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Three Spine Alpha Helical Protein Chain with Cubic-Quintic Nonlinearity

7.1 Introduction

In chapter 6, we considered a model of alpha helical proteins in which the dynamics is found to be governed by continuum and discrete fourth order NLS equations. Having identified a higher order integrable model, we now look for more integrable models by incorporating quadrupole - quadrupole type interaction between the next neighbouring molecules. As we are interested in cubic-quintic type nonlinearity in the system we choose the parameters so that the resulting dynamics is governed by a set of three coupled cubic-quintic NLS equation. We obtain the linear eigen value problem of the above model through AKNS formalism and
study the energy transfer properties using Darboux transformation.

7.2 Model and Equations of Motion

We consider a model representing homogeneous alpha helical protein chain with dipole - dipole interactions, quadrupole - quadrupole interactions and nearest and next nearest neighbour molecular excitations and interspine coupling. The energy associated with the above description of the alpha helical protein molecule can be accommodated in the following Hamiltonian

$$H = H_{31} + H_{32} + H_{33} + H_{34},$$

(7.1)

where $H_{31} = H_{21}$, $H_{33} = H_{23}$ and $H_{34} = H_{24}$.

The internal molecular excitations with quadrupole - quadrupole type coupling between the adjacent unit cells with the amide - I excitation energy $H_{32}$ is given by

$$H_{32} = \sum_{n,\alpha,\rho} \left\{ B_{n,\alpha}^\dagger \left( E_1 B_{n,\alpha} B_{n,\alpha}^\dagger B_{n,\alpha} B_{n,\alpha} + J_1 B_{n,\alpha} B_{n+\rho,\alpha}^\dagger B_{n+\rho,\alpha} B_{n+\rho,\alpha} 
\right.
\right.
\left.
\right.
\left.
\left. 
- J_2 [B_{n+1,\alpha}^\dagger B_{n+1,\alpha+1} B_{n,\alpha} - B_{n-1,\alpha} B_{n-1,\alpha}^\dagger B_{n-1,\alpha+1}]
\right.
\right.
\left.
\right.
\left.
\right. 
- J_3 [B_{n+1,\alpha}^\dagger B_{n+1,\alpha-1} B_{n+1,\alpha} - B_{n-1,\alpha} B_{n-1,\alpha}^\dagger B_{n-1,\alpha-1}]
\right.
\right.
\left.
\right.
\left.
\right. 
- J_4 [B_{n+1,\alpha}^\dagger B_{n+1,\alpha+1} B_{n,\alpha} - B_{n-1,\alpha} B_{n-1,\alpha}^\dagger B_{n-1,\alpha+1}]
\right.
\right.
\left.
\right.
\left.
\right. 
- J_5 [B_{n+1,\alpha}^\dagger B_{n+1,\alpha-1} B_{n,\alpha} - B_{n-1,\alpha} B_{n-1,\alpha}^\dagger B_{n-1,\alpha-1}]
\right) \}. \quad (7.2)$$

The quadrupole - quadrupole type coupling between the adjacent unit cells is represented by the parameters $J_1$, $J_2$, $J_3$, $J_4$ and $J_5$. 

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7.2 Model and Equations of Motion

The Hamiltonian for the collective excitations of the coherent state can be written as

\[
\langle H \rangle = \sum_{n,\alpha,\rho} \left\{ a_{n+1,\alpha}^* (E_0 + W) a_{n,\alpha} + E_1 a_{n,\alpha}^* a_{n,\alpha} - J_0 a_{n+1,\alpha} - \right.
\]

\[
- L a_{n,\alpha} - J_1 a_{n,\alpha}^* a_{n+1,\alpha} - J_2 [a_{n+1,\alpha}^* a_{n+1,\alpha}] + J_3 [a_{n+1,\alpha}^* a_{n+1,\alpha}] + J_4 [a_{n+1,\alpha}^* a_{n+1,\alpha}] + J_5 [a_{n+1,\alpha}^* a_{n+1,\alpha}]
\]

\[
- \left. \right\} (\chi_1 a_{n,\alpha}^* a_{n,\alpha} + \chi_2 a_{n+1,\alpha}^* a_{n+1,\alpha}) + (\chi_3 a_{n,\alpha}^* a_{n-1,\alpha} + \chi_4 a_{n+1,\alpha}^* a_{n+2,\alpha}) + (\chi_5 a_{n,\alpha}^* a_{n+1,\alpha} + \chi_6 a_{n,\alpha}^* a_{n+2,\alpha}) + (\chi_7 a_{n,\alpha}^* a_{n+1,\alpha} + \chi_8 a_{n,\alpha}^* a_{n+2,\alpha}) + \eta_1 a_{n,\alpha} + \eta_2 (a_{n+1,\alpha} - a_{n-1,\alpha}) \right) \right\}.
\]

Having constructed the Hamiltonian for the collective excitations of the coherent state involving different interactions, now in order to understand the underlying dynamics, we derive the equations of motion for the dynamic variables \(a_{n,\alpha}, b_{n,\alpha}\), by using the Hamiltonian (7.3) in the equation of motion (4.14) and
after using the commutation relations (2.13) and (2.14):

\[ \frac{i\hbar}{dt} a_{n,\alpha} = \sum_{\rho} \{(E_0 + W)a_{n,\alpha} - J_{01}a_{n+\rho,\alpha} - La_{n,\alpha+\rho} - 2J_1 a_{n,\alpha} a_{n+\rho,\alpha} \]

\[ - J_2[a_{n+1,\alpha} a_{n+1,\alpha+1} + a_{n+1,\alpha+1} a_{n+1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha+1} a_{n+1,\alpha+1} \]

\[ - J_3[a_{n+1,\alpha} a_{n-1,\alpha+1} a_{n+1,\alpha-1} a_{n-1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha-1} a_{n-1,\alpha+1} \]

\[ - J_4[a_{n+1,\alpha} a_{n,\alpha+1} a_{n+1,\alpha+1} a_{n+1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha+1} a_{n,\alpha+1} \]

\[ - J_5[a_{n+1,\alpha} a_{n+1,\alpha-1} a_{n-1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha-1} a_{n-1,\alpha+1} \]

\[ + a_{n,\alpha}[\chi_1(b_{n+1,\alpha} - b_{n-1,\alpha}) + \chi_2(b_{n+1,\alpha-1} - b_{n-1,\alpha-1}) \]

\[ + \chi_3(b_{n+1,\alpha+1} - b_{n-1,\alpha+1}) + a_{n+1,\alpha}[\chi_4(b_{n+2,\alpha} - b_{n,\alpha}) \]

\[ + \chi_5(b_{n+2,\alpha-1} - b_{n,\alpha-1}) + \chi_6(b_{n+2,\alpha+1} - b_{n,\alpha+1}) \]

\[ + a_{n-1,\alpha}[\chi_7(b_{n,\alpha} - b_{n-2,\alpha}) + \chi_8(b_{n,\alpha-1} - b_{n-2,\alpha-1}) \]

\[ + \chi_9(b_{n,\alpha+1} - b_{n-2,\alpha+1}) + a_{n,\alpha}(2\chi_{10}|a_{n,\alpha}|^2 + \chi_{11}|a_{n,\alpha+1}|^2 \]

\[ + \chi_{12}|a_{n,\alpha-1}|^2(b_{n+1,\alpha} - b_{n-1,\alpha}) + \chi_{11}|a_{n,\alpha-1}|^2 b_{n+1,\alpha-1} \]

\[ - b_{n-1,\alpha-1} + \chi_{12}|a_{n,\alpha+1}|^2 a_{n,\alpha}(b_{n+1,\alpha+1} - b_{n-1,\alpha+1}) \]

\[ + [\eta_1 a_{n,\alpha} + \eta_2(a_{n+1,\alpha} - a_{n-1,\alpha})][(c_{n,\alpha} + \rho - 2c_{n,\alpha})] \].

(7.4)

The equations of motion for the dynamical variables \( b_{n,\alpha} \) and \( c_{n,\alpha} \) are the same as given in Eqs. (6.8) and (4.11) respectively. Using Eq. (4.12) in Eq. (7.4), we get

\[ \frac{i\hbar}{dt} a_{n,\alpha} = \sum_{\rho} \{(E_0 + W)a_{n,\alpha} - J_{01}a_{n+\rho,\alpha} - La_{n,\alpha+\rho} - 2J_1 a_{n,\alpha} a_{n+\rho,\alpha} \]

\[ - J_2[a_{n+1,\alpha} a_{n+1,\alpha+1} a_{n+1,\alpha} + a_{n+1,\alpha+1} a_{n+1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha+1} a_{n+1,\alpha+1} \]

\[ - J_3[a_{n+1,\alpha} a_{n-1,\alpha+1} a_{n+1,\alpha-1} a_{n-1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha-1} a_{n-1,\alpha+1} \]

\[ - J_4[a_{n+1,\alpha} a_{n,\alpha+1} a_{n+1,\alpha+1} a_{n+1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha+1} a_{n,\alpha+1} \]

\[ - J_5[a_{n+1,\alpha} a_{n+1,\alpha-1} a_{n-1,\alpha} - a_{n-1,\alpha} a_{n-1,\alpha-1} a_{n-1,\alpha+1} \]

\[ + a_{n,\alpha}[\chi_1(b_{n+1,\alpha} - b_{n-1,\alpha}) + \chi_2(b_{n+1,\alpha-1} - b_{n-1,\alpha-1}) \]

\[ + \chi_3(b_{n+1,\alpha+1} - b_{n-1,\alpha+1}) + a_{n+1,\alpha}[\chi_4(b_{n+2,\alpha} - b_{n,\alpha}) \]

\[ + \chi_5(b_{n+2,\alpha-1} - b_{n,\alpha-1}) + \chi_6(b_{n+2,\alpha+1} - b_{n,\alpha+1}) \]

\[ + a_{n-1,\alpha}[\chi_7(b_{n,\alpha} - b_{n-2,\alpha}) + \chi_8(b_{n,\alpha-1} - b_{n-2,\alpha-1}) \]

\[ + \chi_9(b_{n,\alpha+1} - b_{n-2,\alpha+1}) + a_{n,\alpha}(2\chi_{10}|a_{n,\alpha}|^2 + \chi_{11}|a_{n,\alpha+1}|^2 \]

\[ + \chi_{12}|a_{n,\alpha-1}|^2(b_{n+1,\alpha} - b_{n-1,\alpha}) + \chi_{11}|a_{n,\alpha-1}|^2 b_{n+1,\alpha-1} \]

\[ - b_{n-1,\alpha-1} + \chi_{12}|a_{n,\alpha+1}|^2 a_{n,\alpha}(b_{n+1,\alpha+1} - b_{n-1,\alpha+1}) \]

\[ + [\eta_1 a_{n,\alpha} + \eta_2(a_{n+1,\alpha} - a_{n-1,\alpha})][(c_{n,\alpha} + \rho - 2c_{n,\alpha})] \].
\[ -J_3[a_{n+1,a}a_{n,a-1}a_{n+1,a-1} - a_{n-1,a}a_{n,a-1}a_{n-1,a-1}] + a_{n,a}[\chi_1(b_{n+1,a} - b_{n-1,a}) + \chi_2(b_{n+1,a-1} - b_{n-1,a-1})] \\
+ \chi_3(b_{n+1,a+1} - b_{n-1,a+1}) + a_{n+1,a}(\chi_4(b_{n+2,a} - b_{n,a}) \\
+ \chi_5(b_{n+2,a-1} - b_{n,a-1}) + \chi_6(b_{n+2,a+1} - b_{n,a+1})] + a_{n-1,a}[\chi_7(b_{n,a} - b_{n-2,a}) + \chi_8(b_{n,a-1} - b_{n-2,a-1}) \\
+ \chi_9(b_{n,a+1} - b_{n-2,a+1})] + a_{n,a}(2\chi_{10}|a_{n,a}|^2 + \chi_{11}|a_{n,a+1}|^2 \\
+ \chi_{12}|a_{n,a-1}|^2(b_{n+1,a} - b_{n-1,a}) + \chi_{11}|a_{n,a-1}|^2a_{n,a}(b_{n+1,a-1} - b_{n,a-1}) \\
- b_{n-1,a-1} + \chi_{12}|a_{n,a+1}|^2a_{n,a}(b_{n+1,a+1} - b_{n-1,a+1}) + A[\eta_1a_{n,a} \\
+ \eta_2(a_{n+1,a} - a_{n,a})][\eta_1(|a_{n,a+\rho}|^2 - 2|a_{n,a}|^2) + \eta_2((a_{n,a+\rho}^* \\
(a_{n+1,a+\rho} - a_{n-1,a} - a_{n-1,a}-a_{n-1,a}))]. \tag{7.5} \]

In the continuum limit,

\[ i\hbar \frac{da_{\alpha}}{dt} = (E_0 + W - 2J_0)a_{\alpha} - J_0e^2a_{\alpha,xx} - La_{n,a+\rho} - J_1[2|a_{\alpha}|^2a_{\alpha} + e^2|a_{\alpha}|^2a_{\alpha,xx} \\
+ 2e^2a_{\alpha,x}a_{\alpha,x}a_{\alpha} + e^2a_{\alpha,xx}a_{\alpha}^2] - 2eJ_2a_{\alpha+1}^*(a_{\alpha}a_{\alpha+1,x} + a_{\alpha+1}a_{\alpha,x}) \\
- 2eJ_3a_{\alpha-1}^*(a_{\alpha}a_{\alpha-1,x} + a_{\alpha-1}a_{\alpha,x}) - 2eJ_4a_{\alpha+1}^*(a_{\alpha}a_{\alpha+1,x} \\
+ a_{\alpha+1}a_{\alpha,x}) - 2eJ_5a_{\alpha-1}(a_{\alpha}a_{\alpha-1,x} + a_{\alpha-1}a_{\alpha,x}) + 2\delta a_{\alpha}b_{\alpha,x} \\
+ 2\delta a_{\alpha}b_{\alpha-1,x} + 2\delta_2a_{\alpha}b_{\alpha+1,x} + 2e(\chi_{10}|a_{\alpha}|^2 + \chi_{11}|a_{\alpha+1}|^2) \]
7.2 Model and Equations of Motion

\[ + \chi_{12}|a_{\alpha-1}|^2 a_{\alpha} b_{\alpha,x} + 2\epsilon (\chi_{11}|a_{\alpha-1}|^2 b_{\alpha-1,x} + \chi_{12}|a_{\alpha+1}|^2 b_{\alpha+1,x}) a_{\alpha} \\
+ [\eta_{1} a_{\alpha} + 2\epsilon\eta_{2} a_{\alpha,x}] [\eta_{1} (|a_{\alpha+r}|^2 - 2 |a_{\alpha}|^2) + \epsilon\eta_{2} (a_{\alpha+r}^* a_{\alpha+r,x} - a_{\alpha}^* a_{\alpha,x})], \]  

(7.6)

\[ m \frac{d^2 b_{\alpha}}{dt^2} = \beta [2\epsilon (\delta_1 |a_{\alpha}|^2 + \delta_2 |a_{\alpha+1}|^2 + \delta_3 |a_{\alpha-1}|^2 + \chi_{10}|a_{\alpha}|^4 + \chi_{11}|a_{\alpha}|^2 |a_{\alpha+1}|^2 \\
+ \chi_{12}|a_{\alpha}|^2 |a_{\alpha-1}|^2) - \epsilon^2 (2\delta_3 a_{\alpha} a_{\alpha}^* + 2\delta_4 a_{\alpha+1} a_{\alpha+1}^* \\
+ 2\delta_5 a_{\alpha-1} a_{\alpha-1}^* )]. \]  

(7.7)

where \( \delta_1 = \chi_1 + \chi_4 + \chi_7, \delta_2 = \chi_2 + \chi_5 + \chi_8, \delta_3 = \chi_3 + \chi_6 + \chi_9. \) Defining a new variable \( \gamma_{3,\alpha} = -\epsilon b_{\alpha,x} \) and the wave variable \( \xi = x - v_1 t, \) Eq. (7.7) can be solved for \( \gamma_{3,\alpha} \) to give

\[ \gamma_{3,\alpha} = \beta [2\epsilon (\delta_1 |a_{\alpha}|^2 + \delta_2 |a_{\alpha+1}|^2 + \delta_3 |a_{\alpha-1}|^2 + 2\chi_{10}|a_{\alpha}|^4 + 2\chi_{11}|a_{\alpha}|^2 |a_{\alpha+1}|^2 \\
+ 2\chi_{12}|a_{\alpha}|^2 |a_{\alpha-1}|^2) - \epsilon^2 (2\delta_3 a_{\alpha} a_{\alpha}^* + 2\delta_4 a_{\alpha+1} a_{\alpha+1}^* \\
+ 2\delta_5 a_{\alpha-1} a_{\alpha-1}^* )]. \]  

(7.8)

where \( \delta_4 = \chi_4 - \chi_7, \delta_5 = \chi_5 - \chi_8 \) and \( \delta_6 = \chi_6 - \chi_9. \) Introducing Eq. (7.8) into Eq. (7.6) and after making the transformation

\[ a_{\alpha} = \epsilon q_{\alpha} \exp \left( \frac{-i\epsilon \delta_0}{2\hbar} \tau \right), \quad \xi \rightarrow X, \quad \text{and} \quad \tau \rightarrow \frac{\epsilon^2}{\hbar} T, \]  

(7.9)

we get

\[ i q_{\alpha, T} = -q_{\alpha, xx} + \beta_1 |q_{\alpha}|^2 q_{\alpha} + \beta_2 |q_{\alpha+1}|^2 q_{\alpha} + \beta_3 |q_{\alpha-1}|^2 q_{\alpha} + \beta_4 |q_{\alpha}|^4 q_{\alpha} \]
\[ + \beta_5 |q_a|^2 |q_{a+1}|^2 q_a + \beta_6 |q_a|^2 |q_{a-1}|^2 q_a + \beta_7 |q_{a+1}|^2 |q_{a-1}|^2 q_a \]
\[ + \beta_8 |q_{a-1}|^2 q_a + \beta_9 |q_{a+1}|^4 q_a + \beta_{10} q_{a,x}^2 q_{a,x} + \beta_{11} q_{a} q_{a+1} q_{a+1,x} \]
\[ + \beta_{12} q_a q_{a-1} q_{a-1,x} + \beta_{13} |q_a|^2 q_{a,x} + \beta_{14} |q_{a+1}|^2 q_{a,x} \]
\[ + \beta_{15} |q_{a-1}|^2 q_{a,x} + \beta_{16} q_{a,x} q_{a,x}^\ast + \beta_{17} q_{a+1} q_{a,x} q_{a+1,x} \]
\[ + \beta_{18} q_{a-1} q_{a,x} q_{a,x}^\ast + \beta_{19} |q_{a}|^2 q_a + \beta_{20} |q_{a+1}|^2 q_a + \beta_{21} |q_{a-1}|^2 q_a \]
\[ + \beta_{22} q_{a,x}^2 + \beta_{23} q_{a+1} q_{a,x} \]
\[ + \beta_{25} q_{a+1} q_{a+1,x} + \beta_{26} q_{a} q_{a+1,x} q_{a+1} + \beta_{27} q_{a} q_{a-1,x} q_{a-1} \]
\[ + \beta_{28} q_{a} q_{a} q_{a+1} q_{a+1} + \beta_{29} q_{a} q_{a} q_{a-1} q_{a-1} \]
\[ + 2 q_{a,x}^2 + \beta_{30} |q_a|^2 q_{a,x} \]  

with \( \delta_0 = E_0 + W - 2 J_0 \), \( \beta_1 = -4 \beta(\delta_1^2 + \delta_2^2 + \delta_3^2) - 2 \eta_1^2 - 2 J_1 \), \( \beta_2 = \beta_3 = -4 \beta(\delta_1 \delta_2 + \delta_2 \delta_3 + \delta_3 \delta_1) + \eta_2^2 \), \( \beta_4 = -12 \beta \chi_1 1 \delta_1 \), \( \beta_5 = -8 \beta(\delta_1 \chi_{11} + \delta_3 \chi_{12}) \), \( \beta_6 = -8 \beta(\delta_3 \chi_{10} + \delta_2 \chi_{11} + \delta_1 \chi_{12}) \), \( \beta_7 = -12 \beta \delta_3 \chi_{10} - 8 \beta \delta_2 \chi_{11} \), \( \beta_8 = -4 \beta(\delta_2 \chi_{10} + \delta_1 \chi_{11} + \delta_3 \chi_{12}) \), \( \beta_9 = -4 \beta(\delta_3 \chi_{10} + \delta_2 \chi_{11} + \delta_1 \chi_{12}) \), \( \beta_{10} = -4 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 4 \epsilon J_2 \), \( \beta_{11} = -4 \beta(\delta_1 \delta_5 + \delta_2 \delta_6 \chi_{12}) + 6 \eta_1 \eta_2 - 4 \epsilon J_3 \), \( \beta_{12} = -4 \beta(\delta_2 \delta_4 + \delta_3 \delta_5 + \delta_6) + 2 \eta_1 \eta_2 - 2 \epsilon J_4 \), \( \beta_{13} = -4 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 2 \epsilon J_5 \), \( \beta_{14} = -2 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 2 \epsilon J_6 \), \( \beta_{15} = -2 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 2 \epsilon J_7 \), \( \beta_{16} = 8 \beta \epsilon^2 (\delta_1^2 + \delta_2^2 + \delta_3^2) - 2 \epsilon J_1 \), \( \beta_{17} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \), \( \beta_{18} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \), \( \beta_{19} = -4 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) \), \( \beta_{20} = -4 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 2 \epsilon J_2 \), \( \beta_{21} = -4 \beta(\delta_1 \delta_4 + \delta_2 \delta_5 + \delta_3 \delta_6) - 2 \epsilon J_3 \), \( \beta_{22} = 4 \beta \epsilon^2 (\delta_1^2 + \delta_2^2 + \delta_3^2) - 2 \epsilon J_1 \), \( \beta_{23} = -2 \epsilon J_2 \), \( \beta_{24} = -2 \epsilon J_3 \), \( \beta_{25} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \), \( \beta_{26} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \), \( \beta_{27} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \), \( \beta_{28} = 4 \beta \epsilon^2 (\delta_1 \delta_5 + \delta_2 \delta_6 + \delta_3 \delta_6) \)
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\[ \delta_5 \delta_6 + \delta_4 \delta_6 \), \ \beta_{29} = 2 \epsilon^2 q_2^2, \ \beta_{30} = 2 \epsilon^2 J_1. \]

Putting \( \alpha = 1, 2, 3 \), Eq. (7.10) becomes

\[ i \dot{q}_{1,T} = -q_{1,xx} + \beta_1 |q_1|^2 q_1 + \beta_2 |q_2|^2 q_1 + \beta_3 |q_3|^2 q_1 + \beta_4 |q_1|^4 q_1 \]
\[ + \beta_5 |q_1|^2 |q_2|^2 q_1 + \beta_6 |q_1|^2 |q_3|^2 q_1 + \beta_7 |q_2|^2 |q_3|^2 q_1 \]
\[ + \beta_8 |q_1|^4 q_1 + \beta_9 |q_2|^4 q_1 + \beta_{10} q_1^2 q_1^* + \beta_{11} q_1 q_2 q_2^* \]
\[ + \beta_{12} q_1 q_3 q_3^* + \beta_{13} |q_1|^2 q_1, x + \beta_{14} |q_2|^2 q_1, x \]
\[ + \beta_{15} |q_3|^2 q_1, x + \beta_{16} q_1, x q_1^* + \beta_{17} q_2 q_1, x q_2^* \]
\[ + \beta_{18} q_3, x q_3^* + \beta_{19} |q_1|^2 q_1 + \beta_{20} |q_2|^2 q_1 + \beta_{21} |q_3|^2 q_1 \]
\[ + \beta_{22} q_1^2 q_1, x + \beta_{23} q_1 q_2 q_2, x + \beta_{24} q_1^2 q_3, x \]
\[ + \beta_{25} q_1 q_3 q_3^* + \beta_{26} q_1 q_2, x q_2^* + \beta_{27} q_1 q_3, x q_3^* \]
\[ + \beta_{28} q_3, x q_3^* + \beta_{29} (q_1, x q_2 q_2^* + q_1, x q_3 q_3^* \]
\[ + 2 q_1^* q_1^2 q_1, x \] \( (7.11) \)

\[ i \dot{q}_{2,T} = -q_{2,xx} + \beta_1 |q_2|^2 q_2 + \beta_2 |q_3|^2 q_2 + \beta_3 |q_1|^2 q_2 + \beta_4 |q_2|^4 q_2 \]
\[ + \beta_5 |q_2|^2 |q_3|^2 q_2 + \beta_6 |q_2|^2 |q_3|^2 q_2 + \beta_7 |q_3|^2 |q_1|^2 q_2 \]
\[ + \beta_8 |q_1|^4 q_2 + \beta_9 |q_3|^4 q_2 + \beta_{10} q_2^2 q_2^* + \beta_{11} q_2 q_3 q_3^* \]
\[ + \beta_{12} q_2 q_1 q_1^* + \beta_{13} |q_2|^2 q_2 + \beta_{14} |q_3|^2 q_2, x \]
\[ + \beta_{15} |q_1|^2 q_2, x + \beta_{16} q_2, x q_2^* + \beta_{17} q_3 q_2, x q_3^* \]
\[ + \beta_{18} q_2, x q_1^* + \beta_{19} |q_2|^2 q_2 + \beta_{20} |q_3|^2 q_2 + \beta_{21} |q_1|^2 q_2 \]
\[ + \beta_{22} q_2^2 q_2^* + \beta_{23} q_2 q_3 q_3^* + \beta_{24} q_2 q_1 q_1, x \]
\[ 
\chi = \frac{1}{625} \chi_1^2, \quad \chi_2 = \chi_3 = -2 \chi_4, \quad \chi_4 = 4 \chi_5, \quad \chi_5 = \chi_6, \quad \chi_6 = -\chi_7, \quad \chi_7 = -\chi_8, \\
\chi_8 = -\chi_9, \quad \chi_9 = \frac{-\rho_1^2}{125 \chi_1}, \quad \chi_{11} = \chi_{12}, \quad \chi_{12} = \frac{-1}{5 \chi_1} (\rho_1 + \rho_2 + \rho_3), \quad \chi_{11} = -\frac{1}{163} (\frac{8}{3} \rho_1^2 - 3 \rho_2^2 - 3 \rho_3^2 + 2 \rho_1 \rho_3 + 2 \rho_1 \rho_2 + 6 \rho_2 \rho_3), \quad \tau_2 = \zeta_3 = \rho_1, \quad \tau_3 = \zeta_1 = \rho_2, \quad \tau_1 = \zeta_2 = \rho_3,
\]

Eqs. (7.11)-(7.13) reduce to
\[ 
iq_{4,T} = -q_{3,xx} + \beta_1 |q_3|^2 q_3 + \beta_2 |q_1|^2 q_3 + \beta_3 |q_2|^2 q_3 + \beta_4 |q_3|^4 q_3 + \beta_5 |q_3|^2 q_3 + \beta_6 |q_3|^2 |q_2|^2 q_3 + \beta_7 |q_1|^2 |q_2|^2 q_3 + \beta_8 |q_2|^3 q_3 + \beta_9 |q_1|^4 q_3 + \beta_{10} |q_3|^2 q_3 + \beta_{11} q_3 q_1 q_3^* + \beta_{12} q_3 q_2 q_2^* + \beta_{13} |q_3|^2 q_3 + \beta_{14} |q_1|^2 q_3 + \beta_{15} |q_2|^2 q_3 + \beta_{16} q_3 q_3 q_3^* + \beta_{17} q_1 q_3 q_3^* + \beta_{18} q_3 q_3 q_2 q_2^* + \beta_{19} |q_3|^2 q_3 + \beta_{20} |q_1|^2 q_3 + \beta_{21} |q_2|^2 q_3 + \beta_{22} q_3 q_3 q_3^* + \beta_{23} q_3 q_1 q_1^* + \beta_{24} q_3 q_2 q_2^* + \beta_{25} q_3 q_1 q_1^* + \beta_{26} q_3 q_1 q_1^* + \beta_{27} q_3 q_2 q_2^* + \beta_{28} q_3 q_1 q_1^* + \beta_{29} (q_3 q_1 q_1^* + q_3 q_2 q_2^*) + 2q_3^* q_3^* + \beta_{30} |q_3|^2 q_3^*.
\]
7.2 Model and Equations of Motion

\[-\rho_1|q_1|^2 + (\zeta_2 - \rho_2)|q_2|^2 + (\zeta_3 - \rho_3)|q_3|^2]|q_3|^2q_1 - 2i[(\rho_1|q_1|^2 + \rho_2|q_2|^2)\rho_3|q_3|^2]q_1 + 2i(\rho_1q_1^*q_{1,x} + \rho_2q_2^*q_{2,x} + \rho_3q_3^*q_{3,x})q_1 = 0, \quad (7.14)\]

\[iq_{2,t} + q_{2,xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2)q_2 + (\tau_1|q_1|^2 + \tau_2|q_2|^2 + \tau_3|q_3|^2)^2q_1 + 2\tau_1[(\rho_1 - \tau_1)|q_1|^2 + (\rho_2 - \tau_2)|q_2|^2 + (\rho_3 - \tau_3)|q_3|^2]|q_1|^2q_2 + 2\tau_3[(\zeta_1 - \tau_1)|q_1|^2 + (\zeta_2 - \tau_2)|q_2|^2 + (\zeta_3 - \tau_3)|q_3|^2]|q_3|^2q_2 - 2i[(\tau_1|q_1|^2 + \tau_2|q_2|^2 + \tau_3|q_3|^2])q_2 = 0, \quad (7.15)\]

\[iq_{3,t} + q_{3,xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2)q_3 + (\zeta_1|q_1|^2 + \zeta_2|q_2|^2 + \zeta_3|q_3|^2)^2q_3 + 2\zeta_1[(\rho_1 - \zeta_1)|q_1|^2 + (\rho_2 - \zeta_2)|q_2|^2 + (\rho_3 - \zeta_3)|q_3|^2]|q_1|^2q_3 + 2\zeta_2[(\tau_1 - \zeta_1)|q_1|^2 + (\tau_2 - \zeta_2)|q_2|^2 + (\tau_3 - \zeta_3)|q_3|^2]|q_2|^2q_3 - 2i[(\zeta_1|q_1|^2 + \zeta_2|q_2|^2 + \zeta_3|q_3|^2)]q_3 = 0. \quad (7.16)\]

Eqs. (7.14)-(7.16) are a set of coupled equations which represent the dynamics of alpha helical proteins with cubic-quintic nonlinear couplings, which can be proved to be a set of completely integrable three coupled cubic-quintic NLS equations. To check the complete integrability of the Eqs. (7.14)-(7.16) we follow the AKNS procedure and construct the Lax pair of operators associated with the three coupled cubic-quintic NLS equations. The detailed calculations are presented in the following section.
7.3 Lax pair Using AKNS formalism

The integrability property of the two coupled cubic - quintic NLS equations has been established by Radhakrishnan et al [69] and in this context they constructed the Lax pair of operators for the two coupled cubic - quintic NLS equations and its soliton solutions by Hirota's bilinearization method. For deriving the Lax pair of operators for the 3-coupled cubic - quintic NLS equation, we consider the \((4 \times 4)\) linear eigen value problem as given in Eqs. (6.23). Here \(L\) and \(M\) are given by the \(4 \times 4\) matrices respectively as

\[
L = \begin{pmatrix}
-\iota \lambda & q_1 & q_2 & q_3 \\
-q_1^* & -\iota \theta_{1,x} + \iota \lambda & 0 & 0 \\
-q_2^* & 0 & -\iota \theta_{2,x} + \iota \lambda & 0 \\
-q_3^* & 0 & 0 & -\iota \theta_{3,x} + \iota \lambda \\
\end{pmatrix} \tag{7.17}
\]

and

\[
M = [M_{l,m}], \quad l, m = 1, 2, 3, 4, \tag{7.18}
\]

in which \(M_{l,m}\) are functions of \(q_n, q_n^*\ (n=1,2,3)\) and \(t\). Applying the compatibility condition \(V_{XT} = V_{TX}\) and equating the coefficients of \(V_1, V_2, V_3\) and \(V_4\) independently to zero with the assumption that \(\xi_t = 0\), we have from Eqs. (7.17)-(7.18) the following relations

\[
M_{1,1,X} = \sum_{n=1}^{3} [q_n M_{n+1,1} + q_n^* M_{1,n+1}], \tag{7.19}
\]

\[
M_{1,j,X} = \sum_{n=1}^{3} [q_{j-1,T} + q_n M_{n+1,j} - q_{j-1} M_{1,1} - 2i \lambda M_{1,j} + i \theta_{j-1,x} M_{1,j}], \tag{7.20}
\]

\[
M_{s,1,X} = \sum_{n=1}^{3} [-q_{s-1,T} + q_n^* M_{n+1} - q_{s-1}^* M_{1,1} + 2i \lambda M_{s,1} - i \theta_{s-1,x} M_{s,1}], \tag{7.21}
\]

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7.3 Lax pair Using AKNS formalism

\[
M_{s,j,X} = -q_{j-1}M_{s,1} - q_{s-1}^* M_{1,j} - i\theta_{j-1,xt} \quad \text{for} \quad s = j,
\]

\[
= -q_{j-1}M_{s,1} - q_{s-1}^* M_{1,j} + i(\theta_{j-1,x} - \theta_{s-1,x})M_{s,j} \quad \text{for} \quad s \neq j
\]

(7.22)

\[s, j = 2, 3, 4.\] We now expand \(M_{l,m}\) in powers of \(\lambda\) as

\[
M_{l,m} = \sum_{n=0}^{3} \lambda^n M_{l,m}^n.
\]

(7.23)

Substituting Eq. (7.23) in Eqs. (7.19)-(7.23) and equating the coefficients of like powers of \(\lambda\), we obtain the following results:

\[
M_{1,1} = -2i\lambda^2 + i(|q_1|^2 + |q_2|^2 + |q_3|^2),
\]

(7.24)

\[
M_{1,j} = 2q_j - 1\lambda + \theta_{j-1,x} q_{j-1} + iq_{j-1,x},
\]

(7.25)

\[
M_{s,1} = -2q_{s-1}^* \lambda - \theta_{s-1,x} q_{s-1}^* + iq_{s-1,x},
\]

(7.26)

\[
M_{s,j} = 2i\lambda^2 - i|q_{j-1}|^2 - i\theta_{j-1,t}, \quad \text{for} \quad s = j,
\]

\[
= -iq_{s-1}^* q_{j-1} \quad \text{for} \quad s \neq j
\]

(7.27)

with \(\theta_{1,x} = \rho_1|q_1|^2 + \rho_2|q_2|^2 + \rho_3|q_3|^2, \quad \theta_{2,x} = \tau_1|q_1|^2 + \tau_2|q_2|^2 + \tau_3|q_3|^2, \quad \theta_{3,x} = \zeta_1|q_1|^2 + \zeta_2|q_2|^2 + \zeta_3|q_3|^2\). Also the compatibility condition reduces to a set of completely integrable three coupled cubic - quintic NLS equations. Once the Lax pair of operators are known, we can construct the multisoliton solutions of Eqs. (7.14)-(7.16).
7.4 Solitary Wave Solution

Based on Darboux transformation method we derive the soliton solutions for the set of three coupled integrable cubic-quintic NLS equation. To proceed, we introduce the following transformation

\[ \hat{V} = DV = (\lambda I - S)V; \]  

(7.28)

where \( D \) is the Darboux matrix and \( I \) is a 4x4 identity matrix. From Eqs. (6.23) and (7.28), the N-soliton solutions for the set of three coupled integrable cubic-quintic NLS equation are expressed as

\[ q^n_1 = q_1 + 2i(\Gamma_1)_{12}, \]  

(7.29)

\[ q^n_2 = q_2 + 2i(\Gamma_1)_{13}, \]  

(7.30)

\[ q^n_3 = q_3 + 2i(\Gamma_1)_{14}. \]  

(7.31)

Here \( \Gamma_1 \) is defined as

\[ (\Gamma_1)_{pq} = \frac{\det(M_q^{(p)})}{\det(W_n)}, \quad (1 \leq p \leq 4), \]  

(7.32)

where \( M_q^{(p)} \) can be obtained by replacing the \( q \)-th row of \( W_n \) by the \( p \)-th row of \( B \). In order to generate the soliton solution, we take \( q_1 = q_2 = q_3 = 0 \) as the trivial solution. Hence the basic solution is

\[ h = \begin{pmatrix} c_{11}e^{-i\lambda x - 2i\lambda^2 T} \\ c_{21}e^{i\lambda x + 2i\lambda^2 T} \\ c_{31}e^{i\lambda x + 2i\lambda^2 T} \\ c_{41}e^{i\lambda x + 2i\lambda^2 T} \end{pmatrix} \]
where \( c_{11}, c_{21}, c_{31} \) and \( c_{41} \) are all arbitrary constants. For \( n = 1 \), Eqs. (7.14)-(7.16) give the one soliton solution

\[
q_1^{(1)} = 2\left(\frac{c_{11}}{c_{11}}\right)\lambda_1 Sech[2\lambda_1 X + 8\lambda_R \lambda_1 T]exp\left(-2i\lambda_R X + 4i(\lambda_1^2 - \lambda_R^2)T\right),
\]
(7.33)

\[
q_2^{(1)} = 2\left(\frac{c_{31}}{c_{11}}\right)\lambda_1 Sech[2\lambda_1 X + 8\lambda_R \lambda_1 T]exp\left(-2i\lambda_R X + 4i(\lambda_1^2 - \lambda_R^2)T\right),
\]
(7.34)

\[
q_3^{(1)} = 2\left(\frac{c_{41}}{c_{11}}\right)\lambda_1 Sech[2\lambda_1 X + 8\lambda_R \lambda_1 T]exp\left(-2i\lambda_R X + 4i(\lambda_1^2 - \lambda_R^2)T\right).
\]
(7.35)

This also governs the solitary wave solution of the well known Manakov model. The possibility of the interrelation between Manakov model and the cubic - quintic model was established in [69] which are given by

\[
q_1 = q_{1,M}e^{i\int (\rho_1|q_{1,M}|^2 + \rho_2|q_{2,M}|^2 + \rho_3|q_{3,M}|^2)dt},
\]
(7.36)

\[
q_2 = q_{2,M}e^{i\int (\tau_1|q_{1,M}|^2 + \tau_2|q_{2,M}|^2 + \tau_3|q_{3,M}|^2)dt},
\]
(7.37)

\[
q_3 = q_{3,M}e^{i\int (\zeta_1|q_{1,M}|^2 + \zeta_2|q_{2,M}|^2 + \zeta_3|q_{3,M}|^2)dt},
\]
(7.38)

where \( q_{1,M}, q_{2,M} \) and \( q_{3,M} \) are the exact solution of Manakov model [87]. From these relation we get the solitary wave solution for our resultant integrable three coupled cubic-quintic NLS equations which are plotted in the Figure 7.1. Proceeding in a similar way, one can construct the multisoliton solutions of our proposed model.

## 7.5 Conclusion

In this chapter we have analyzed the formation of localized modes in three spine alpha helical protein with higher order nonlinear couplings by proposing a model...
Figure 7.1: Space-time plot of solitary wave solution \( (a)|q_1|^2 \), \( (b)|q_2|^2 \) and \( (c)|q_3|^2 \) for the three coupled cubic - quintic NLS equation at \( t = 0.5 \text{sec} \).

Hamiltonian which includes internal molecular excitations, dipole - dipole interaction and nonlinear coupling between the internal molecular excitations with the displacements. In addition, we have included the quadrupole - quadrupole type interaction between the adjacent unit cells. As we are interested in the higher order behaviour of the system, we have also incorporated the nonlinear couplings which gives rise to change in energy of the amide -I bond at different levels caused by the stretching of the helix between two nearest neighbours as well as next nearest neighbour unit cells. In order to maintain the localized excitations along the hydrogen bonding spines in the alpha helical proteins with interspine coupling, new higher order nonlinear couplings between higher order molecular excitations and displacements have also been included. Under coherent representation, in the continuum limit, the collective excitations are found to be governed by a set of perturbed three coupled NLS equations. Specifically, we
look for integrable models so that the elementary excitations can be represented in terms of soliton modes and we were able to identify an integrable model for specific choice of parameters. The integrable model proposed here also leads to a set of integrable three coupled cubic - quintic NLS equations which we have derived through AKNS formalism. We have obtained the linear eigen value problem (Lax pair) for the three coupled cubic - quintic NLS equations which may be useful to solve them for multisolitons using IST technique and to obtain the integrability properties. We have constructed the solitary wave solutions using Darboux transformation technique. The plot of $|a_j|^2, j = 1, 2, 3$ shows that the presence of cubic-quintic nonlinearity modifies only the amplitude of the soliton but does not affect its velocity.