CHAPTER V

ORTHOGONAL GRAPHS

In this chapter, we define orthogonality among graphs of the same order and investigate the orthogonality of several classes of graphs of the same order.

5.1. INTRODUCTION: It is well known that there are many binary operations in the set of graphs like $\cup$, $+$, etc. We wish to define a function on the set of graphs of the same order, which behaves like an inner product in vector spaces. If we consider the set of all square matrices of order $n$ over $\mathbb{C}$, it forms a vector space over $\mathbb{C}$. It has an inner product defined as follows. If $A$, $B$ are two square matrices of order $n$, $<A, B> = \text{Trace} (AB^T)$. This naturally leads to a definition of inner product in the set of all graphs of the same order. Let $G_1$, $G_2$ be two graphs of the same order say $n$. Consider specific ordering of the vertices of $G_1$ as well as $G_2$. With respect to this ordering, we can find $\text{Trace}(A(G_1).A(G_2))$. The inner product $<G_1, G_2>$ with respect to a particular choice of labeling of vertices is defined as $<G_1, G_2> = \text{Trace}(A(G_1).A(G_2))$. (Since the adjacency matrix is symmetric we take $A(G_2) (=A(G_2)^T)$). Change of ordering of the vertices result in a change
of the value of \( < G_1, G_2 > \). For example, let \( G_1 \) be the cycle \( C_4 \) formed by vertices \( v_1, v_2, v_3, v_4 \). Let \( v_1, v_2, v_3, v_4 \) be an ordering of the vertices. Then

\[
A(G_1) = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

Let \( G_2 \) be the graph \( K_2 \cup K_2 \) formed by the vertices \( u_1, u_2, u_3, u_4 \) where \( u_1 \) and \( u_3 \) are adjacent and \( u_2 \) and \( u_4 \) are adjacent. For the ordering \( u_1, u_3, u_4, u_2 \),

\[
A(G_2) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
A(G_1) \cdot A(G_2) = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\]

Trace\((A(G_1) \cdot A(G_2))\) = 4. If we take the ordering \( u_1, u_2, u_3, u_4 \) then

\[
A(G_2) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

In this case

\[
A(G_1) \cdot A(G_2) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Trace\((A(G_1) \cdot A(G_2))\) = 0. Thus inner product depends upon the ordering of the vertices in the graphs considered.
5.1.1. Definition: Let \( \mathcal{G}_n \) denote the set of all graphs with order \( n \). Let \( G_1, G_2 \in \mathcal{G}_n \). If there exists an ordering of vertices of \( G_1, G_2 \) such that

\[
< G_1, G_2 > = \text{Trace}(A(G_1)A(G_2)) = 0 \text{ where } A(G_1) \text{ and } A(G_2) \text{ are the adjacent matrices of } G_1 \text{ and } G_2
\]

then \( G_1 \) is said to be orthogonal to \( G_2 \) and this is denoted by \( G_1 \perp G_2 \).

5.1.2. Remark: Orthogonality is a symmetric relation in \( \mathcal{G}_n \).

5.1.3. Proposition: Let \( G \in \mathcal{G}_n \). Then \( G \) and \( \overline{G} \) are orthogonal.

Proof:

\[
A(G) = \begin{pmatrix}
0 & a_{12} & \ldots & a_{1n} \\
a_{21} & 0 & \ldots & a_{2n} \\
& \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \ldots & 0
\end{pmatrix}
\]

\[
A(\overline{G}) = \begin{pmatrix}
0 & b_{12} & \ldots & b_{1n} \\
b_{21} & 0 & \ldots & b_{2n} \\
& \cdots & \cdots & \cdots \\
b_{n1} & b_{n2} & \ldots & 0
\end{pmatrix}
\]

If \( a_{ij} = 1 \) then \( b_{ij} = 0 \) and vice versa (\( i \neq j \), \( i, j = 1, 2, \ldots, n \)).

Let \( C_{ij} = (i, j)^{th} \) element of \( A(G)A(\overline{G}) \)

Then \( C_{ii} = a_{i1}b_{1i} + a_{i2}b_{2i} + \ldots + a_{in}b_{ni} \)

\[
= a_{i1}b_{1i} + a_{i2}b_{2i} + \ldots + a_{in}b_{ni}
\]

\[
= 0 \text{ (since } a_{ij}b_{ij} = 0)\]

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Therefore \( \text{Trace}(A(G)A(\overline{G})) = 0 \) and hence \( G \) and \( \overline{G} \) are orthogonal.

5.1.4. **Proposition**: Any graph is orthogonal to any spanning subgraph of its complement.

**Proof**: The adjacency matrix of a spanning subgraph of the complement where the ordering of the vertices of the subgraph is the same as the ordering of the complement differs from the adjacency matrix of the complement in non diagonal entries such that some of the non diagonal entries which are 1 in the adjacency matrix of the complement may be 0 in the adjacency matrix of the spanning subgraph. Hence the diagonal entries of the product of the adjacency matrix of the given graph and that of the spanning subgraph are all 0 and hence the proposition.

5.1.5. **Proposition**: Let \( G_1 \) be a graph. Consider an ordering of vertices of \( G_1 \) and \( \overline{G}_1 \) such that \( G_1 \perp \overline{G}_1 \). Let \( G_2 \) be a graph on \( V(G_1) \) such that \( G_2 \perp G_1 \). Then \( G_2 \) is a spanning subgraph of \( G_1 \).

**Proof**: Let for a suitable ordering of vertices of \( G_1 \) and \( G_2 \), \( G_1 \perp G_2 \). Then \( \text{Trace}(A(G_1)A(G_2)) = 0 \). Therefore each diagonal entry of \( A(G_1)A(G_2) = 0 \). Let \( C_{ij} = (i, j)^{th} \) element of \( A(G_1)A(G_2) \).
Then \( C_{ii} = a_{i1}b_{i1} + a_{i2}b_{i2} + \ldots + a_{in}b_{in} \)

\[ = a_{i1}b_{i1} + a_{i2}b_{i2} + \ldots + a_{in}b_{in} \]

\[ = 0 \]

Therefore \( a_{ij}b_{ij} = 0 \) for every \( j = 1 \) to \( n \). If \( a_{ij} = 1 \), then \( b_{ij} = 0 \). If \( a_{ij} = 0 \), then \( b_{ij} \) can be either 0 or 1. If we take \( b_{ij} = 1 \) whenever \( a_{ij} = 0 \) we get the adjacency matrix of the complement. Thus if \( v_i \) and \( v_j \) are not adjacent in \( G_1 \), \( v_i \) and \( v_j \) can be non adjacent in \( G_2 \) or adjacent in \( G_2 \). If \( v_i \) and \( v_j \) are adjacent in \( G_1 \), then \( v_i \) and \( v_j \) are non adjacent in \( G_2 \). Thus \( G_2 \) is a spanning subgraph of \( \overline{G_1} \).

5.1.6. Proposition: The maximum size of a graph orthogonal to a given graph is \( nc_2 - p \) where \( n \) is the number of vertices and \( p \) is the number of edges of the given graph. The maximum size is reached in the complement.

Proof: Obvious.

5.1.7. Proposition: If a graph \( G \) contains a dominating vertex then any graph perpendicular to \( G \) contains at least one isolated vertex.

Proof: A dominating vertex of a graph becomes an isolated vertex of the complement. Hence the proposition.
5.1.8. **Remark**: Any graph orthogonal to a star, wheel or complete graph is disconnected. Hence the domination numbers of a graph with a dominating vertex and its orthogonal are different.

5.1.9. **Remark**: Any graph with a dominating vertex cannot be orthogonal to itself.

5.1.10. **Proposition**: $C_n (n \geq 5)$ is orthogonal to itself with respect to different orderings of the vertices.

**Proof**: Let $v_1, v_2, \ldots, v_n$ be the vertices of $C_n$ considered in this order. Consider $C_n$. Remove the edges $v_1v_3, v_3v_5, v_4v_6, \ldots, v_nv_2, v_1v_4, v_2v_5, \ldots, v_nv_3, \ldots$ so that the total number of edges removed is $(n^2 - 5n + 2)/2$. The resulting graph is $C_n$ and keeping the ordering of $C_n$ the same for the resulting graph we get that $C_n$ is orthogonal to itself.

5.1.11. **Corollary**: $C_n$ is orthogonal to $P_n (n \geq 5)$.

**Proof**: In the above construction remove $v_2v_4$ also then we get the corollary.

5.1.12. **Corollary**: $P_n$ is orthogonal to $P_n (n \geq 5)$.

5.1.13. **Proposition**: Any subdivision of a star is orthogonal to itself.
5.1.14. Definition: Let \( \mathcal{G}_n \) denote the set of all digraphs with order \( n \).

Let \( G_1, G_2 \in \mathcal{G}_n \). If there exists an ordering of vertices of \( G_1, G_2 \) such that
\[
< G_1, G_2 > = \text{Trace}(R(G_1)R(G_2)^T) = 0
\]
then \( G_1 \) is said to be orthogonal to \( G_2 \). Here \( R(G) \) stands for reachability matrix of \( G \).

5.1.15. Theorem: A necessary and sufficient condition for two digraphs \( G_1, G_2 \) on the same set of vertices \( v_1, v_2,\ldots, v_n \) is that if whenever \( v_i \) is reachable from \( v_j \) in \( G_1 \) then \( v_i \) is not reachable from \( v_j \) in \( G_2 \).

Proof: Let \( G_1, G_2 \) be two digraphs with \( p \) vertices. The reachability matrix \( R = (r_{ij}) \) is the \( p \times p \) matrix with \( r_{ij} = 1 \) if \( v_j \) is reachable from \( v_i \) and 0 otherwise. If \( v_i \) is reachable from \( v_j \) in \( G_1 \) and \( v_i \) is not reachable from \( v_j \) in \( G_2 \) then
\[
\text{Trace}(R(G_1)R(G_2)^T) = 0.
\]
Therefore \( G_1 \perp G_2 \).

Given \( G_1 \) is perpendicular to \( G_2 \). Therefore \( v_i \) is reachable from \( v_j \) in \( G_1 \) and \( v_i \) is not reachable from \( v_j \) in \( G_2 \). Hence the result follows immediately.

5.1.16. Theorem: In the case of an intree \( G_1 \) and outtree \( G_2 \),
\[
\text{Trace}(R(G_1)R(G_2)^T) = 0.
\]
Proof: An outtree is a digraph with a source having no semicycles; an intree is its dual. Whenever \( v_i \) is reachable from \( v_j \) in an outtree, then \( v_j \) is not reachable from \( v_i \) in an tree. Therefore \( \text{Trace}(R(G_1).R(G_2)^T) = 0. \)

5.1.17. Remark: In a tournament, \( \text{Trace}(R(G_1).R(G_2)^T) = 0. \)

5.1.18. Theorem: If \( G_1 \) and \( G_2 \) are strongly connected digraphs then \( G_1 \) is not orthogonal to \( G_2 \) (where the matrices associated are distance matrices).

Proof: A digraph is strongly connected if every pair of points are mutually reachable. Then we get, the non-zero entries in the diagonal of \( D(G_1).D(G_2)^T \). Therefore \( G_1 \) is not orthogonal to \( G_2 \).

5.1.19. Result: A complete symmetric digraph is orthogonal to a digraph \( G \) if and only if \( G \equiv nk_1 \).