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Introduction to Nonlinear Dynamics

1.1 Nonlinearity

The change of state of physical systems as a function of time is their evolution, whose study constitutes the subject of dynamics. Obviously changes take place due to the interplay of forces, simple and complex, which act on the systems. The nature of evolution of different physical systems depends upon the nature of the forces acting on them and on their initial state. A system is said to be linear/nonlinear, if it is subjected to a linear/nonlinear force.

Basically linear systems are generally gradual and gentle; their smooth and regular behaviour is met in various physical phenomena such as slowly flowing streams, small vibrations of a pendulum, electrical circuits that operate under normal conditions, engines working at low power, slowly reacting chemicals and
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so on. In contrast, nonlinear systems can exhibit regular as well as complicated and irregular behaviours depending upon various factors. One example is given below.

Consider a pendulum placed in air medium. The restoring force is proportional to $\sin \theta$ which is nonlinear in $\theta$. Its equation of motion is

$$\frac{d^2 \theta}{dt^2} + \alpha \frac{d\theta}{dt} + \frac{g}{L} \sin \theta = 0,$$  \hspace{1cm} (1.1)

where $\alpha$ is the damping coefficient. For small displacements $\sin \theta \approx \theta$ and the pendulum is a linear system. In this approximation its equation of motion is a linear differential equation which is given by

$$\frac{d^2 \theta}{dt^2} + \alpha \frac{d\theta}{dt} + \frac{g}{L} \theta = 0.$$ \hspace{1cm} (1.2)

When the bob is subjected to a weak periodic external force, in the limit $t \to \infty$ the bob exhibits periodic oscillations with the frequency of the applied force. However if the initial displacement is sufficiently large then one has to analyse the full nonlinear equation with an additional term representing the external force. It not only shows oscillatory behaviour but also shows rotational motion and other complicated periodic and irregular oscillations as well, depending upon the strength of the applied force.

Typically nonlinear problems are in the form of nonlinear evolution equations [1-20] which describe how some variables or a set of variables evolve in time from the given initial state. They are often characterized by nonlinear ordinary or
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partial differential equations depending on whether the system is finite or infinite dimensional. Also these dynamical systems are mainly concerned with the study of integrable and nonintegrable systems. Integrating the equations of motion completely, obtaining analytic solutions and finding acceptable constants of motion or integrals of motion or invariants of such nonlinear systems seem to be rare. The existence of such integrable nonlinear dynamical systems often means the existence of very regular motion, but it is not precluded in nonintegrable systems. Systems governed by nonlinear ordinary differential equations in general exhibit aperiodic and chaotic behaviour with solutions having complicated singularity structure in the complex time plane. The irregular and unpredictable time evolution of many nonlinear systems is typically called chaos. However very often the variation of particular physical property may depend in a continuous fashion on both space and time variables. Typical examples include the vibrations of a string, propagation of electromagnetic waves, waves on the surface of water and so on. In these cases the dynamics of the underlying systems are governed by partial differential equations and they are often considered to be infinite dimensional (in phase space). Among the nonlinear continuous systems the special case is the nonlinear dispersive wave systems [21-74], because these systems exhibit an exciting type of wave solution of finite energy, having remarkable stability properties called soliton as represented in Figure 1.1. Thus soliton and chaos form the two major building blocks of nonlinear dynamics.
1.2 Soliton

Soliton refers to a localized solution of (1+1) dimensional nonlinear partial differential equation. Here we introduce the soliton concept via its remarkable and beautiful historical path. In 1830’s the Scottish Naval Architect, John Scott Russell, was carrying out investigations on the shapes of the hulls of ships and speeds and forces needed to propel them for Union Canal Company so that they can make safe steam navigation. In August 1834, riding on a horse back, Scott Russell observed the "Great Wave of Translation" in the Union canal connecting the Scottish cities of Edinburgh and Glasgow, where he was carrying out his experiments. He reported his observations to the British association in his 1838 "Report on Waves" in the following delightful description [75].

*I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled
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Figure 1.2: Evolution of an initial elevation of water into a solitary wave.

forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1838, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Scott Russel also performed some laboratory experiments generating solitary
waves by dropping a weight at one end of a water channel as shown in Figure 1.2. He was able to deduce empirically that the volume of water in the wave is equal to the volume of water displaced and further that the speed $c$ of the solitary wave is obtained from the relation

$$c^2 = g(h + a), \quad (1.3)$$

where $a$ is the amplitude of the wave, $h$ is the undisturbed depth of water and $g$ is the acceleration due to gravity. A consequence of Eq. (1.3) is that taller waves travel faster. Russel also made many other observations and experiments on solitary waves. In particular he tried to generate waves of depression by raising the weight from the bottom of the channel initially. He found however that an initial depression becomes a train of oscillatory waves whose lengths increase and amplitudes decrease with time.

Boussinesq [76] and Rayliegh [77] independently assumed that a solitary wave has a length much greater than the depth of the water and thereby deduced Russel’s empirical formula for $c$ from the equations of motion of an inviscid incompressible fluid. They further showed essentially that the wave height above the mean level $h$ is given by

$$\zeta(x,t) = a \text{sech}^2[(x - ct)/b], \quad (1.4)$$

where $b^2 = 4h^2(h + a)/3a$ for any positive amplitude $a$. 
In 1895 Korteweg and de Vries [78] developed this theory and found an equation governing the two-dimensional motion of weakly nonlinear long waves:

$$\frac{\partial \zeta}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{h}} \left( \zeta \frac{\partial \zeta}{\partial x} + \frac{2}{3} \alpha \frac{\partial \zeta}{\partial x} + \frac{2}{3} \sigma \frac{\partial^3 \zeta}{\partial x^3} \right),$$  \hspace{1cm} (1.5)

where $\alpha$ is a small but otherwise arbitrary constant, $\sigma = \frac{1}{3} h^3 - Th/g\rho$, and $T$ is the surface tension of the liquid of density $\rho$. Note that in the approximations used to derive this equation one considers long waves propagating in the direction of increasing $x$. A similar equation, with $-\partial \zeta/\partial t$, may be applied to waves propagating in the opposite direction. Eq. (1.5) was brought into the nondimensional form by making the transformation $t = \frac{1}{2} \sqrt{\frac{g}{h\rho}} \tau$, $x = -\sigma^{-\frac{1}{2}} \xi$, $u = \frac{1}{2} \eta + \frac{1}{3} \alpha$. Hence they obtained

$$u_t + 6uu_x + u_{xxx} = 0,$$  \hspace{1cm} (1.6)

where subscripts denote partial differentiations. This is essentially the original form of the Korteweg de Vries equation or the KdV equation. Not until more than half a century later, the KdV equation and its solitary wave received their rightful recognition by physicists and mathematicians. The breakthrough came in an entirely different context this time in the study of wave propagation in nonlinear lattices. In early 1950s, Fermi, Pasta and Ulam [79] considered the dynamics of a chain of weakly coupled nonlinear oscillators to check the widely held concepts of ergodicity and equipartition of energy in irreversible statistical mechanics. Much to their surprise, the system did not approach energy equipartition that is the
energy did not spread throughout the normal modes, but returned almost peri-
odically to the originally excited mode and a few nearby modes. This remarkable
near recurrence phenomenon, known now a days as the FPU problem, was further
examined and it was confirmed that the nonlinear terms did not guarantee the
approach of the system to thermal equilibrium.

The entirely unexpected results of FPU experiments motivated Zabusky and
Kruskal [80] in 1965 to re investigate this problem. They considered the following
initial value problem in a periodic domain

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \delta \frac{\partial^3 u}{\partial t^3} = 0,
\]

(1.7)

where \( u(2, t) = u(0, t), u_x(2, t) = u_x(0, t), u_{xx}(2, t) = u_{xx}(0, t) \) for all \( t \), and

\[
u(x, 0) = \cos \pi x, \quad 0 \leq x \leq 2.
\]

(1.8)

Putting \( \delta = 0.022 \), Zabusky and Kruskal computed \( u \) for \( t > 0 \). Figure 1.3
shows the solutions of the KdV equation with \( \delta = 0.022 \) and \( u(x, 0) = \cos \pi x \)
for \( 0 \leq x \leq 2 \). The dotted curve gives \( u \) at \( t = 0 \), the broken curve gives \( u \)
at \( t = 1/\pi \) and the continuous curve gives \( u \) at \( t = 3.6/\pi \). Note the formation
of eight more-or-less distinct solitary waves, whose crests lie close to a straight
line. They found that the solution breaks up into a train of eight solitary waves,
each like a sech-squared solution, that these waves move through one another as
the faster ones catch up the slower ones and that finally the initial state recurs.
A remarkable quality of these solitary waves was that they could collide with
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Each other and yet preserve their shapes and speeds after the collision. This particle-like nature led Zabusky and Kruskal to name such waves solitons.

A ‘soliton’ is not precisely defined, but is used to describe any solution of a nonlinear equation or system which (i) represents a wave of permanent form; (ii) is localized, decaying or becoming constant at infinity; and (iii) may interact strongly with other solitons so that after the interaction it retains its form, almost as if the principle of superposition were valid.

The observation of soliton interaction in an ocean is given in Figure 1.4. Mathematically, when solitons are present in (1+1) dimensional nonlinear partial differential equations, the solution of the cauchy initial value problem asymptotically separates into N-distinct exponentially localized states.

Figure 1.3: Experimental observation of soliton.
1.3 Methods for Solving Nonlinear Differential Equations

Soon after the discovery of soliton by numerical experiment, the question was then, can one solve nonlinear partial differential equations analytically. Given a nonlinear partial differential equation, there is no general way of knowing whether it has soliton solutions or not or how the soliton solutions can be found. The following are some methods that were developed and applied successfully to particular cases.
1.3 Methods for Solving Nonlinear Differential Equations

1.3.1 The Inverse Scattering Transform (IST) Method

The Inverse Scattering Transform method may be considered as a nonlinear Fourier transform method. The procedure was originally developed by Gardner, Greene, Kruskal and Miura [81]. The analysis proceeds in three steps similar to the case of Fourier transform method applicable for linear dispersive systems: (i) Direct scattering transform analysis, (ii) Analysis of time evolution of scattering data and (iii) Inverse scattering transform (IST) analysis.

The method is schematically shown in Figure 1.5. The details are given below.

Figure 1.5: Schematic diagram of the inverse scattering transform method.
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1.3.1.1 Direct Scattering Analysis and Scattering Data at $t = 0$

The given information is the initial data $u(x, 0)$, which has the property that it vanishes sufficiently fast as $x \to \pm \infty$. Now considering the Schrödinger spectral problem at $t = 0$,

$$
\psi_{xx} + [\lambda - u(x, 0)]\psi = 0, \quad u \xrightarrow{|x| \to \infty} 0. \tag{1.9}
$$

it is well known from linear spectral theory that the system (1.9) admits

(1) a finite number of bound states with eigenvalues

$$
\lambda = -\kappa_n^2, \quad n = 1, 2, ..., N \tag{1.10}
$$

and normalization constants $C_n(0)$ of the associated eigen states and

(2) a continuum or scattering states with the continuous eigenvalues,

$$
\lambda = k^2, \quad -\infty < k < \infty. \tag{1.11}
$$

They are further characterized by the reflection coefficients $R(k, 0)$ and transmission coefficients $T(k, 0)$ such that

$$
|R(k, 0)|^2 + |T(k, 0)|^2 = 1. \tag{1.12}
$$

Thus from given potential (initial data) $u(x, 0)$ with the boundary conditions $u \to 0$ as $x \to \pm \infty$, one can carry out a direct scattering analysis of Eq. (1.9) to obtain the scattering data at $t = 0$:

$$
S(0) = \kappa_n, C_n(0), R(k, 0), T(k, 0), \quad n = 1, 2, ..., N, \quad -\infty < k < \infty. \tag{1.13}
$$
1.3 Methods for Solving Nonlinear Differential Equations

1.3.1.2 Time Evolution of Scattering Data

Now as the potential \( u(x, t) \) evolves from its initial value \( u(x, 0) \) so that it satisfies the KdV equation and the corresponding scattering data given by Eq. (1.13) evolve from \( S(0) \) to \( S(t) \). In order to understand this one can use the time evolution equation of the eigen function

\[
\psi_t = -4 \frac{\partial^3 \psi}{\partial x^3} + 3(u \frac{\partial}{\partial x} + \frac{\partial}{\partial x}) \psi. \tag{1.14}
\]

Since the scattering data is intimately associated with the asymptotic \((x \to \pm \infty)\) behaviour of the eigenfunction, where the potential \( u(x, t) \to 0 \), it is enough if we confine the analysis to this region. Thus as \( x \to \pm \infty \), Eq. (1.14) can be written as

\[
\psi_t = -4 \frac{\partial^3 \psi}{\partial x^3}, \quad x \to \pm \infty. \tag{1.15}
\]

It may be shown that

\[
R(k, t) = R(k, 0)e^{-8ik^3 t}, \tag{1.16}
\]

\[
T(k, t) = T(k, 0), \tag{1.17}
\]

\[
\kappa_n(t) = \kappa_0, \quad n = 1, 2, \ldots, N \tag{1.18}
\]

\[
C_n(t) = C_n(0)e^{4\kappa_n^3 t}. \tag{1.19}
\]

Thus at an arbitrary future instant of time ’t’, the scattering data \( S(t) \) corresponding to the potential \( u(x, t) \) evolves from \( S_n(0) \) of the initial data \( u(x, 0) \):

\[
S(t) = \{\kappa_n(t) = \kappa_n(0), C_n(t) = C_n(0)e^{4\kappa_n^3 t}, n = 0, 1, 2, \ldots, N, \}
\]
1.3 Methods for Solving Nonlinear Differential Equations

\[ R(k, t) = R(k, 0)e^{-8i k^3 t}; -\infty < k < \infty \}. \quad (1.20) \]

1.3.1.3 Inverse Scattering Analysis

Now given the scattering data \( S(t) \) as in Eq.(1.20) at time \( t \), one can invert the data and obtain uniquely the potential \( u(x, t) \) of the Schrödinger spectral problem, in which the time variable \( t \) enters only as a parameter. It can be done by solving a linear, Volterra type singular, integral equation called Gelfand-Levitan-Marchenko integral equation [82]. The scattering data \( S(t) \) given by Eq.(1.20) is given as input into this integral equation, which when solved gives the solution \( u(x, t) \) of the KdV equation. The linear integral equation reads

\[
K(x, y, t) + F(x+y, t) + \int_x^\infty F(y+z, t)K(x, z, t)dz = 0, \quad y > x
\]

(1.21)

where

\[
F(x+y, t) = \sum_{n=1}^N C_n^2(t)e^{-\kappa_n(x+y)} + \frac{1}{2\pi} \int_{-\infty}^\infty R(k, t)e^{ik(x+y)}dk.
\]

(1.22)

Note that in the above equation the time variable \( t \) enters only as a parameter and that all the information about the scattering data are contained in the function \( F(x+y, t) \). Solving Eqs.(1.21) and (1.22) we finally obtain the potential

\[
u(x, t) = -2 \frac{d}{dx} K(x, x + 0, t).
\]

(1.23)

Thus the initial value problem of the KdV equation stands solved. For solving the general initial value problem, one has to solve the Gelfand-Levitan-Marchenko integral equation, with the full set of scattering data \( S(t) \). However for the special
but important class of the so-called reflectionless potentials characterized by the condition
\[
R(k, t) = 0, \quad (1.24)
\]
it is possible to solve fully the Gelfand - Levitan - Marchenko integral equation. Then if there are \(N\) bound states, the corresponding solution, \(u(x, t)\), turns out to be the \(N\)-soliton solution.

The discovery of IST opened the door to theoretical and analytical studies of completely integrable systems which possess multi soliton solutions [83-93].

1.3.2 Linear Scattering Problems and Associated Nonlinear Evolution Equations

Following the development of the method of inverse scattering to solve the initial value problem for the KdV equation by Gardner, Greene, Kruskal and Miura, it was then of considerable interest to determine whether the method would be applicable to other physically important nonlinear evolution equations. The method of inverse scattering is highly nontrivial and was thought by some to be a fluke, a clever transformation analogous to the Cole - Hopf transformation which linearizes Burger’s equation
\[
u_t - 2uu_x - u_{xx} = 0. \quad (1.25)
\]
If we make the transformation \(u = -\phi_x/\phi\), then \(\phi(x, t)\) satisfies the linear heat equation \(\phi_t - \phi_{xx} = 0\). However Zakharov and Shabat [94] proved that the method
indeed was no fluke by extending Lax’s ideas [95] in order to relate the nonlinear Schrödinger (NLS) equation

\[ iu_t + u_{xx} + \kappa u^2 u^* = 0, \]  

(1.26)

where * denotes the complex conjugate and \( \kappa \) is a constant to a certain linear scattering problem. They showed that if

\[ L = i \begin{pmatrix} 1 + k & 0 \\ 0 & 1 - k \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \]  

(1.27)

\[ M = ik \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} \frac{-iuu^*}{1+k} & u_x^* \\ u_x & \frac{iuu^*}{1-k} \end{pmatrix}, \]  

(1.28)

with \( \kappa = 2/(1 - k^2) \), then \( L \) and \( M \) satisfy Lax’s equation

\[ L_t + [L, M] = 0, \]  

(1.29)

if and only if \( u(x, t) \) satisfies the NLS equation (1.26). Eq. (1.29) called, the Lax’s equation contains a nonlinear evolution equation if \( L \) and \( M \) are correctly chosen. If a nonlinear partial differential equation arises as the compatibility condition of two such operators \( L \) and \( M \), then Eq. (1.29) is called the Lax representation of the partial differential equation and \( L \) and \( M \) are the associated Lax pair of operators. The Lax condition indeed plays a very important role in soliton theory, not only for these equations but also for the other soliton systems as well. In fact it is now an accepted fact that existence of a Lax pair is indeed a decisive hallmark of integrable systems.
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1.3.3 AKNS formalism

Ablowitz, Kaup, Newell and Segur [96] developed a procedure, which showed that the initial value problem for a remarkable large class of physically interesting nonlinear evolution equations could be solved by this method [97-116]. Given a suitable scattering problem, this method allows one to derive the nonlinear evolution equations solvable by IST with that scattering problem. To illustrate the method consider two linear equations

\[ v_x = Xv, \]  
\[ v_t = Tv, \]  
\[ (1.30) \]
\[ (1.31) \]

where \( v \) is an \( n \)-dimensional vector and \( X \) and \( T \) are \( nxn \) matrices. If we require that Eqs. (1.30) and (1.31) are compatible, that is requiring that \( v_{xt} = v_{tx} \), then \( X \) and \( T \) must satisfy

\[ X_t - T_x + [X, T] = 0. \]  
\[ (1.32) \]

This equation (1.32) and Lax’s equation (1.29) are similar; Eqs. (1.30) and (1.31) are somewhat more general as they allow eigenvalue dependence other than \( Lv = \lambda v \).

As an example, consider the 2x2 scattering problem given by

\[ v_{1,x} = -ikv_1 + q(x, t)v_2, \]  
\[ (1.33) \]
\[ v_{2,x} = ikv_2 + r(x, t)v_1 \]  
\[ (1.34) \]
and the most general linear time dependence

\begin{align}
v_{1,t} &= Av_1 + Bv_2, \quad \text{(1.35)} \\
v_{2,t} &= Cv_1 + Dv_2, \quad \text{(1.36)}
\end{align}

where \( A, B, C \) and \( D \) are scalar functions of \( q(x,t), r(x,t) \) and \( k \), independent of \((v_1, v_2)\). Essentially, we just specify that

\[
X = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

in Eqs. (1.30) and (1.31). Note that if there were any \( x \)-derivatives on the right-hand side of Eqs.(1.35) and (1.36) then they can be eliminated by use of Eqs. (1.33) and (1.34). Further, when \( r = -1, \) Eqs. (1.33) and (1.34) reduce to the Schrödinger scattering problem

\[
v_{2,xx} + (k^2 + q)v_2 = 0, \quad \text{(1.38)}
\]

which is equivalent to Eq. (1.9), with \( k^2 = -\lambda \).

This procedure provides a simple technique which allows us to find nonlinear evolution equations expressible in the form of Eq. (1.32). From Eqs. (1.33) and (1.35) we find

\begin{align}
v_{1,t} &= -ik v_{1,t} + q v_2 + q v_{2,t}, \quad \text{(1.39)} \\
&= -ik(A v_1 + B v_2) + q v_2 + q(C v_1 + D v_2), \quad \text{(1.40)} \\
v_{1,x} &= A_x v_1 + Av_{1,x} + B_x v_2 + B v_{2,x}, \quad \text{(1.41)} \\
&= A_x v_1 + A(-ik v_1 + q v_2) + B_x v_2 + B(ik v_2 + r v_1). \quad \text{(1.42)}
\end{align}
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According to the compatibility condition, we get

\[ A_x = qC - rB, \quad (1.43) \]
\[ B_x + 2ikB = q_t - (A - D)q. \quad (1.44) \]

Similarly from Eqs. (1.34) and (1.36) we get

\[ v_{2,xt} = ikv_{2,t} + r_tv_1 + rv_{1,t}, \quad (1.45) \]
\[ = -ik(Cv_1 + Dv_2) + r_tv_1 + r(Av_1 + Bv_2), \quad (1.46) \]
\[ v_{2,tx} = C_xv_1 + Cv_{1,x} + D_xv_2 + Dv_{2,x}, \quad (1.47) \]
\[ = C_xv_1 + C(-ikv_1 + qv_2) + D_xv_2 + D(ikv_2 + rv_1). \quad (1.48) \]

Using again the compatibility condition, we obtain

\[ C_x - 2ikC = r_t + (A - D)r, \quad (1.49) \]
\[ (-D)_x = qC - rB. \quad (1.50) \]

Therefore from Eqs. (1.43) and (1.50), without loss of generality we may assume \( D = -A \) and hence it is seen that \( A, B \) and \( C \) necessarily satisfy the compatibility conditions

\[ A_x = qC - rB, \quad (1.51) \]
\[ B_x + 2ikB = q_t - 2Aq, \quad (1.52) \]
\[ C_x - 2ikC = r_t + 2Ar. \quad (1.53) \]

We now solve Eqs. (1.51)-(1.53) for \( A, B \) and \( C \). In general, this can only be done if another condition is satisfied, this condition being the evolution equation. Since
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$k$, the eigenvalue is a free parameter, we may find solvable evolution equations by seeking finite power series expansions for $A$, $B$ and $C$:

\[ A = \sum_{j=0}^{n} A_j k^j, \quad B = \sum_{j=0}^{n} B_j k^j, \quad C = \sum_{j=0}^{n} C_j k^j. \]  

(1.54)

Using Eq. (1.54) in Eqs. (1.51)-(1.53) and equating coefficients of powers of $k$, we obtain $3n+5$ equations. There are $3n+3$ unknowns, $A_j, B_j, C_j, j = 0, 1, ..., n$ and so we also obtain two nonlinear evolution equations for $r$ and $q$. Now let us consider some examples. For $n = 2$, $A, B$ and $C$ are quadratic in $k$, i.e.,

\[ A = A_2 k^2 + A_1 k + A_0, \]  

(1.55)

\[ B = B_2 k^2 + B_1 k + B_0, \]  

(1.56)

\[ C = C_2 k^2 + C_1 k + C_0. \]  

(1.57)

Substitute Eqs. (1.55)-(1.57) in Eqs. (1.51)-(1.53) and equate powers of $k$. The coefficients of $k^3$ immediately give $B_2 = C_2 = 0$. At order $k^2$, we obtain $A_2 = a = constant$, $B_1 = iaq$, $C_1 = iar$. At order $k^1$, we obtain $A_1 = b = constant$.

For simplicity we set $b = 0$, then $B_0 = -\frac{1}{2}aq_x$ and $C_0 = \frac{1}{2}ar_x$. Finally at order $k^0$, we get $A_0 = \frac{1}{2}arq + c$ with $c$, a constant. In addition we obtain the following evolution equations

\[ -\frac{1}{2}aq_{xx} = q_t - aq^2r, \]  

(1.58)

\[ \frac{1}{2}ar_{xx} = r_t + aqr^2. \]  

(1.59)

If in Eqs. (1.58) and (1.59) we set $r = \mp q^*$ and $a = 2i$, we get the NLS equation

\[ iq_t = q_{xx} \pm 2q^2q^*. \]  

(1.60)
Setting $r = \mp q^*$, we find

\begin{align*}
A &= 2ik^2 \pm iq^*, \\
B &= 2qk + iq_x, \\
C &= \mp2q^*k \pm iq_x^*
\end{align*}

(1.61) (1.62) (1.63)

satisfy Eqs. (1.61)-(1.63) provided that $q(x,t)$ satisfies the NLS equation (1.60).

The same technique can be extended to find the Lax pair of operators for other solitonic systems. Also many other interesting nonlinear evolution equations can be obtained for suitable choices of $A$, $B$ and $C$.

### 1.3.4 Hirota’s Bilinearization Method

In 1971, R. Hirota introduced a new ”direct method” for constructing multisoliton solutions to integrable nonlinear evolution equations [117-126]. The idea is to make a transformation into new variables, so that in these new variables multisoliton solutions appear in a particularly simple form. The equations turned out to be quadratic in the new dependent variables and all derivatives appeared as Hirota’s bilinear derivatives, this is called ”Hirota bilinear form”.

(i) The first step of this method is to make suitable transformations of nonlinear partial differential and difference equations which provide that the equations are in quadratic form in dependent variables. This new form is called bilinear form. To find such a transformation is not easy for some equations and sometimes it requires the introduction of new dependent and sometimes even independent
variables.

(ii) As a second step we introduce a special differential operator called Hirota D - operator which is used to write the bilinear form of the equation as a polynomial of D - operator which we call the Hirota bilinear form. Unfortunately there is no systematic way to construct the Hirota bilinear form for given nonlinear partial differential and difference equations. In fact, some equations may not be written in the Hirota bilinear form but perhaps in trilinear or multilinear forms. Here we can conjecture that all completely integrable nonlinear partial differential and difference equations can be put into the Hirota bilinear form. On the other hand, the converse is not true, that is, there exist some equations which are not integrable but have Hirota bilinear forms.

(iii) The last step of the Hirota method is using the perturbation expansion, which becomes finite as we will see, in the Hirota bilinear form and analyzing the coefficients of the perturbation parameter and its powers separately. At that point the information we gain makes us to reach to multisoliton solutions if the equation is integrable.

The Hirota direct method has taken an important role in the study of integrable systems. Most equations (even nonintegrable ones) having Hirota bilinear form possess automatically one and two soliton solutions. When we come to the three soliton solutions we come across a very restrictive condition. Actually this condition is not sufficient to search the integrability of an equation but it can be
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used as a powerful tool for this purpose.

The bilinearization involves the Hirota’s operator defined as

\[
D^m_x D^n_t (a,b) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x,t)b(x',t') \bigg|_{x=x',t=t'}.
\]  

(1.64)

For example here we give the procedure for finding soliton solutions to KdV equation (1.6) using Hirota’s bilinearization method.

We use the transformation

\[
u = 2 \frac{\partial^2}{\partial x^2} \log F
\]  

(1.65)

in Eq. (1.6) and it reduces to the bilinear form

\[
FF_{xt} - F_x F_t + FF_{xxxx} - 4F_x F_{xxx} + 3F_{xx} = 0.
\]  

(1.66)

Expanding \( F \) in a formal power series in terms of a small parameter \( \epsilon \) as

\[
F = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ...
\]  

(1.67)

and substituting Eq. (1.67) in Eq. (1.66) and equating each power of \( \epsilon \) separately to zero, we get a system of linear partial differential equations as

\[
O(\epsilon^0) : \quad 0 = 0, \quad (1.68)
\]

\[
O(\epsilon) : \quad f^{(1)}_{xt} + f^{(1)}_{xxxx} = 0, \quad (1.69)
\]

\[
O(\epsilon^2) : \quad f^{(2)}_{xt} + f^{(2)}_{xxxx} = f^{(1)}_x f^{(1)}_t + 4f^{(1)}_{xxx} f^{(1)}_x - 3(f^{(1)}_{xx})^2, \quad (1.70)
\]

\[
O(\epsilon^3) : \quad f^{(3)}_{xt} + f^{(3)}_{xxxx} = f^{(1)}_x f^{(2)}_t + f^{(2)}_x f^{(1)}_t - f^{(1)}_x f^{(2)}_t - f^{(2)}_{xt} f^{(1)} + 4f^{(2)}_{xxx} f^{(1)}_x - f^{(1)}_{xx} f^{(2)}_x. \quad (1.71)
\]
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To solve these linear PDEs we can easily write the solution as

\[ f^{(i)} = \sum_{i=1}^{N} e^{\eta_i}, \quad \eta_i = k_i x - \omega_i t + \eta_0^{(i)}, \quad \omega_i = k_i^3, \quad \eta_0^{(i)} = \text{constant}. \quad (1.72) \]

(i) One Soliton Solution

For example, for \( N = 1 \)

\[ f^{(1)} = e^{\eta_1}, \quad \eta_1 = k_1 x - \omega_1 t + \eta_0^{(1)} \quad (1.73) \]

with \( \omega_1 = k_1^3 \) and

\[ f^{(2)}_{xt} + f^{(2)}_{xxxx} = 0. \quad (1.74) \]

So we can choose \( f^{(2)} = 0 \). Then one can easily prove that all \( f^{(i)} = 0, \ i \geq 3 \).

Thus the solution to (1.66) becomes

\[ F_1 = 1 + e^{\eta_1}. \quad (1.75) \]

Using this in the transformation (1.65), we finally obtain

\[ u(x, t) = \frac{k_1^2}{2} \text{sech}^2 \frac{1}{2} (k_1 x - k_1^3 t + \eta_0^{(1)}). \quad (1.76) \]

This gives the exact one soliton solution.

(ii) Two Soliton Solution

Proceeding in a similar way for \( N = 2 \), one has

\[ f^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad \eta_1 = k_1 x - \omega_1 t + \eta_0^{(1)}, \quad \eta_2 = k_2 x - \omega_2 t + \eta_0^{(2)}. \quad (1.77) \]
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Substituting Eq. (1.77) in the resultant linear partial differential equations and solving, we obtain

\[ f^{(2)} = e^{m + \eta_2 + A_{12}}, \quad e^{A_{12}} = \frac{(k_1 - k_2)}{(k_1 + k_2)}^2. \]  (1.78)

The solution of Eq. (1.66) becomes

\[ F = 1 + e^m + e^{\eta_2} + e^{m + \eta_2 + A_{12}}. \]  (1.79)

Eq. (1.65) then gives the two soliton solution of KdV equation as

\[ u = \frac{1}{2} (k_2^2 - k_1^2) \left[ \frac{k_2 \cosech^2(\eta_2/2) + k_1^2 \sech^2(\eta_1/2)}{(k_2 \coth(\eta_2/2) - k_1 \tanh(\eta_1/2))^2} \right]. \]  (1.80)

(iii) N-Soliton Solutions

One can proceed as before for the general case with the choice

\[ f^{(1)} = e^{\eta_1} + e^{\eta_2} + \ldots + e^{\eta_N} \]  (1.81)

and then solve successively for \( f^{(2)}, \ldots, f^{(N)} \) and finally obtain \( F \) and \( u \) which give the N-soliton solution.

1.3.5 Darboux Transformation

The Darboux transformation, as the representative among various methods in soliton theory provides an effective procedure for constructing exact multisoliton solutions for integrable nonlinear evolution equations and understanding the nonlinear mechanisms in different physical contexts [127,128]. To proceed, we
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introduce the following transformation to Eqs. (1.30) and (1.31)

\[ \tilde{v} = Dv = (\lambda I - S)v, \]  \hspace{1cm} (1.82)

where \( \tilde{v} \) is the vector eigen function which has the same dimension as \( v \). The central task is to construct the matrix \( S \) based on the solutions of the linear eigenvalue problems (1.30) and (1.31), which take the form

\[ S = H\Lambda H^{-1}, \quad H = \begin{pmatrix} h_1 & -h_2^* \\ h_2 & h_1 \end{pmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2). \]  \hspace{1cm} (1.83)

\( D \) is the Darboux matrix, while \( \tilde{v} \) also satisfies the same linear eigenvalue problems (1.30) and (1.31) which can be written as

\[ \tilde{v}_x = \tilde{X}\tilde{v}, \]  \hspace{1cm} (1.84)

\[ \tilde{v}_t = \tilde{T}\tilde{v}. \]  \hspace{1cm} (1.85)

The compatibility condition now becomes

\[ \tilde{X}_t - \tilde{T}_x + [X, T] = 0. \]  \hspace{1cm} (1.86)

Then the Darboux matrix \( D \) is required to satisfy

\[ D_x = \tilde{X}D - DX, \]  \hspace{1cm} (1.87)

\[ D_t = \tilde{T}D - DT. \]  \hspace{1cm} (1.88)

According to the knowledge about the Darboux transformation of higher degree, the \( n \)-times iteration of the Darboux matrix is in the form

\[ D_n(x, t, \lambda) = \lambda^n I + \sum_{k=1}^{n} \Gamma_k \lambda^{n-k}. \]  \hspace{1cm} (1.89)
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which is the product of $n$ Darboux transformation of degree one namely

$$D_n(x, t, \lambda) = (\lambda I - S_n)(\lambda I - S_{n-1})(\lambda I - S_1).$$  \hfill (1.90)$$

Hence from Eqs. (1.89) and (1.90), it is found that

$$\Gamma_1 = -(S_1 + S_2 + ... + S_n).$$  \hfill (1.91)$$

To this state, the $n$-times Darboux transformation for eigenfunction is given as

$$v_n(x, t, \lambda) = \left(\lambda^n I + \sum_{k=1}^{n} \Gamma_k \lambda^{n-k}\right).$$  \hfill (1.92)$$

The following step is to compute out the matrix $\Gamma_1$ in order to generate the new solutions from Eq. (1.92). Because $h_j (j = 1, 2)$ is the column solution of linear systems (1.30) and (1.31) with $\lambda = \lambda_j$, it satisfies the linear algebraic equations $D_n(x, t, \lambda_j)h_j = 0$, i.e.,

$$\sum_{k=1}^{n} \Gamma_k \lambda_j^{n-k}h_j = -\lambda_j^n h_j, \quad j = 1, 2$$  \hfill (1.93)$$

which can be rewritten in a matrix form

$$(\Gamma_1, \Gamma_2, ..., \Gamma_n)W_n = B$$  \hfill (1.94)$$

with

$$W_n = \begin{pmatrix} k_1^{n-1}h_1 & k_1^{n-1}h_2 & ... & k_1^{n-1}h_{2n} \\ k_1^{n-2}h_1 & k_1^{n-2}h_2 & ... & k_1^{n-2}h_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ h_1 & h_2 & ... & h_{2n} \end{pmatrix}$$  \hfill (1.95)$$
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and

\[ B = \left( \begin{array}{cccc} k_1^n h_1 & k_2^n h_2 & \cdots & k_{2n}^n h_{2n} \end{array} \right). \]  

(1.96)

With the use of Cramer’s rule, \( \Gamma_1 \) can be solved from the linear algebraic equations

\[ (\Gamma_1)_{pq} = \frac{\text{det}(M_q^{(p)})}{\text{det}(W_n)}, \quad (1 \leq p \leq 2), \]  

(1.97)

where \( M_q^{(p)} \) can be obtained by replacing the \( q \)-th row of \( W_n \) by the \( p \)-th row of \( B \).

1.3.6 Perturbation Methods

Any real system, due to the existence of boundaries, defects, impurities, dissipation, external fields, etc., usually experiences some kind of perturbation or when soliton behaviour in the system is being observed or measured, the measurement itself is bound to alter the state of the system to some extent. Such systems are often described by perturbed nonlinear equations which are not integrable. Perturbation theory comprises mathematical methods that are used to find an approximate solution to such a problem which cannot be solved exactly, by starting from the exact solution of a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a small term to the mathematical description of the exactly solvable problem. Some of the perturbation methods are given below:
1.3 Methods for Solving Nonlinear Differential Equations

1.3.6.1 Sine-Cosine Function Method

To obtain explicit travelling and solitary wave solutions of nonlinear evolution equations, we establish the algorithm of sine-cosine function method [129]. Consider a nonlinear partial differential equation of the form

\[ P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \]  

which describes the dynamical wave solution \( u(x, t) \). It is useful to summarize the main steps of this method.

(i) To find the traveling wave solution of Eq. (1.98), we introduce the wave variable \( \xi = t - \delta x \), so that \( u(x, t) = u(\xi) \) and

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2}, \quad \frac{\partial}{\partial x} = -\delta \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial x^2} = \delta^2 \frac{\partial^2}{\partial \xi^2}.
\]

Hence Eq. (1.98) becomes the ordinary differential equation (ODE)

\[ P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \]

(ii) We then integrate Eq. (1.101) as many times as possible, setting the constant of integration zero.

(iii) The solution of Eq. (1.101) is assumed to be of the form

\[ u(x, t) = \lambda \sin^\beta(\mu \xi) \]

or in terms of a cosine function as

\[ u(x, t) = \lambda \cos^\beta(\mu \xi), \]

29
where \( \lambda, \mu \) and \( \beta \) are the parameters that will be determined.

(iv) We substitute Eq. (1.102) or (1.103) in the reduced equation (1.101), balance the terms of the sine functions when Eq. (1.102) is used, or balance the cosine functions when Eq. (1.103) is used to find \( \beta \). Then we solve the resulting system of algebraic equations by using the symbolic computations to obtain all possible values of the parameters \( \lambda \) and \( \mu \).

The main advantage of this method is that it can be applied directly to most types of differential equations. Another important advantage of the method is that it is capable of greatly reducing the size of computational work.

### 1.3.6.2 Tanh Method for Partial Differential Equations

Recently, variants of the tanh method have been successfully applied to many nonlinear polynomial systems of PDE’s in any number of independent variables [130] and the detailed algorithm is given here.

Consider a system of partial differential equations

\[
\Delta(u(x), u'(x), u''(x), \ldots, u^{(m)}(x)) = 0. 
\]

(1.104)

We seek solutions in the travelling frame of reference

\[
T = \tanh \xi, \quad \xi = c_1 x + c_2 t + \delta, 
\]

(1.105)

where the components \( c_1 \) and \( c_2 \) of the wave vector \( x \) and the phase \( \delta \) are constants. Based on the identity \( \cosh^2 \xi - \sinh^2 \xi = 1 \),

\[
tanh' \xi = \sech^2 \xi = 1 - \tanh^2 \xi, 
\]

(1.106)
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\[ \tanh^3 \xi = -2 \tanh \xi + 2 \tanh^3 \xi, \text{etc.} \]  \hspace{1cm} (1.107)

Therefore, the first and consequently all higher-order derivatives are polynomial in \( T \). Thus repeatedly applying the chain rule

\[ \frac{\partial}{\partial t} = \frac{\partial \xi}{\partial x_j} \frac{d}{dT} \frac{d}{dT} = c_j(1 - T^2) \frac{d}{dT}, \]  \hspace{1cm} (1.108)

Eq. (1.104) transforms into a coupled system of nonlinear ODEs

\[ \Gamma(T, U(T), U'(T), ...) = 0. \]  \hspace{1cm} (1.109)

Since we seek polynomial solutions

\[ U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j, \]  \hspace{1cm} (1.110)

the leading exponents \( M_i \) must be determined before the \( a_{ij} \) can be computed. Substituting \( U_i(T) \) in Eq. (1.109), the coefficients of every power of \( T \) in every equation must vanish. In particular, the highest degree terms must vanish. Since the highest degree terms only depend on \( T^{M_i} \) in Eq. (1.110), it suffices to substitute \( U_i(T) = T^{M_i} \) in Eq. (1.109). In the resulting polynomial system, equating every two possible highest exponents in every component gives a linear system for determining the \( M_i \). The linear system is then solved for \( M_i \).

If one or more exponents \( M_i \) remain undetermined, we assign strictly positive integer value to the free \( M_i \), so that every Eq. (1.109) has at least two different terms with equal highest exponents in \( T \).

To generate the system for the unknown coefficients \( a_{ij} \) and wave parameters \( c_j \), substitute Eq. (1.110) in Eq. (1.109) and set the coefficients of \( T_j \) to zero. The
resulting nonlinear algebraic system for the unknown \( a_{ij} \) is parameterized by the wave parameters \( c_j \) and the parameters in (1.104) if any.

### 1.3.6.3 Tanh - Method for Discrete Differential Equations (DDEs)

While there has been considerable work done on finding exact solutions to PDEs, little work is being done to symbolically compute exact solutions of DDEs. Here we present an adaptation of the tanh method to solve nonlinear polynomial differential difference equations [130]. Given a system of \( M \) polynomial DDEs,

\[
\Delta(u_{n+p_1}(x),...,u_{n+p_k}(x),...,u'_{n+p_1}(x),...,u'_{n+p_k}(x),...,u_{r}(x),...,u_{r}(x)) = 0,
\]

(1.111)

where the dependent variable \( u \) has \( M \) components \( u_i \), the continuous variable \( x \) has \( N \) components \( x_i \), the discrete variable \( n \) has \( Q \) components \( n_j \), the \( k \) shift vectors \( p_i \) and \( u^{(r)}(x) \) denotes the collection of mixed derivative terms of order \( r \). We assume that any arbitrary coefficients that parameterize the system are strictly positive.

Using the properties of hyperbolic tangent

\[
T_n = \tanh(\xi_n),
\]

(1.112)

where

\[
\xi_n = \sum_{i=Q}^{n} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = d.n + c.x + \delta.
\]

(1.113)