not an independent point of $D$. Therefore $y$ is adjacent to some $y_1 \in D$. Therefore $y'$ is adjacent to $y_1 \in D$).

Therefore $\gamma(S(G)) \leq |D_1| = |D| + q = \gamma(G) + q \leq 2\gamma(G)$. ■

7.2.7. **Lemma:** Let $D$ be a minimum dominating set of $G$ with the maximum number of edges. Let $\{S_1, S_2, \ldots, S_r\}$ be the star decomposition of $D$. Let $V(S_i) = t_i$, $i = 1, 2, \ldots, r$. Then

$$\gamma(S(G)) \leq \gamma(G) + (n - t_1 - t_2 - \cdots - t_r).$$

**Proof:** For each star $S_i$ with center $x$ and claws $y_1, y_2, \ldots, y_{i-1}$, the corresponding points of $T_i' = \{x', y_1', y_2', \ldots, y_{i-1}'\}$ are dominated by $x$, $y_1$, $y_2$, $\ldots$, $y_{i-1}$. Therefore $D_1 = D \cup T$ where $T$ is the set of points in $V_{i}'$ not belonging to $T_i'$ is a dominating set of $S(G)$.

Therefore $\gamma(S(G)) \leq |D_1| = |D| + |T| = \gamma(G) + (n - t_1 - t_2 - \cdots - t_r)$. ■
7.2.8. **Theorem:** Let $G$ be a connected graph. Let $\gamma_{td}$ denotes the cardinality of minimum total dominating set. Then $\gamma(S(G)) = \gamma_{td}$.

**Proof:** Let $D$ be a total dominating set with minimum cardinality for $G$. Therefore $D$ dominates $S(G)$. Therefore $\gamma(S(G)) \leq |D| = \gamma_{td}$.

Suppose $T$ is a dominating set of $S(G)$ with $|T| < \gamma_{td}$. Without loss of generality, $T \subseteq V_1$. Therefore $T$ is not a total dominating set of $G$. Hence $T$ has an independent point. Therefore $T$ cannot dominate $S(G)$, which is a contradiction. Therefore $\gamma(S(G)) = \gamma_{td}$.

7.2.9. **Corollary:** $\gamma_{td}(S(G)) = \gamma_{td}(G) = \gamma(S(G))$

**Proof:** Let $D$ be a total dominating set of $G$ with minimum cardinality. Therefore every point of $V(G)$ is adjacent to some point of $D$. Let $u' \in V'$. Then $u$ is adjacent to some point $x$ of $D$. Therefore $u'$ is adjacent to $x$. Therefore $D$ is a total dominating set of $S(G)$. Therefore

$\gamma_{td}(S(G)) \leq \gamma_{td}(G) = \gamma(S(G)) \leq \gamma_{td}(S(G))$.

Therefore $\gamma_{td}(S(G)) = \gamma_{td}(G) = \gamma(S(G))$.  

7.2.10. Theorem: Let G be a connected graph then \( \gamma_i(S(G)) = 2\gamma_i(G) \)

Proof: Let D be a minimum independent dominating set of G. Then the points corresponding to the elements of D will not be dominated by points of D. Therefore \( D \cup D' \) where \( D' = \{ v : v \in D \} \) is an independent dominating set of \( S(G) \). Therefore \( \gamma_i(S(G)) \leq 2\gamma_i(G) \).

Suppose \( D_1 \) is a minimum independent dominating set of \( S(G) \). Let \( D_1 \cap V(G) = \{ u_1, u_2, ..., u_k \} \). Then \( \{ u_1, u_2, ..., u_k \} \) dominates \( N'(u_1), N'(u_2), ..., N'(u_k) \). \( D_1 \) contains \( u_1', u_2', ..., u_k' \). Let \( u \in V(G) \) and \( u \) not adjacent to \( u_1, u_2, ..., u_k \). Then \( u' \) is not adjacent to \( u_1, u_2, ..., u_k \). Therefore \( u' \) must belong to \( D_1 \). Therefore for every point \( u \) not adjacent to \( u_1, u_2, ..., u_k \), the corresponding points \( u' \in D_1 \) and \( u \notin \{ u_1', u_2', ..., u_k' \} \).

Let \( I \) be the cardinality of minimum independent set of \( V(G) - (\{ u_1, u_2, ..., u_k \} \cup N(u_1) \cup ... \cup N(u_k)) \), then these \( I \) points are not adjacent to \( \{ u_1', u_2', ..., u_k' \} \) and not adjacent to the
corresponding points of these 1 points we have to take the corresponding 1 points. Also we have to take atleast some other 1 points in V' to dominate these 1 independent points and other left out points of V'. For : Let this minimum independent dominating set be \( \{y_1, y_2, \ldots, y_1\} \). For each \( y_i \) there exists atleast one \( z_i \in V_i \) such that \( y_i \) dominates \( z_i \) and \( z_1, z_2, \ldots, z_i \) are distinct. For, if there exist less than 1 points to be dominated by \( y_1, y_2, \ldots, y_1 \) then some of the points \( y', y_2, \ldots, y' \) are isolated. Otherwise less than \( (1 - 1) \) points will be adjacent to \( u_1, u_2, \ldots, u_i \) and hence the domination number will be less than 1. If there are isolated points in \( y_1, y_2, \ldots, y_i \) then they cannot be dominated by any point in \( D_1 \cap V' \). So those isolated points must belong to \( \{u_1, u_2, \ldots, u_k\} \), a contradiction. Now \( z'_1, z'_2, \ldots, z'_i \) have to be taken in \( D_1 \), since \( u_1, u_2, \ldots, u_k \) cannot dominate \( z'_1, z'_2, \ldots, z'_i \).

Therefore \( |D_1| \geq 2k + 2l = 2(k + l) \geq 2 \gamma_i(G) \).

That is, \( \gamma_i(S(G)) \geq 2 \gamma_i(G) \). Hence \( \gamma_i(S(G)) = 2 \gamma_i(G) \).

\[ \square \]
We conclude with the following open problems.

1. Let $\Gamma_1$ be the set of all graphs on a set $V$ and $\Gamma_2$ be the set of all topologies on $V$. Let $\phi : \Gamma_1 \rightarrow \Gamma_2$ and $\psi : \Gamma_2 \rightarrow \Gamma_1$. Determine the relation between $\gamma(G)$ and $\gamma(\psi \circ \phi(G))$.

2. Determine $\phi$ and $\psi$ so that $\psi \circ \phi$ preserves certain specified parameters of $G$. 